Math Q & Soln Ed Times (3) 1

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Application of this result to the second part of the question gives relations of form

$$y_1 = \eta_1 + P_1 \sin(pt+q_1),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$y_n = \eta_n + P_n \sin(pt+q_n).$$

Hence $\bar{y} = 1/n\, \Sigma y_r = Y + P\sin(pt+q)$ .... (2),

where $Y = \bar{\eta}$.

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The result can be generalized. For, if ξ_r varies periodically, though not simple harmonically, about the value $\xi_r = 0$, we may write

$$\xi_1 = \sum_{s=1}^{\infty} (A_s \sin spt + A'_s \cos spt),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\xi_r = \sum_{s=1}^{\infty} (R_s \sin spt + R'_s \cos spt),$$

where p and the coefficients A_s, A'_s, ... are determined from particular circumstances. If, then, $y = f(x_1, \ldots, x_n)$ and $b_r \equiv \partial f/\partial a_r$,

therefore
$$y = y_0 + b_1 \sum_{s=1}^{\infty} (A_s \sin spt + A'_s \cos spt) + \ldots$$
$$+ b_r \sum_{s=1}^{\infty} (R_s \sin spt + R'_s \cos spt) + \ldots$$
$$= y_0 + \sum_{s=1}^{\infty} \{(b_1 A_s + \ldots + b_r R_s + \ldots) \sin spt$$
$$+ (b_1 A'_s + \ldots + b_r R'_s + \ldots) \cos spt\} \quad \ldots\ldots \ (4),$$

or $y = y_0 + Y$, where Y is a periodic function given by the Fourier series on the right-hand side of (4). It can also be shown that y is periodic if $x_1, \ldots, x_n$ are periodic, whether or not $\xi_1, \ldots, \xi_n$ are small.

Approximations to certain Square Roots and the Series of Numbers connected therewith.

By ALBERT TARN, B.Sc. Lond.

If P = the sum of the positive terms in the expansion of $(x-y)^n$ and Q that of the negative, so that $(x-y)^n = P - Q$ and $(x+y)^n = P + Q$, it follows that

$$\frac{Q}{P} = \frac{(x+y)^n - (x-y)^n}{(x+y)^n + (x-y)^n} = \frac{1 - [(x-y)/(x+y)]^n}{1 + [(x-y)/(x+y)]^n},$$

which, as n increases, tends to unity as limiting value.

Now, if we expand $(\sqrt{x}+1)^n$ or $(\sqrt{x}-1)^n$, we get the terms alternately rational and irrational, and we can write the results as

$$(\sqrt{x}+1)^n = a\sqrt{x} + b,$$

and
$$(\sqrt{x}-1)^n = a\sqrt{x} - b \quad \text{or} \quad b - a\sqrt{x},$$

according as n is odd or even. In either case, our previous reasoning shows that $b/(a\sqrt{x})$ or $(a\sqrt{x})/b$ tends towards unity as n increases, or b/a tends towards the value of $\sqrt{x}$. If, therefore, we

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can easily determine a and b, we can by this means obtain a series of approximations to $\sqrt{x}$.

Writing these approximations in the form b/a, we can, in the case of certain square roots, obtain successive values of b and a with comparative facility, and the series of numbers so obtained possess many remarkable relations. When, however, x is an odd number, the values of a and b are all divisible by powers of 2, and it is then desirable to reduce them to the simplest ratio. Such series we shall therefore speak of as "reduced series."

Also, $(\sqrt{2}-1)^n$ and $(\sqrt{3}-1)^n$ both tend to become infinitesimal as n increases, so that in these cases $a\sqrt{x}$ and b must tend to actual equality,* and therefore $(\sqrt{2}+1)^n$ and $(\sqrt{3}+1)^n$ must tend to $2b_n$, or a whole number, where b_n is the nth term of the *unreduced* series.

The series of values for a and b for certain surds are given below, a_n and b_n being the nth terms of the two series.

√2 Series.—

a	1, 2, 5, 12, 29, 70, 169, 408, 985, ... ✓
b	1, 3, 7, 17, 41, 99, 239, 577, 1393, ... 1333

These terms are connected by the relations

$$a_n = 2a_{n-1} + a_{n-2}, \quad b_n = 2b_{n-1} + b_{n-2}.$$

The ratio b/a is alternately less and greater than $\sqrt{2}$, and the 9th terms give a result agreeing with $\sqrt{2}$ to the 6th place of decimals.

The ratios of the various terms of both series to the preceding terms approximate to $\sqrt{2}+1$; thus both $\frac{985}{408}$ and $\frac{1393}{577}$ agree with $\sqrt{2}+1$ to 5 places of decimals.

By doubling the terms of the b series, we obtain approximations to $(\sqrt{2}+1)^n$. Thus, taking the 8th term, we have

$$2 \times 577 = 1154,$$

and
$$(\sqrt{2}+1)^8 = 577 + 408\sqrt{2} = 1153{\cdot}999 \text{ to 3 places.}$$

It will be noticed that $a_n = a_{n-1} + b_{n-1}$. In the above series, however, we have also the relation

$$b_n = a_n + a_{n-1} \quad \text{and} \quad b_{2n} = (2a_n)^2 + (-1)^n.$$

Also
$$b_n^2 = 2a_n^2 + (-1)^n;$$

whence also
$$b_{2n} = 2b_n^2 + (-1)^{n-1} \quad \text{and} \quad \sum_1^n b_n = a_{n+1} - 1.$$

3 *Series.*—The values of a and b for these series, beginning with the second, being all divisible by powers of 2, we give the reduced series, indicating above them the factors by which they must be multiplied in order to give the values of $(\sqrt{3}+1)^n$.

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* This is not implied in the statement that $b/(a\sqrt{x})$ or $(a\sqrt{x})/b$ tends towards unity as n increases.

Factor	2 4 8 16 32	
a	1, 1, 3, 4, 11, 15, 41, 56, 153, 209, ...	← 2530
b	1, 2, 5, 7, 19, 26, 71, 97, 265, 362, ...	← 2531

Thus $(\sqrt{3}+1)^3 = 10+6\sqrt{3}$, $(\sqrt{3}+1)^8 = 1552+896\sqrt{3}$.

We note that

$$a_{2n} = a_{2n-1}+a_{2n-2}, \qquad b_{2n} = b_{2n-1}+b_{2n-2};$$

and

$$a_{2n+1} = 2a_{2n}+a_{2n-1}, \qquad b_{2n+1} = 2b_{2n}+b_{2n-1}:$$

and approximations to $\sqrt{3}$ are obtained by dividing terms of the b series by corresponding terms of the a series. Thus

$$\tfrac{362}{209} = 1{\cdot}73206 \text{ nearly.}$$

Again, doubling the terms of the b series and multiplying by the factor, we get approximations to $(\sqrt{3}+1)^n$. Thus

$$2\times 362\times 32 = 23168,$$

and $\quad (\sqrt{3}+1)^{10} = 32(362+209\sqrt{3}) = 23167{\cdot}956.$

The approximation is not so close as in the $\sqrt{2}$ series, because $\sqrt{3}-1$ is a larger fraction than $\sqrt{2}-1$.

The following relations between terms of the two series may be noted: $\quad b_{2n} = a_{2n}+a_{2n-1}, \quad b_{2n+1} = a_{2n+1}+2a_{2n}$.

$\sqrt{5}$ Series.—The two series of numbers associated with this surd are by far the most remarkable and interesting, but the series given below are reduced, and

$$b_n + a_n\sqrt{5} = 2\times[\tfrac{1}{2}(\sqrt{5}+1)]^n.$$

The series are as follows:—

a	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...	✓
b	1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...	✓

and are very easily written out, since each term beginning with the 3rd is the sum of the two preceding.

This is also characteristic of successive powers of $\frac{1}{2}(\sqrt{5}+1)$, since it is a root of the equation $x^2 = x+1$, and, therefore, of the equation $x^n = x^{n-1}+x^{n-2}$.

Thus $\quad [\tfrac{1}{2}(\sqrt{5}+1)]^2 = \tfrac{1}{2}(\sqrt{5}+1)+1 = \tfrac{1}{2}(3+\sqrt{5}).$

$\quad [\tfrac{1}{2}(\sqrt{5}+1)]^3 = \tfrac{1}{2}(3+\sqrt{5})+\tfrac{1}{2}(\sqrt{5}+1) = \tfrac{1}{2}(4+2\sqrt{5})$, &c.

Further, since $\frac{1}{2}(\sqrt{5}-1)$ is a fraction less than unity, and $-\frac{1}{2}(\sqrt{5}-1)$ is also a root of the same equation, so that successive powers of this quantity are obtained in the same fashion, it readily follows that b_n and $a_n\sqrt{5}$ must tend to equality. Hence $[\frac{1}{2}(\sqrt{5}+1)]^n$ being equal to $\frac{1}{2}(b_n+a_n\sqrt{5})$, it follows that b_n itself must be an approximation to the value of $[\frac{1}{2}(\sqrt{5}+1)]^n$.

Thus $\quad b_{10} = 123$

and $\quad [\tfrac{1}{2}(\sqrt{5}+1)]^{10} = \tfrac{1}{2}(123+55\sqrt{5}) = 122{\cdot}992.$

There are many peculiar relations between the terms of these two series, *i.e.*,

$$(1)\quad b_n = a_{n-1}+a_{n+1}; \qquad (2)\quad a_n = \tfrac{6}{5}b_{n-2}-\tfrac{1}{5}b_{n-4}.$$

Thus $\quad 55 = \frac{6}{5}$ of $47-\frac{1}{5}$ of $7 = \frac{1}{5}(282-7) = \frac{275}{5}$;

and, in connexion with this, we obtain a very close approximation to the nth term of the a series, *i.e.*,

$$\tfrac{6}{5}[\tfrac{1}{2}(\sqrt{5}+1)]^{n-2}-\tfrac{1}{5}[\tfrac{1}{2}(\sqrt{5}+1)]^{n-6}.$$

Thus, applied to determine a_{10}, we get 55·0036.

$$(3)\quad \sum_1^n a_n+1 = a_{n+2}, \qquad \sum_1^n b_n+3 = b_{n+2}.$$

Hence, approximately,

$$\sum_1^n a_n = \tfrac{6}{5}[\tfrac{1}{2}(\sqrt{5}+1)]^{n}-\tfrac{1}{5}[\tfrac{1}{2}(\sqrt{5}+1)]^{n-4}-1.$$

$$(4)\quad a_{2n+1} = a_{n+1}^2+a_n^2, \quad a_{2n} = a_{n+1}^2-a_{n-1}^2, \quad a_{2n} = a_nb_n,$$

and $\quad a_{2n+1} = a_nb_{n+1}+(-1)^n.$

Further, $\quad (5)\quad b_{2n} = b_n^2+(-1)^{n-1}\times 2.$

Thus $\quad b_{26} = 521^2+2.$

It will be found that b_{26}/a_{26} agrees with $\sqrt{5}$ very closely, *i.e.*, to 10 places of decimals.

$$(6)\quad a_n^2 = a_{n-1}.a_{n+1}+(-1)^{n-1}, \quad b_n^2 = b_{n-1}.b_{n+1}+(-1)^n\times 5,$$

and, finally, $\quad b_n^2 = 5a_n^2+(-1)^n\,4.$

Thus the a series approximates very closely to a geometrical progression, the common ratio being $\frac{1}{2}(\sqrt{5}+1)$; yet the b series gives the actual approximations to $[\frac{1}{2}(\sqrt{5}+1)]^n$, the differences between b_n and $[\frac{1}{2}(\sqrt{5}+1)]^n$ to 3 places of decimals curiously involving terms of the a series in the decimal figures.

Series for $\sqrt{6}$ and $\sqrt{10}$, *inter alia*, may be written out as follows:—

$\sqrt{6}$.

a	1, 2, 9, 28, 101, 342, 1189, 4088, ...	← 2532
b	1, 7, 19, 73, 241, 847, 2899, 10033, ...	← 2533

where $\quad a_n = 2a_{n-1}+5a_{n-2}, \quad b_n = 2b_{n-1}+5b_{n-2}.$

Also $\quad a_n = a_{n-1}\cdot b_{n-1}, \quad a_{2n} = 2a_nb_n, \quad b_n = a_n+5a_{n-1}.$

$\sqrt{10}$.

a	1, 2, 13, 44, 205, 806, 3457, ...	← 2534
b	1, 11, 31, 161, 601, 2651, 10711, ...	← 2535

in which $\quad a_n = a_{n-1}+b_{n-1}, \quad b_n = b_{n-1}+10a_{n-1}.$

Also $\quad a_{2n} = 2a_nb_n, \quad b_n = a_n+9a_{n-1}.$

The approximations, however, as x increases, are less close than those of the above series.

$\sqrt{7}$ *Series.*—

a	1, 1, 5, 8, 31, 55, 203, 368, 1345, 2449 ← 2536
b	1, 4, 11, 23, 79, 148, 533, 977, 3553, 6484 ← 2537

X

cont →

a	8933, 16280, 59359, 108199, ...
b	23627, 43079, 157039, 286276, ...

Relations:

$$a^{2n+1} = 2a_{2n} + 3a_{2n-1}\,; \quad a_{2n} = a_{2n-1} + 3a_{2n-2}\,;$$
$$b_{2n+1} = 2b_{2n} + 3b_{2n-1}\,; \quad b_{2n} = b_{2n-1} + 3b_{2n-2}.$$

18076. (W. N. Bailey.)—The internal bisector of the angle between the tangents from a point P to a conic passes through a fixed point A. Show that the locus of P is the cubic which passes through A, the foci of the conic, the feet of the perpendiculars from A on the axes, and the feet of the normals from A. Show also that A is a double point of the cubic, the tangents there being at right angles, and that the asymptote is parallel to the line joining A to the centre of the conic. Sketch the curve.

Solutions (I) *by* Prof. J. Nanson ; (II) *by* C. E. Wright.

(I) The line AP is clearly a tangent at P to one of the two confocals through P. We therefore require the locus of the points of contact of tangents from A to the confocals. That this locus is a cubic with a double point at A is readily seen. For *one* confocal can be drawn to touch an arbitrary line through A, and the point of contact can be *at* A only when the arbitrary line touches at A one of the *two* confocals through A. The cubic locus is clearly also the locus of the feet of normals from A to the confocals, and therefore obviously passes through *all* the points mentioned. A sketch of the curve is given in the solution of Quest. 17853, *Educational Times*, September, 1915.

(II) The tangents drawn from (x, y) to the conic (ellipse) $x^2/a^2 + y^2/b^2 = 1$ make angles γ_1, γ_2 with the x-axis given by

$$\tan\gamma_1 + \tan\gamma_2 = \frac{-2xy}{a^2 - x^2}, \qquad \tan\gamma_1\tan\gamma_2 = \frac{b^2 - y^2}{a^2 - x^2}.$$

If θ is the slope of the internal bisector of the angle between them,

$$\tan 2\theta = \frac{-2xy}{(a^2 - b^2) - (x^2 - y^2)}.$$

If this bisector passes through A (h, k),

$$\tan\theta = (y - k)/(x - h)\,;$$

and therefore $\dfrac{2(y-k)(x-h)}{(x-h)^2 - (y-k)^2} = \dfrac{-2xy}{(a^2-b^2) - (x^2-y^2)}$,

or $(x-h)(y-k)\{a^2 - b^2 - (x^2 - y^2)\} + xy\{(x-h)^2 - (y-k)^2\} = 0.$

This is the equation of the required locus; it clearly reduces to the third degree in x, y, and passes along through the points (h, k), $(h, 0)$, $(0, k)$, $[\pm\sqrt{(a^2 - b^2)}, 0]$. By definition of the locus, it must pass through the feet of the normals from A to the conic: this can also be verified from the equation. The terms of the third degree are $(kx - hy)(x^2 - y^2)$, and the asymptote is easily shown to be

$$y/k - x/h = (a^2 - b^2)/(h^2 + k^2).$$

Referred to A as origin, the terms of the second degree become $xy\,(a^2 - b^2 - h^2 + k^2) + hk\,(x^2 - y^2)$, and hence the nodal tangents are at right angles.

If either h or k is zero, the locus reduces to one of the axes, and a circle (the above cubic is circular).

When $h = 0$, the locus is $x = 0$, and the circle

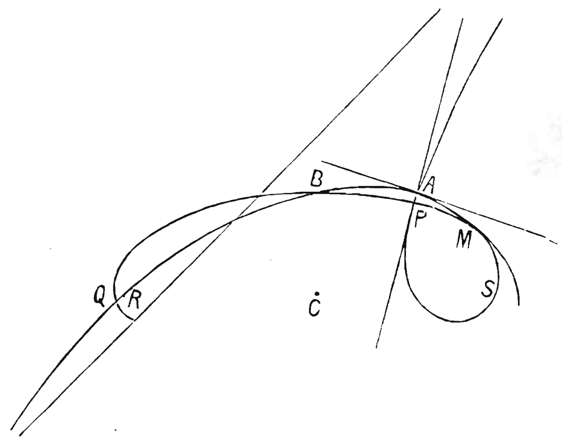
$$kr^2 + y\,(c^2 - k^2) - kc^2 = 0$$
$$(\text{where } r^2 = x^2 + y^2,\ c^2 = a^2 - b^2).$$

As k varies, these circles envelope the foci, as is clear from the equation; or by forming the envelope, which becomes

$$r = \pm c e^{\pm i\theta},$$

of which the only real points are $r = \pm c$, $\theta = 0$.

A rough sketch of the cubic in a particular case is appended.



$a = 2$, $b = 1$, $h = k = 1$.

P, Q, feet of normals from A (only two being real).

R, S, foci. B, M, other intersections of conic and cubic.

18082. (W. F. Beard, M.A. Suggested by Question 18027.)—If the sides of a triangle reflect the opposite corners on to a straight line, the nine-point centre lies on the circum-circle.