COLLEGIATE ARTICLES

Papers Whose Reading Does Not Presuppose Graduate Training

KUMMER NUMBERS

by P. A. Piza

Let ${}_nD_c$ be integers defined by the following relations:

$${}_nD_1 = 1. \qquad {}_nD_{c>n} = 0. \qquad {}_nD_{c<1} = 0. \qquad {}_nD_c = c({}_{n-1}D_c + {}_{n-1}D_{c-1}).$$

A short table of D-numbers is as follows:

n \ c	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1)	1									
(2)	1	2								
(3)	1	6	6							
(4)	1	14	36	24						
(5)	1	30	150	240	120					
(6)	1	62	540	1560	1800	720				
(7)	1	126	1806	8400	16800	15120	5040			
(8)	1	254	5796	40824	126000	191520	141120	40320		
(9)	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
(10)	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

D-numbers have been found by the author to possess remarkable properties, among them the following:

$$x^t - (x-1)^t + \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x-1}{\alpha-1}.$$

(1)..
$$x^t = \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x}{\alpha}.$$

$$\sum_{\beta=1}^{x} \beta^t = \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{x+1}{\alpha+1}.$$

With this last formula it is possible to compute the sum of the tenth powers of the first one thousand integers, it being the 32-digit number

91,409,924,241,424,243,424,241,924,242,500

in a simpler manner than was done by Jacob Bernoulli in ARS CONJECTAND I with the first use that he made of his famous fractionary Bernoullian Numbers*.

The development of

$$\sum_{\beta=1}^{1000} \beta^{10} = \sum_{\alpha=1}^{10} \binom{1001}{\alpha+1} {}_{10}D_\alpha$$

is as follows:

*See SOURCE BOOK OF MATHEMATICS page 90.

1	$\binom{1001}{2}$ =	500,500
+1022	$\binom{1001}{3}$	170,333,163,000
+55980	$\binom{1001}{4}$	2,327,832,672,165,000
+818520	$\binom{1001}{5}$	6,786,929,139,064,074,000
+5103000	$\binom{1001}{6}$	7,023,889,581,003,395,100,000
+16435440	$\binom{1001}{7}$	3,215,573,813,620,290,802,680,000
+29635200	$\binom{1001}{8}$	720,412,779,053,311,333,189,200,000
+30240000	$\binom{1001}{9}$	81,107,697,233,553,078,668,580,000,000
+16329600	$\binom{1001}{10}$	4,344,777,125,406,971,318,118,493,440,000
+3628800	$\binom{1001}{11}$	86,983,315,783,400,173,257,685,393,920,000
		91,409,924,241,424,243,424,241,924,242,500

Professor Jekuthiel Ginsburg detected in these D-coefficients the following relation:

(2).. $\quad {}_nD_c = c!\,{}_{n-1}T_{n-c}$,

where the ${}_{n-1}T_{n-c}$ are the Stirling Numbers of the second kind.

The object of this paper is to present another family of very interesting numbers derivable from D-coefficients, which family of numbers Prof. Oystein Ore has called Kummer Numbers in a recent letter to the writer, because he has observed that they happen to be the coefficients of the Kummer polynomials $P_i(x,y)$.*

Let us consider the following numerical instance of (1):

$$5^5 = 1\binom{5}{4} + 30\binom{5}{3} + 150\binom{5}{2} + 240\binom{5}{1} + 120\binom{5}{0}.$$

By repeated application of the rule of formation of binomial coefficients $\binom{n}{c}=\binom{n-1}{c}+\binom{n-1}{c-1}$, the above numerical function can be successively expressed as follows:

$$5^5 = 1\binom{5}{4} + 1\binom{5}{3} + 29\binom{5}{2} + 121\binom{5}{1} + 119\binom{5}{0}$$
$$+ 29\binom{5}{3} + 121\binom{5}{2} + 119\binom{5}{1} + 1\binom{5}{0}$$
$$= 1\binom{6}{4} + 29\binom{6}{3} + 121\binom{6}{2} + 119\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{6}{4} + 1\binom{6}{3} + 28\binom{6}{2} + 93\binom{6}{1}$$
$$+ 28\binom{6}{3} + 93\binom{6}{2} + 26\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{7}{4} + 28\binom{7}{3} + 93\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{7}{4} + 1\binom{7}{3} + 27\binom{7}{2}$$
$$+ 27\binom{7}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{8}{4} + 27\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{8}{4} + 1\binom{8}{3}$$
$$+ 26\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}$$
$$= 1\binom{9}{4} + 26\binom{8}{3} + 66\binom{7}{2} + 26\binom{6}{1} + 1\binom{5}{0}.$$
$$5^5 = 1\binom{9}{5} + 26\binom{8}{5} + 66\binom{7}{5} + 26\binom{6}{5} + 1\binom{5}{5}.$$

*See Dickson's History of the Theory of Numbers', volume II, pages 741 and 7– citations 76 and 180. This reference is due to Prof. Ore.

We have thus met the symmetrical set of Kummer Numbers 1, 26, 66, 26, 1, ich is a partition of 120 = 5!, in combined products with a *vertical* set of ljacent binomial coefficients, to obtain a partition of the fifth power of an teger.

By similar treatment of the consecutive rows of D-numbers, we get other mmetrical sets of these Kummer Numbers, one such set of t numbers for each ponent t, as follows:

For $t=1$: 1
For $t=2$: 1, 1
For $t=3$: 1, 4, 1
For $t=4$: 1, 11, 11, 1
For $t=5$: 1, 26, 66, 26, 1
For $t=6$: 1, 57, 302, 302, 57, 1.

Note that the sum of each set is t!
Kummer Numbers are defined by the following relations:

$$ {}_tK_1 = 1 = {}_tK_t \, . \qquad {}_tK_{c>t} = 0 = {}_tK_{c<1} \, . $$
$$ {}_tK_c = c \cdot {}_{t-1}K_c + (t+1-c) \cdot {}_{t-1}K_{c-1} \, . $$

A short table of Kummer Numbers is as follows:

t \ c	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1)	1									
(2)	1	1								
(3)	1	4	1							
(4)	1	11	11	1						
(5)	1	26	66	26	1					
(6)	1	57	302	302	57	1				
(7)	1	120	1191	2416	1191	120	1			
(8)	1	247	4293	15619	15619	4293	247	1		
(9)	1	502	14608	88234	156190	88234	14608	502	1	
(10)	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

295 460 498

Eulerian nos

As a consequence of the summation relations of D-coefficients, these Kummer bers have the following properties:

$$ x^t - (x-1)^t = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{x\ \ t-1-\alpha}{t-1} \, . $$
$$ x^t = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{x+t-\alpha}{t} \, . $$
$$ \sum_{\beta=1}^{x} \beta^t = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{x+t+1-\alpha}{t+1} \, . $$

$$\sum_{\beta=1}^{x} \sum_{\gamma=1}^{\beta} \gamma^t = \Sigma^2 x^t = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{x+t+2-\alpha}{t+2} ,$$

where the superscript 2 in Σ^2 means that there are to be two successive summations of the t-th powers of the first x integers.

In general, for m successive summations of the powers, we will have

$$(3).. \qquad \Sigma^m x^t = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{x+t+m-\alpha}{t+m} ,$$

where m may be any positive or negative integer or zero.

When $m = 0$:

$$\Sigma^0 x^t = x^t ,$$

and the superscript zero means that no summation nor subtraction is to be performed on x^t.

When $m = -1$:

$$\Sigma^{-1} x^t = x^t - (x-1)^t ,$$

where the superscript $m = -1$ means a negative summation or the difference between x^t and $(x-1)^t$.

Also

$$\Sigma^{-2} x^t = x^t - 2(x-1)^t + (x-2)^t ,$$

where the superscript -2 means the difference of the differences between x^t, $(x-1)^t$ and $(x-2)^t$.

$$\Sigma^{-3} x^t = x^t - 3(x-1)^t + 3(x-2)^t - (x-3)^t .$$

$$\Sigma^{-t} x^t = t!$$

Thus the successive sums and differences of the t-th powers of the integers up to x can be formulated in one general and concise notation and summation (3).

If the Fermat equation $x^t + y^t = z^t$ is ever possible in integers, it can be expressed in terms of our summations, as follows:

$$\sum_{\alpha=1}^{t} {}_tD_\alpha \left[\binom{x}{\alpha} + \binom{y}{\alpha} \right] = \sum_{\alpha=1}^{t} {}_tD_\alpha \binom{z}{\alpha} .$$

$$\sum_{\alpha=1}^{t} {}_tK_\alpha \left[\binom{x+t-\alpha}{t} + \binom{y+t-\alpha}{t} \right] = \sum_{\alpha=1}^{t} {}_tK_\alpha \binom{z+t-\alpha}{t} .$$

San Juan, Puerto Rico