Sum of cubes of the first "n" natural numbers

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We will demonstrate here the Nicomachus's theorem on the sum of the cubes of the first n natural numbers, using the manipulation of a three-dimensional geometric model.

Introduction

In the number theory, the sum of the first n cubes is given by the *square of the n-th triangular number*, that is,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2$$

This identity is sometimes called *Nicomachus's theorem*, named after the Greek mathematician of the Hellenistic age, Nicomachus of Gerasa, which proved it arithmetically. Many mathematicians have studied this equality, demonstrating it in many different ways. The idea of visually demonstrate the Nicomachus's identity is not new. Roger B. Nelsen, in his work *Proofs without Words* (1993) presents seven different versions. The advantage of visual demonstrations is to provide sometimes, as in the present work, a graphic evidence of the solution.

Proposition

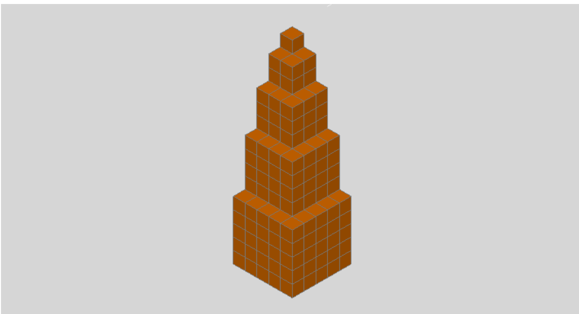
The sum of the cubes of the first n natural numbers is given by the square of the nth triangular number:

$$\sum_{k=1}^{n} k^3 = T_n^{\,2} = \left(\sum_{k=1}^{n} k \right)^2$$

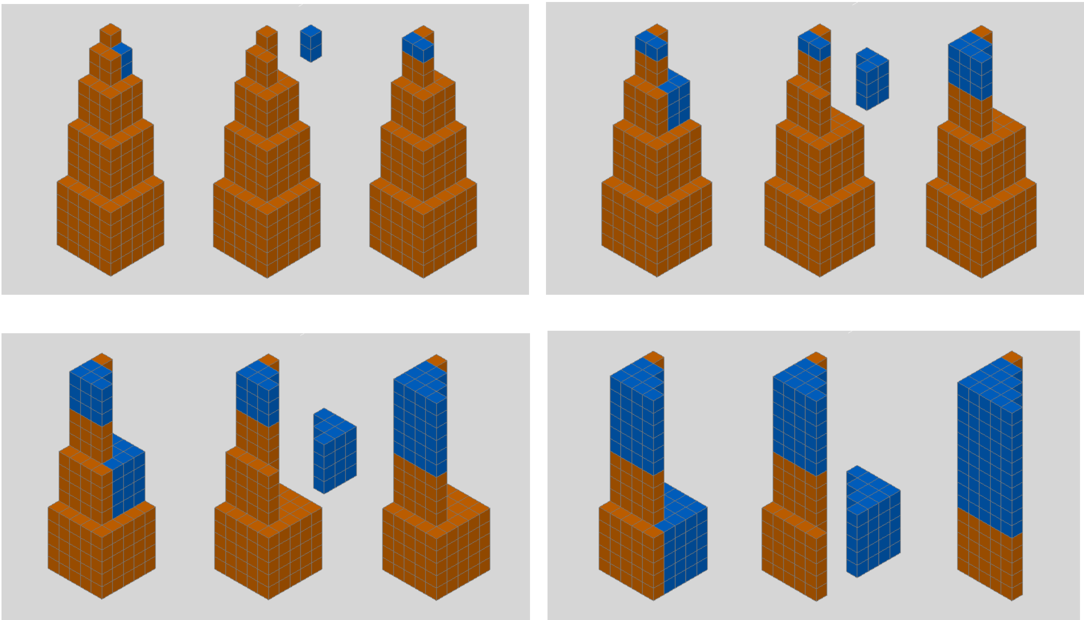
where the triangular number is the sum of the first n natural numbers.

Proof

We build a three-dimensional geometric model that represents the sum of the cubes of the first *5* natural numbers, using cubic bricks of unit volume, in the following way:

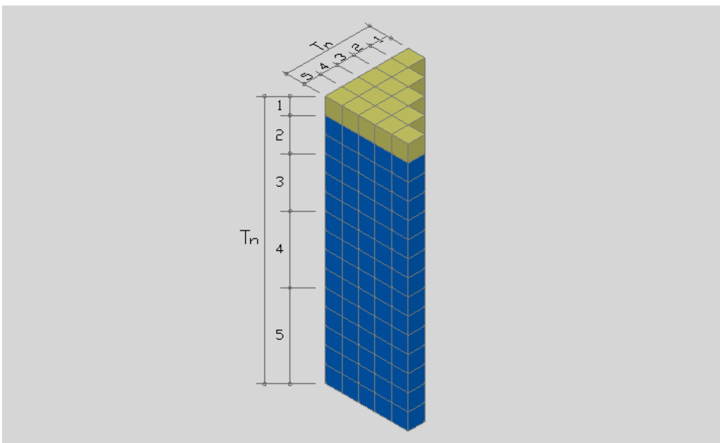


In an attempt to obtain a figure equivalent to this model, which gives evidence of the identity to prove, we operate on the model an inductive transformation, moving the unit cubes as follows:



The inductivity of the process lies in the fact that, in each cube of the sum, the unit cubes to move are neatly arranged in *1 +2 +3 + ... + (k-1)* columns, each of height *k*.

The final result of the transformation has been always, for any *n*, a pseudo-parallelepiped whose base is a geometrical representation of the triangular number *Tn*, and whose height is the number *Tn* itself, which remains unchanged during the transformation.



It is thus evident that, the total number of unit cubes, which gives the sum of the cubes of the first n natural numbers, is given by:

$$\sum_{k=1}^{n} k^3 = T_n{}^2 = \left(\sum_{k=1}^{n} k \right)^2$$

that is, the identity that one wanted to prove.

Links

1 - Animated proof: http://youtu.be/XM2plLoJkRk

2 - http://en.wikipedia.org/wiki/Squared_triangular_number

References

Nelsen, Roger B. (1993), *Proofs without Words*, Cambridge University