A000537, Wolfdieter Lang, Jan 12 2013
Ibn al-Haytham's way to prove a recurrence for the sums of the k-th power of the first positive integers S(k,n) := sum(j^k,j=1..n).
See the H. K. Strick reference given in A000537, p. 13.
Here written for the fourth power (k=4), but one can take any power k>=1 instead.

Start with

```
sum(j^3,j=1..n)*(n + 1) =

(1^3 + 2^3 + …  + n^3) (n + 1) =

1^3*( 1 + 1 + 1 + 1 + … +  1)   [there are n+1 1s in the bracket]
    +2^3*(2 + 1 + 1 + … +  1)   [there are n-1 1s in the bracket]
        +3^3*(3 + 1 + … +  1)   [there are n-2 1s in the bracket]
…
…
       +(n-1)^3*((n-1) +1 +1)
                  +n^3*(n +1)
```

Now sum the diagonals. In the first diagonal put 1^3*1 = 1^4 in the others put 1^3*1=1^3.

```
= 1^3*1 + 2^3*2 + 3^3*3 + … + n^3*n    [sum(j^4,j=1..n) = S(4,n)]
+ 1^3*1 + 2^3*1 + 3^3*1 + … + n^3*1    [sum(j^3,j=1..n) = S(3,n)]
+ 1^3*1 + 2^3*1 + … +(n-1)^3*1         [sum(j^3,j=1..n-1)=S(3,n-1)]
…
…
+ 1^3*1 + 2^3*1                        [sum(j^3,j=1..2) = S(3,2)]
+ 1^3*1                                [sum(j^3,j=1..1) = S(3,1)]

= S(4,n) + sum(S(3,m),m=1..n).
```

Therefore: S(4,n) = S(3,n)*(n+1) – sum(S(3,m),m=1..n), n >= 1.
One can cancel the S(3,n) terms:

```
            S(4,n) = S(3,n)*n – sum(S(3,m),m=1..n-1), n >= 1,
```

where the undefined sum for n=1 is put to 0.

As mentioned above, one can take any power k >= 1 in this calculation, thus

```
S(k,n) = S(k-1,n)*(n+1) – sum(S(k-1,m),m=1..n)
       = S(k-1,n)*n – sum(S(k-1,m),m=1..n-1),  n>=1, k>=1.
```

--------------------------- e.o.f. -----------------------------