Channel Reconstruction via Quadratic Programming in Massive MIMO Networks

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***Abstract*—We consider the problem of channel reconstruction in FDD networks wherein each base-station (BS) employs a transmit array comprising of a multitude of antenna elements and simultaneously serves multiple different users (a.k.a. Massive MIMO networks). In channel reconstruction the BS seeks to reconstruct the instantaneous true channel seen by each user as accurately as possible, which is both critical and challenging. It is challenging since the true high-dimensional instantaneous channel must be recovered by the BS from quantized low-dimensional observations while fully exploiting other available side information. It is a critical problem in that the viability of FDD Massive MIMO directly depends on whether an effective implementable reconstruction scheme can be found. We propose a unified framework for channel reconstruction that combines instantaneous quantized feedback, long-term statistical subspace information as well other auxiliary estimates. Interestingly, the resulting problem is an NP-hard non-convex quadratically constrained quadratic programming (QCQP) problem that has received wide attention in diverse areas but hitherto lacks an efficient algorithm that meets our required stringent complexity limits. We propose a novel approach based on the K-best methodology that is well suited for implementation and demonstrate that it offers a superior performance and complexity tradeoff.**

I. INTRODUCTION

The advent of Massive-MIMO has galvanized downlink (DL) MU-MIMO which in current 4G cellular systems has failed to realize its potential. Massive MIMO with its larger antenna aperture size promises orders of magnitude improvement in spectral efficiency [1] provided timely channel state information (CSI) can be accurately obtained at each base-station (BS). The latter requirement is feasible over TDD networks that can leverage channel reciprocity but is onerous for the more prevalent FDD networks. Indeed, over many scenarios the loss due to inaccurate CSI in FDD massive MIMO systems can be very significant [2]. Consequently, realizing massive MIMO benefits in FDD or more generally with limited training observations has become an important area of research [3], [4].

The fundamental problem in acquiring CSI in FDD networks is that of provisioning resources for training and feedback. Since each user's DL channel is high-dimensional due to the size of the BS array as well as dynamic, a naive approach of periodically sending an orthogonal pilot through each antenna will consume unaffordable overhead (48% for an array size of 64 [5]). Consequently pre-beamformed pilot transmission has been standardized under which each user observes and estimates one or more low dimesional projections of its channel. These estimates are then separately quantized and reported back to the BS. In this context, we note that recovery of high-dimensional vector(s) (or a common support) from multiple low dimensional noisy (albeit unquantized) observations is closely related to the multiple measurement vectors problem that is actively being studied under compressive sensing. Prominent works in this area typically adopt a grid-based approach with a finite dictionary (of structured atoms) and model the unknown vector(s) as a linear combination of a small number of atoms. However, incomplete knowledge of the dictionary can be quite detrimental [6]. On the other hand, gridless approaches assume a continuum of atoms with all atoms again having an amenable structure [7]. The latter assumption can only be justified for uniform arrays. We note here that a non-negligible fraction of massive MIMO deployments will use irregular arrays (due to a host of reasons such as form factor, realized patterns etc.). Consequently, there is a need for reconstruction algorithms that do not pre-suppose certain array geometries.

In this paper we consider a practical Massive MIMO FDD downlink and propose a novel formulation for channel reconstruction that can utilize disparate feedback and side-information such as fine time-scale quantized feedback as well as coarse time-scale subspace information. It applies to arbitrary array configurations and entails minimizing a weighted sum of squared ℓ_2 norms subject to quadratic non-convex constraints. Our problem thus belongs to the class of non-convex QCQP that is an active research area [8], [9]. Other important features of our work are as follows.

- We show that our non-convex optimization can be efficiently and optimally solved in certain important special cases.
- We propose an effective sub-optimal approach to address the general case of our problem, based on the K-best methodology to decompose the original problem into smaller sub-problems. Here K is a tunable parameter that controls the performance and complexity tradeoff. We note that K-best methodology has been used with considerable success in multiantenna multi-user detection problems [10]. A key aspect of our K-best based approach is that closed form solutions are derived to optimally solve the constituent non-convex sub-problems. We believe our K-best based approach will also be useful in other applications involving such non-convex QCQP.
- We also derive an alternate reconstruction scheme that builds upon a popular linear model by addressing some of the challenges posed by quantized feedback.
- We then evaluate our schemes under practically meaningful scenarios. We show that gain in reconstruction quality arising from our proposed formulation translates to substantial throughput gains (over 30% in one example). Moreover, directly adopting a conventional approach as an alternate reconstruction scheme is quite ill suited for quantized feedback. While an enhancement we propose improves the alternate reconstruction scheme, it still remains quite inferior to our proposed scheme based on our novel formulation and K-best methodology.

II. PROBLEM FORMULATION

We consider a narrowband downlink wherein a base-station (BS) having N_t transmit antennas communicates with a set of users that are equipped with one receive antenna each. Thus, the channel vector seen by any generic user in the downlink in each slot, which is a time interval (1ms or less) over which that user's channel can be assumed to be constant, is modeled as an $N_t \times 1$ complex-valued vector[1] $\mathbf{h} \in \mathbb{C}^{N_t \times 1}$. Since reciprocity does not hold in FDD, the BS has to rely on user feedback along with possibly other available information, to accurately reconstruct the channel vector of each user being served by it in every slot. Our focus is on a stand-alone scenario in which there is no slot-level coordination among different cells so that the inter-cell interference (ICI) seen by a served user from any other interfering BS cannot be controlled by its serving BS. Each user reports an SINR representing a ratio of the desired signal strength from its serving BS and the ICI it perceives. Then, without loss of generality, we can normalize the channel vector of each user using the average ICI strength it perceives, and pose the reconstruction problem for that user as that of determining its corresponding normalized channel vector. This normalization gives us an equivalent problem (considered henceforth) in which SINRs reported by the users represent effective channel gains in presence of unit variance noise at their receivers. Next, suppose that the following disparate information is available at the BS, using which it has to reconstruct an approximation of the (normalized) channel $\mathbf{h}$.

• *Direct Feedback:* Conforming to the LTE standard, we suppose that beamformed reference symbols (pilots) are employed periodically by the BS to enable the user to estimate a low-dimensional projection of its channel vector. In particular, we let $\mathbf{B}_i,\ 1 \leq i \leq M$ denote M pre-beamforming matrices. Each matrix $\mathbf{B}_i$ is of size $N_t \times m$, where m is much smaller than N_t. In each slot, the user estimates $\mathbf{B}_i^{\dagger}\mathbf{h},\ 1 \leq i \leq M$ and reports its feedback, which for each i comprises of a quantized unit-norm vector $\mathbf{g}_i \in \mathbb{C}^{m \times 1}$, referred to here as PMI vector and an effective gain $\gamma_i \in I\!R_+$ which is a quantized scalar. We note here that while our formulation allows for any quantization codebook from which each PMI vector is selected, in practise it must be one among those that have already been defined. This essentially restricts m to be either 4 or 8. In all these defined codebooks the codewords satisfy a *constant magnitude property* i.e., all elements of each codeword vector have the same magnitude. Hence, the magnitudes of the individual elements of any reported PMI vector will convey no information. As shown in the sequel, this can have a major impact on the performance of some reconstruction schemes. On the other hand, the number of feedback reports, M, obtained by the BS is a design parameter and allows for trading off reconstruction accuracy and signaling overhead. Furthermore, for any given m, M, the choice of pre-beam matrices $\{\mathbf{B}_i\}_{i=1}^{M}$ is by itself an important design choice. However, this choice is subject to several practical constraints and is made from a finite collection that is common for all served users. In this work, we assume that a set of M chosen pre-beam matrices has been provided as input, but we place no restriction on the choice of such a set.

• *Subspace Information:* The BS can have information via longer-term statistical estimation about the subspace in which the instantaneous channel vector $\mathbf{h}$ is likely to be in. Let $\mathbf{U}_{\rm sub} \in \mathbb{C}^{N_t \times S}$ be a semi-unitary matrix whose column-span equals the said subspace. Clearly such subspace information is useful if the instantaneous $\mathbf{h}$ belongs to it with high probability and if its dimension S is much smaller than N_t. This is indeed possible over relevant massive MIMO scenarios. For later use, we let $\mathbf{P}_{\rm sub} \triangleq \mathbf{U}_{\rm sub}\mathbf{U}_{\rm sub}^{\dagger}$ and $\mathbf{P}_{\rm sub}^{\perp} \triangleq \mathbf{I} - \mathbf{U}_{\rm sub}\mathbf{U}_{\rm sub}^{\dagger}$ denote two orthogonal projection matrices. We also let $\mathcal{R}(\mathbf{P}_{\rm sub})$ ($\mathcal{R}(\mathbf{P}_{\rm sub}^{\perp})$) denote the range or column span of $\mathbf{P}_{\rm sub}$ ($\mathbf{P}_{\rm sub}^{\perp}$).

• *Auxiliary Estimate:* We suppose that another estimate of the true channel, denoted here by $\breve{\mathbf{h}}$, can also be available. This estimate can be obtained for instance via another set of possibly non-beamformed and sparser pilots. We assume that the first element of $\breve{\mathbf{h}}$ is real-valued and non-negative. This results in no loss of generality as detailed below.

We formulate our re-construction problem as

$$\min_{\mathbf{z} \in \mathbb{C}^{N_t \times 1}} \{ w_1 \|\mathbf{P}_{\rm sub}\mathbf{z}\|^2 + w_2 \|\mathbf{P}_{\rm sub}^{\perp}\mathbf{z}\|^2 + w_3 \|\breve{\mathbf{h}} - \mathbf{z}\|^2 \}$$
$$\text{s.t. } |\mathbf{g}_i^{\dagger}\mathbf{B}_i^{\dagger}\mathbf{z}|^2 \geq \gamma_i \ \ i = 1, \cdots, M, \qquad \text{(P1)}$$

where the $w_1, w_2, w_3 \in I\!R_+$ are given weights. Some comments on our formulation are in order:

• Note that our formulation has a robust flavor since it aims to minimize weighted sum of projected channel squared ℓ_2 norms (or energies) and deviation from an available reference, subject to certain constraints. A weight is assigned to each side-information at hand which reflects its reliability. Specifically, if the subspace information is highly reliable then we can simply set $w_1 = 1$ and $w_2 >> 1$ to be a large value. This will force the resulting reconstructed channel to lie in the desired subspace. On the other hand setting $w_1 = w_2$ ignores the subspace information. Moreover, setting a large value of w_3 forces the reconstruction to approach $\breve{\mathbf{h}}$ and doing so is clearly beneficial when the auxiliary estimate is indeed reliable.

• Notice that our formulation is applicable to arbitrary transmit antenna array geometries. This is quite useful in practise since arrays with irregular spacing are being found promising.

• A key problem in modeling PMI feedback is that the specific quantization rule implemented by the user is not completely known to the BS. In LTE networks this quantization rule is left as an implementation issue and only one aspect has been standardized. Specifically, the only requirement any such rule must meet is that *if the BS chooses to transmit to only that user employing its reported PMI and the corresponding pre-beam matrix, then the SINR observed by that user must exceed the reported one*, whenever the channel and interference conditions do not change. The constraints we formulate in (P1) precisely capture this requirement.

• Notice that for the purpose of scheduling and transmitting data it suffices to know the channel up-to a phase term, i.e., any reconstruction $\hat{\mathbf{z}}$ and $\exp(j\phi)\hat{\mathbf{z}}$ will have the same impact on system performance for any phase term $\exp(j\phi)$, where $j = \sqrt{-1}$. Since PMI codebooks were designed considering only a single PMI feedback, each PMI vector also conveys information up-to a phase term. Indeed, given any prebeam matrix choice, the user would obtain and feedback the same PMI vector for any channel $\mathbf{h}$ as well as $\exp(j\phi)\mathbf{h}$, for any phase term $\exp(j\phi)$. This introduces a major issue of *phase ambiguity* while combining multiple PMI feedback for reconstruction. Similarly, $\breve{\mathbf{h}}$ can also be an estimate of $\mathbf{h}$ upto a phase term. (P1) is inherently tailored to incorporate such

[1]We drop the user and slot indices since they are unimportant for our purposes. Each user's channel changes across slots but the channel statistics remain constant across several slots.

phase ambiguity. In particular, we simply choose the phase term of $\breve{\mathbf{h}}$ as the reference and have ensured that the other terms in the objective as well as all constraints are invariant to multiplying the vector of variables, $\mathbf{z}$, by any phase term.

• (P1) has a quadratic and convex objective but non-convex constraint sets. Notable works have shown that such non-convex QCQP is NP-hard in general and can be optimally solved in a tractable manner only for $M \leq 3$ [8], [11] or when all Grammians involved are also Toeplitz [12]. Unfortunately, for our purpose Toeplitz condition need not be met while even existing optimal methods (for $M \leq 3$) are unsuitable since they involve solving large-dimensional semi-definite programs (SDPs). On the other hand, most of the existing sub-optimal techniques that we are aware of either also involve SDP solvers or are iterative in nature and entail input instance dependent complexity. Key exceptions are [13], [14] which adopt a successive approach to address the constraints. However, they consider a simpler multicast beamforming setup without subspace or auxiliary side-information and also do not propose and develop the K-best methodology.

We now proceed to derive a sub-optimal approach to solve (P1) that has a tunable and deterministic complexity, i.e., our method is non-iterative and its complexity does not depend on the input instance but only on problem dimensions and choice of certain tuning parameter. Before that, we detail a specific case in which an optimal solution of (P1) can be obtained analytically. We will then use the insights garnered and develop an effective approach for the general case.

III. An Optimally Solvable case of (P1)

The specific case we consider here that can be optimally solved, is one where $w_3 = 0$ and all but one of the vectors $\{\mathbf{B}_i\mathbf{g}_i\}_{i=1}^M$ are mutually orthogonal. Further, each one of these $M-1$ vectors belongs to either $\mathcal{R}(\mathbf{P}_{\text{sub}})$ or $\mathcal{R}(\mathbf{P}_{\text{sub}}^{\perp})$. Then, without further loss of generality, let us adopt a labeling of feedback for which there exists a $J \in \{0,\cdots,M-1\}$ such that each $\mathbf{B}_i\mathbf{g}_i,\ i = 1\cdots,J$ lies in $\mathbf{P}_{\text{sub}}$, whereas each $\mathbf{B}_i\mathbf{g}_i,\ i = J+1\cdots,M-1$ lies in $\mathcal{R}(\mathbf{P}_{\text{sub}}^{\perp})$. There is no restriction on $\mathbf{B}_M\mathbf{g}_M$. Next, letting $\mathbf{D}_1 = \text{diag}\{1/\sqrt{\gamma_1},\cdots,1/\sqrt{\gamma_J},1/\sqrt{\gamma_M}\}$ and $\mathbf{D}_2 = \text{diag}\{1/\sqrt{\gamma_{J+1}},\cdots,1/\sqrt{\gamma_{M-1}},1/\sqrt{\gamma_M}\}$, obtain two QR decompositions

$$\mathbf{P}_{\text{sub}}[\mathbf{B}_1\mathbf{g}_1,\cdots,\mathbf{B}_J\mathbf{g}_J,\mathbf{B}_M\mathbf{g}_M]\mathbf{D}_1 = \mathbf{QR},$$
$$\mathbf{P}_{\text{sub}}^{\perp}[\mathbf{B}_{J+1}\mathbf{g}_{J+1},\cdots,\mathbf{B}_{M-1}\mathbf{g}_{M-1},\mathbf{B}_M\mathbf{g}_M]\mathbf{D}_2 = \mathbf{VS}, \quad (1)$$

where $\mathbf{Q}$ and $\mathbf{V}$ are both semi-unitary whenever they are non-zero matrices. $\mathbf{R} = [r_{i,j}]$ and $\mathbf{S} = [s_{i,j}]$ are both upper triangular with positive diagonal elements whenever they are non-zero matrices.[2] Let d_q and d_v denote the number of non-zero columns (as well as ranks) of $\mathbf{Q}$ and $\mathbf{V}$, respectively, and note that $J \leq d_q \leq J+1$ while $M-J-1 \leq d_v \leq M-J$. Further, let $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{V}}$ denote two matrices such that

$$\mathcal{R}(\mathbf{P}_{\text{sub}}) = \mathcal{R}(\mathbf{Q}) \oplus \mathcal{R}(\tilde{\mathbf{Q}})\ \&\ \mathcal{R}(\mathbf{P}_{\text{sub}}^{\perp}) = \mathcal{R}(\mathbf{V}) \oplus \mathcal{R}(\tilde{\mathbf{V}})$$

with $\tilde{\mathbf{Q}}^\dagger\tilde{\mathbf{Q}} = \mathbf{I}\ \&\ \tilde{\mathbf{Q}}^\dagger\mathbf{Q} = \mathbf{0}$ ($\tilde{\mathbf{V}}^\dagger\tilde{\mathbf{V}} = \mathbf{I}\ \&\ \tilde{\mathbf{V}}^\dagger\mathbf{V} = \mathbf{0}$) whenever $\tilde{\mathbf{Q}}$ ($\tilde{\mathbf{V}}$) is non-zero.

[2] We adopt the convention that decomposing a zero matrix in (1) yields both $\mathbf{Q}$ and $\mathbf{R}$ ($\mathbf{V}$ and $\mathbf{S}$) to be zero matrices. Also since $\mathbf{R}$ ($\mathbf{S}$) can be rectangular, we use upper-triangular to mean $r_{i,j} = 0$ ($s_{i,j} = 0$) forall $i > j$.

Proposition 1. *In this special case, any optimal solution of (P1) must lie in* $\mathcal{R}(\mathbf{Q}) \oplus \mathcal{R}(\mathbf{V})$.

Proof. We first expand, without loss of generality, any candidate reconstruction as $\mathbf{z} = \mathbf{Q}\boldsymbol{\beta} + \tilde{\mathbf{Q}}\tilde{\boldsymbol{\beta}} + \mathbf{V}\boldsymbol{\alpha} + \tilde{\mathbf{V}}\tilde{\boldsymbol{\alpha}}$. Then, in this special case we can simplify (P1) as follows. Recall that

$$\begin{aligned}\mathbf{B}_i\mathbf{g}_i &= \mathbf{P}_{\text{sub}}\mathbf{B}_i\mathbf{g}_i,\ \forall i = 1,\cdots,J\\ \mathbf{B}_i\mathbf{g}_i &= \mathbf{P}_{\text{sub}}^{\perp}\mathbf{B}_i\mathbf{g}_i,\ \forall i = J+1,\cdots,M-1 \qquad (2)\end{aligned}$$

and that all these $M-1$ vectors are mutually orthogonal. Then, using the two QR decompositions in (1) and expanding $\boldsymbol{\alpha} = [\alpha_1,\cdots,\alpha_{d_v}]^T$ and $\boldsymbol{\beta} = [\beta_1,\cdots,\beta_{d_q}]^T$, we see that

$$\begin{aligned}\mathbf{g}_j^\dagger\mathbf{B}_j^\dagger\mathbf{z} &= \sqrt{\gamma_j}r_{j,j}^\dagger\beta_j,\ \forall j = 1,\cdots,J,\\ \mathbf{g}_j^\dagger\mathbf{B}_j^\dagger\mathbf{z} &= \sqrt{\gamma_j}s_{j-J,j-J}^\dagger\alpha_{j-J},\ \forall j = J+1,\cdots,M-1,\\ \mathbf{g}_M^\dagger\mathbf{B}_M^\dagger\mathbf{z} &= \mathbf{g}_M^\dagger\mathbf{B}_M^\dagger\mathbf{P}_{\text{sub}}\mathbf{z} + \mathbf{g}_M^\dagger\mathbf{B}_M^\dagger\mathbf{P}_{\text{sub}}^{\perp}\mathbf{z}\\ &= \sqrt{\gamma_M}\left(\sum_{j=1}^{d_q} r_{j,J+1}^\dagger\beta_j + \sum_{j=1}^{d_v} s_{j,M-J}^\dagger\alpha_j\right). \qquad (3)\end{aligned}$$

On the other hand the objective in (P1) is equal to $w_1(\|\boldsymbol{\beta}\|^2 + \|\tilde{\boldsymbol{\beta}}\|^2) + w_2(\|\boldsymbol{\alpha}\|^2 + \|\tilde{\boldsymbol{\alpha}}\|^2)$. Thus, since from (3) we can infer that none of the constraints depend on $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}$, it is optimal to set them both to zero. □

With Proposition 1 in hand, we expand the reconstruction we seek as $\mathbf{z} = \mathbf{Q}\boldsymbol{\beta} + \mathbf{V}\boldsymbol{\alpha}$. Invoking (3) we can now formulate the re-construction problem (P1) for this special case as

$$\begin{aligned}&\min_{\boldsymbol{\alpha}\in\mathbb{C}^{d_v\times 1},\boldsymbol{\beta}\in\mathbb{C}^{d_q\times 1}} \{w_1\|\boldsymbol{\beta}\|^2 + w_2\|\boldsymbol{\alpha}\|^2\}\\ &\text{s.t. } \{|s_{j,j}^\dagger\alpha_j|^2 \geq 1\}_{j=1}^{M-J-1};\{|r_{j,j}^\dagger\beta_j|^2 \geq 1\}_{j=1}^{J};\\ &\left|\sum_{j=1}^{d_v} s_{j,M-J}^\dagger\alpha_j + \sum_{j=1}^{d_q} r_{j,J+1}^\dagger\beta_j\right|^2 \geq 1 \qquad \text{(P2)}\end{aligned}$$

We offer our next result.

Proposition 2. *(P2) is equivalent to a convex optimization problem*

Proof. We start by noting that only the last constraint depends on the phases of elements of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ whereas the other constraints and the objective are invariant to them. As a result, without loss of optimality we can suppose that the phase of α_j (β_j) equals that of $s_{j,M-J}$ ($r_{j,J+1}$) for all j. Then, let us define non-negative scalars $\alpha'_j = |\alpha_j s_{j,j}|, j = 1,\cdots,M-J-1$ and $\beta'_j = |\beta_j r_{j,j}|, j = 1,\cdots,J$ with $\alpha'_{M-J} = |\alpha_{M-J}s_{M-J,M-J}|$ whenever $d_s = M-J$ and $\beta'_{J+1} = |\beta_{J+1}r_{J+1,J+1}|$ whenever $d_q = J+1$. We can now equivalently express (P2) as

$$\begin{aligned}&\min_{\boldsymbol{\alpha}'\in\mathbb{R}_+^{d_v\times 1},\boldsymbol{\beta}'\in\mathbb{R}_+^{d_q\times 1}} \left\{\sum_{j=1}^{d_v} w'_j(\alpha'_j)^2 + \sum_{j=1}^{d_q} \breve{w}_j(\beta'_j)^2\right\}\\ &\text{s.t. } \{\alpha'_j \geq 1\}_{j=1}^{M-J-1};\ \{\beta'_j \geq 1\}_{j=1}^{J},\\ &\sum_{j=1}^{d_v}\frac{|s_{j,M-J}|}{|s_{j,j}|}\alpha'_j + \sum_{j=1}^{d_q}\frac{|r_{j,J+1}|}{|r_{j,j}|}\beta'_j \geq 1 \qquad \text{(P2b)}\end{aligned}$$

where we have used $w'_j = w_2/|s_{j,j}|^2\ \forall\ j$ and $\breve{w}_j = w_1/|r_{j,j}|^2\ \forall\ j$. Clearly (P2b) has a convex quadratic objective and convex constraint sets. □

The implication of Proposition 2 is that since Slater's condition clearly holds for (P2b), K.K.T conditions are both necessary and sufficient. Indeed, we can explicitly solve these conditions to recover a globally optimally solution. For simplicity, we outline one procedure whose complexity scales linearly in M and remark that a faster bisection search based one can be readily designed. We start by noting that if $\sum_{j=1}^{M-J-1} \frac{|s_{j,M-J}|}{|s_{j,j}|} + \sum_{j=1}^{J} \frac{|r_{j,J+1}|}{|r_{j,j}|} \geq 1$ then the optimal solution can be trivially obtained. Supposing $\sum_{j=1}^{M-J-1} \frac{|s_{j,M-J}|}{|s_{j,j}|} + \sum_{j=1}^{J} \frac{|r_{j,J+1}|}{|r_{j,j}|} < 1$, from K.K.T conditions we can express optimal $\alpha'_j, \beta'_j \ \forall\, j$ as

$$\begin{aligned} \alpha'_j(\hat{\lambda}) &= \max\left\{1, \frac{\hat{\lambda}|s_{j,M-J}|}{|s_{j,j}|2w'_j}\right\} \\ \beta'_j(\hat{\lambda}) &= \max\left\{1, \frac{\hat{\lambda}|r_{j,J+1}|}{|r_{j,j}|2\breve{w}_j}\right\}, \ \forall\, j, \end{aligned} \tag{4}$$

where we have explicitly indicated the dependence on $\hat{\lambda}$, which is an optimal non-negative Lagrange multiplier associated with the last constraint of (P2b) and must satisfy

$$\sum_{j=1}^{d_v} \frac{|s_{j,M-J}|}{|s_{j,j}|}\alpha'_j(\hat{\lambda}) + \sum_{j=1}^{d_q} \frac{|r_{j,J+1}|}{|r_{j,j}|}\beta'_j(\hat{\lambda}) = 1$$

The optimal $\hat{\lambda}$ can be determined by first trying $\hat{\lambda} = \frac{2A}{\sum_{j=1}^{d_v} \frac{|s_{j,M-J}|^2}{w_2} + \sum_{j=1}^{d_q} \frac{|r_{j,J+1}|^2}{w_1}}$ where $A = 1$. Next for this choice of $\hat{\lambda}$ using (4) determine whether any of $\alpha'_j(\hat{\lambda})$ and $\beta'_j(\hat{\lambda})$ is equal to one. If not then we have found the optimal $\hat{\lambda}$. Else, we update $A \to A - \sum_j \frac{|s_{j,M-J}|}{|s_{j,j}|} - \sum_j \frac{|r_{j,J+1}|}{|r_{j,j}|}$ where the two summations correspond to all $\alpha'_j(\hat{\lambda})$ and $\beta'_j(\hat{\lambda})$, respectively, that were found to be one. Next, using this A we recompute $\hat{\lambda} = \frac{2A}{\sum_j \frac{|s_{j,M-J}|^2}{w_2} + \sum_j \frac{|r_{j,J+1}|^2}{w_1}}$ where each summation in the denominator corresponds to all $\alpha'_j(\hat{\lambda})$ and $\beta'_j(\hat{\lambda})$ that were found to exceed one. Using the recomputed $\hat{\lambda}$ we again determine all $\alpha'_j(\hat{\lambda})$ and $\beta'_j(\hat{\lambda})$ via (4) and repeat the process. It can be shown that $\hat{\lambda}$ decreases across the iterations and the process finds an optimal $\hat{\lambda}$ in $O(M)$ iterations.

Remark 1. *We highlight one instance of (P1) that can be formulated as (P2) (hence its optimal solution can be efficiently recovered), which will be evaluated in our numerical experiments. In this case we have $w_1 = w_2$ and $w_3 = 0$ with $M = 3$ & $(\mathbf{B}_1\mathbf{g}_1)^\dagger \mathbf{B}_2\mathbf{g}_2 = \mathbf{0}$. Notice that since $w_1 = w_2$, the subspace information is ignored by (P1). Then, upon setting $\mathbf{P}_{\rm sub} = \mathbf{I}$ with $J = 2$, it is seen (P1) is equivalent to (P2).*

IV. K-best Approach

We now leverage the insights gained from the special case elucidated above to develop an effective sub-optimal algorithm for the general case. First, defining an $N_t \times M$ matrix $\mathbf{C}' = [\mathbf{B}_1\mathbf{g}_1, \cdots, \mathbf{B}_M\mathbf{g}_M]\mathrm{diag}\{1/\sqrt{\gamma_1}, \cdots, 1/\sqrt{\gamma_M}\}$, we extend Proposition 1 to obtain the following one. It can be proved in a similar fashion as Proposition 1.

Proposition 3. *Any optimal solution of (P1) must lie in $\mathcal{R}(\mathbf{P}_{\rm sub}[\mathbf{C}', \breve{\mathbf{h}}]) \oplus \mathcal{R}(\mathbf{P}_{\rm sub}^\perp[\mathbf{C}', \breve{\mathbf{h}}])$.*

Let Π denote any $M \times M$ permutation matrix. For a suitable column permutation of $\mathbf{C}'$, $\mathbf{C} = \mathbf{C}'\Pi$, we specify two QR decompositions, $\mathbf{P}_{\rm sub}[\mathbf{C}, \breve{\mathbf{h}}] = \mathbf{QR}$ and $\mathbf{P}_{\rm sub}^\perp[\mathbf{C}, \breve{\mathbf{h}}] = \mathbf{VS}$, wherein $\mathbf{R}, \mathbf{S}$ are both $(M+1) \times (M+1)$ upper triangular matrices with non-negative diagonal elements. $\mathbf{Q}$ ($\mathbf{V}$) is an $N_t \times (M+1)$ matrix which contains a zero-column corresponding to each diagonal element of $\mathbf{R}$ ($\mathbf{S}$) that is zero. In addition, if the $i^{th}, 1 \leq i \leq M+1$, column of $\mathbf{Q}$ ($\mathbf{V}$) is a zero-column then the i^{th} row of $\mathbf{R}$ ($\mathbf{S}$) has all zeros. Furthermore, the non-zero columns of $\mathbf{Q}$ ($\mathbf{V}$) are mutually orthogonal and unit-norm. Together, these conditions uniquely specify the two decompositions for any given choice of permutation matrix Π. Henceforth, invoking Proposition 3 which assures us that any optimal reconstruction must lie in $\mathcal{R}(\mathbf{Q}) \oplus \mathcal{R}(\mathbf{V})$, since $\mathcal{R}(\mathbf{P}_{\rm sub}[\mathbf{C}', \breve{\mathbf{h}}]) = \mathcal{R}(\mathbf{Q})$ & $\mathcal{R}(\mathbf{P}_{\rm sub}^\perp[\mathbf{C}', \breve{\mathbf{h}}]) = \mathcal{R}(\mathbf{V})$, we will seek an approximately optimal reconstruction $\hat{\mathbf{h}} = \mathbf{Q}\boldsymbol{\beta} + \mathbf{V}\boldsymbol{\alpha}$, $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_{M+1}]^T, \boldsymbol{\beta} = [\beta_1, \cdots, \beta_{M+1}]^T$, which will be feasible (i.e., meets all constraints). Notice that we can express each inequality constraint corresponding to the $i^{th}, 1 \leq i \leq M$ column of $\mathbf{C}$, denoted by $\mathbf{c}_i$, as:

$$\begin{aligned} |\mathbf{c}_i^\dagger \hat{\mathbf{h}}|^2 &= |\mathbf{c}_i^\dagger \mathbf{P}_{\rm sub}\hat{\mathbf{h}} + \mathbf{c}_i^\dagger \mathbf{P}_{\rm sub}^\perp \hat{\mathbf{h}}|^2 \\ &= |\mathbf{c}_i^\dagger \mathbf{P}_{\rm sub}\mathbf{P}_{\rm sub}\hat{\mathbf{h}} + \mathbf{c}_i^\dagger \mathbf{P}_{\rm sub}^\perp \mathbf{P}_{\rm sub}^\perp \hat{\mathbf{h}}|^2 \\ &= \left|\sum_{j=1}^{i} r_{j,i}^\dagger \beta_j + \sum_{j=1}^{i} s_{j,i}^\dagger \alpha_j\right|^2 \geq 1 \end{aligned} \tag{5}$$

Similarly, using the fact that $\|\breve{\mathbf{h}} - \hat{\mathbf{h}}\|^2 = \|\mathbf{P}_{\rm sub}(\breve{\mathbf{h}} - \hat{\mathbf{h}})\|^2 + \|\mathbf{P}_{\rm sub}^\perp(\breve{\mathbf{h}} - \hat{\mathbf{h}})\|^2$ the objective of (P1) can be expressed as [3]

$$w_2\|\boldsymbol{\alpha}\|^2 + w_1\|\boldsymbol{\beta}\|^2 + w_3 \sum_{j=1}^{M+1} \left(|r_{j,M+1} - \beta_j|^2 + |s_{j,M+1} - \alpha_j|^2\right) \tag{6}$$

A. Algorithm Description

We will exploit the triangular structure of $\mathbf{R}, \mathbf{S}$ to determine a suitable permutation in a successive manner via a K-best based algrorithm. Here $K : K \geq 1$ is a design choice that allows us to tradeoff performance gains for complexity reduction. Our algorithm has M steps and in each step we keep up-to K survivors or paths, wherein each survivor is characterized by choice of vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and a matrix $\mathbf{C}$ specifying a set of constraints that have been addressed along that path. Key steps are summarized below:

- In the first step we consider every one of the M possible descendants (i.e., each corresponding to one column of $\mathbf{C}'$ and a constraint). For each such descendant we initialize $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to be empty and $\mathbf{C}$ to be the corresponding column of $\mathbf{C}'$. We then update its $\boldsymbol{\alpha}, \boldsymbol{\beta}$ by solving a successive-step subproblem using an approach that is described in the sequel. This enables us to compute a metric for each descendant and out of these M descendants we keep the first $\min\{M, K\}$ descendants corresponding to the ones with the $\min\{M, K\}$ smallest metrics. These retained descendants then become the survivors for the next step.

[3]Note that for notational convenience we have implicitly assumed $\mathbf{Q}^\dagger\mathbf{Q}$ and $\mathbf{V}^\dagger\mathbf{V}$ to be identity matrices. This however results in no loss of generality due to our construction of $\mathbf{R}, \mathbf{S}$ and the subsequent analysis is applicable to the general case where $\mathbf{Q}^\dagger\mathbf{Q}$ and $\mathbf{V}^\dagger\mathbf{V}$ are diagonal matrices whose diagonal elements are either zero or one.

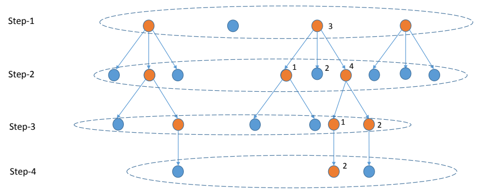

Fig. 1. K-Best Schematic for $M = 4, K = 3$.
Selected descendants are colored orange and chosen permutation is specified.

• At each subsequent step $j : 2 \leq j \leq M-1$, we consider each one of the survivors and expand all of its possible descendants (i.e., columns of $\mathbf{C}'$ corresponding to constraints that have not yet been addressed along that survivor). Then for each descendant, we initialize its $\mathbf{C}, \boldsymbol{\alpha}, \boldsymbol{\beta}$ to be those of its parent survivor. We then add the column of $\mathbf{C}'$ associated with that descendant to $\mathbf{C}$ and update $\boldsymbol{\alpha}, \boldsymbol{\beta}$ by solving the corresponding successive-step subproblem, thereby computing a metric for the descendant of interest. It can be verified that at the j^{th} step we have exactly $(M-j+1)$ descendants for each survivor. Out of all these descendants (across all survivors) we keep the first K descendants corresponding to the ones with the K smallest metrics as survivors. If the total number of descendants is smaller than K we retain all of them. *Notice here that the metric of each descendant maps to a sub-optimal solution of a relaxation of (P1) in which only constraints considered along that descendant are included.* Hence, these retained descendants (which become the survivors for the next step) are deemed most promising candidates thus far.

• On the other hand, at the final step, $j = M$, we obtain metrics for all possible descendants (by solving a successive sub-problem as in previous steps), but now we simply select the descendant with the minimal metric. We then use the updated $\mathbf{C}, \boldsymbol{\alpha}, \boldsymbol{\beta}$ associated with this minimal metric descendant (which we note has addressed all constraints in (P1)) to compute the channel reconstruction.

We have illustrated an example with $M = 4$ and $K = 3$ in Fig. 1. Note that there are $M = 4$ steps and in each of the first $M-1$ steps we retain $K = 3$ descendants, whereas at the last step we select only one descendant. Further, the descendants of any survivor correspond to the remaining constraints that have not been so far addressed along that survivor.

B. Successive Step

To illustrate the sub-problem that needs to be solved, consider any survivor in the i^{th} step so that its associated matrix $\mathbf{C}$ has $i-1$ columns from $\mathbf{C}'$ and we have the decompositions of $\mathbf{P}_{\text{sub}}[\mathbf{C}, \check{\mathbf{h}}] = \mathbf{QR}$ & $\mathbf{P}^{\perp}_{\text{sub}}[\mathbf{C}, \check{\mathbf{h}}] = \mathbf{VS}$ available. Recall that metric computation conducted for this survivor determined vectors $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_{i-1}, \tilde{\alpha}]^T, \boldsymbol{\beta} = [\beta_1, \cdots, \beta_{i-1}, \tilde{\beta}]^T$ such that $\hat{\mathbf{h}} = \mathbf{Q}\boldsymbol{\beta} + \mathbf{V}\boldsymbol{\alpha}$ is a sub-optimal solution to a relaxation of (P1) containing only constraints corresponding to $i-1$ columns of $\mathbf{C}$. Let $\mathbf{c}'$ denote any column from the remaining $M-i+1$ unselected ones in $\mathbf{C}'$ so that it represents a descendant of our survivor. Computing a metric for this descendant entails updating and expanding $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to obtain a solution to a *tighter relaxation of (P1)* in which the additional constraint has been incorporated. The challenge here is to ensure tractability without compromising the quality of the obtained solution too much. The former can be ensured by reducing the number of optimization variables (by fixing others to their previously determined values), whereas the latter pushes us to jointly optimize over as many variables as possible. To meet this challenge, we first obtain the updated decompositions $\mathbf{P}_{\text{sub}}[\mathbf{C}, \mathbf{c}', \check{\mathbf{h}}] = \mathbf{Q}'\mathbf{R}'$ & $\mathbf{P}^{\perp}_{\text{sub}}[\mathbf{C}, \mathbf{c}', \check{\mathbf{h}}] = \mathbf{V}'\mathbf{S}'$ and note that the first $i-1$ columns of $\mathbf{Q}'$ ($\mathbf{V}'$) are identical respectively to the first $i-1$ columns of $\mathbf{Q}$ ($\mathbf{V}$). Similarly, the associated principal submatrix of $\mathbf{R}' = [r'_{i,j}]$ ($\mathbf{S}' = [s'_{i,j}]$) also coincides with that of $\mathbf{R}$ ($\mathbf{S}$). We recall the form of each constraint (5) and the objective (6) and let $\boldsymbol{\alpha}' = [\gamma\alpha_1, \cdots, \gamma\alpha_{i-1}, \alpha, \alpha']^T$ and $\boldsymbol{\beta}' = [\gamma\beta_1, \cdots, \gamma\beta_{i-1}, \beta, \beta']^T$ denote the updated and expanded $\boldsymbol{\alpha}, \boldsymbol{\beta}$, where $\gamma, \alpha, \beta, \alpha', \beta'$ are five complex variables that are determined by solving the following sub-problem:

$$\begin{aligned}
\min_{\gamma,\alpha,\beta,\alpha',\beta'\in\mathbb{C}} \Big\{ & w_1|\beta|^2 + w_2|\alpha|^2 + w_1|\beta'|^2 + w_2|\alpha'|^2 + \\
& w_3(|\beta' - r'_{i+1,i+1}|^2 + |\alpha' - s'_{i+1,i+1}|^2) + \\
& w_3(|\beta - r'_{i,i+1}|^2 + |\alpha - s'_{i,i+1}|^2) + \sum_{j=1}^{i-1}(|\gamma|^2(w_1|\beta_j|^2 + w_2|\alpha_j|^2) \\
& + w_3(|\gamma\beta_j - r'_{j,i+1}|^2 + |\gamma\alpha_j - s'_{j,i+1}|^2))\Big\} \\
& \text{s.t. } |\gamma| \geq 1; \\
& \left|\gamma\sum_{j=1}^{i-1}((r'_{j,i})^{\dagger}\beta_j + (s'_{j,i})^{\dagger}\alpha_j) + (r'_{i,i})^{\dagger}\beta + (s'_{i,i})^{\dagger}\alpha\right|^2 \geq 1
\end{aligned} \tag{7}$$

In formulating (7) *we have made a useful observation that imposing $|\gamma| \geq 1$ ensures that the $i-1$ constraints corresponding to $\mathbf{C}$ that have previously been satisfied by $\boldsymbol{\alpha}, \boldsymbol{\beta}$, remain so even by $\boldsymbol{\alpha}', \boldsymbol{\beta}'$.* Upon inspecting (7) it is evident that we can separately minimize over α', β' and accordingly let $\Gamma = \min_{\alpha',\beta'\in\mathbb{C}}\{w_1|\beta'|^2 + w_2|\alpha'|^2 + w_3(|\beta' - r'_{i+1,i+1}|^2 + |\alpha' - s'_{i+1,i+1}|^2)\}$. The optimal α', β' are immediately seen to be $\alpha' = \frac{w_3 s'_{i+1,i+1}}{w_2 + w_3}, \beta' = \frac{w_3 r'_{i+1,i+1}}{w_1 + w_3}$. Thus, (7) reduces to:

$$\begin{aligned}
\Gamma + \min_{\gamma,\alpha,\beta\in\mathbb{C}} \Big\{ & w_1|\beta|^2 + w_2|\alpha|^2 + w_3(|\beta - r'_{i,i+1}|^2 \\
& + |\alpha - s'_{i,i+1}|^2) + \sum_{j=1}^{i-1}(|\gamma|^2(w_1|\beta_j|^2 + w_2|\alpha_j|^2) + \\
& w_3(|\gamma\beta_j - r'_{j,i+1}|^2 + |\gamma\alpha_j - s'_{j,i+1}|^2))\Big\} \\
& \text{s.t. } |\gamma| \geq 1; \\
& \left|\gamma\sum_{j=1}^{i-1}((r'_{j,i})^{\dagger}\beta_j + (s'_{j,i})^{\dagger}\alpha_j) + (r'_{i,i})^{\dagger}\beta + (s'_{i,i})^{\dagger}\alpha\right|^2 \geq 1
\end{aligned} \tag{8}$$

We propose solving (8) by considering two related sub-problems. The first one is the following one that is obtained upon dropping the magnitude constraint on γ.

$$\begin{aligned}
\min_{\gamma,\alpha,\beta\in\mathbb{C}} \Big\{ & w_1|\beta|^2 + w_2|\alpha|^2 + w_3(|\beta - r'_{i,i+1}|^2 + |\alpha - s'_{i,i+1}|^2) \\
& + \sum_{j=1}^{i-1}(|\gamma|^2(w_1|\beta_j|^2 + w_2|\alpha_j|^2) + \\
& w_3(|\gamma\beta_j - r'_{j,i+1}|^2 + |\gamma\alpha_j - s'_{j,i+1}|^2))\Big\} \\
& \text{s.t. } \left|\gamma\sum_{j=1}^{i-1}((r'_{j,i})^{\dagger}\beta_j + (s'_{j,i})^{\dagger}\alpha_j) + (r'_{i,i})^{\dagger}\beta + (s'_{i,i})^{\dagger}\alpha\right|^2 \geq 1
\end{aligned} \tag{9}$$

Upon defining $\mathbf{x} = [\gamma, \alpha, \beta]^T$ with

$$\begin{aligned}\mathbf{v} &= \left[\sum_{j=1}^{i-1} r'_{j,i}\beta_j^\dagger + \sum_{j=1}^{i-1} s'_{j,i}\alpha_j^\dagger,\ s'_{i,i},\ r'_{i,i}\right]^T, \\ \mathbf{D} &= \text{diag}\Big\{\sum_{j=1}^{i-1}((w_1+w_3)|\beta_j|^2 + (w_2+w_3)|\alpha_j|^2), \\ &\qquad w_2+w_3,\ w_1+w_3\Big\} \\ \mathbf{q} &= w_3\left[\sum_{j=1}^{i-1}(\beta_j^\dagger r'_{j,i+1} + \alpha_j^\dagger s'_{j,i+1}),\ s'_{i,i+1},\ r'_{i,i+1}\right]^T\end{aligned}$$

we see that (9) can be expressed (after dropping constant terms) as

$$\min_{\mathbf{x}\in\mathbb{C}^{3\times1}} \left\{\mathbf{x}^\dagger\mathbf{D}\mathbf{x} - \mathbf{q}^\dagger\mathbf{x} - \mathbf{x}^\dagger\mathbf{q}\right\} \text{ s.t. } \mathbf{x}^\dagger\mathbf{v}\mathbf{v}^\dagger\mathbf{x} \geq 1 \qquad (10)$$

We note that the quadratic programming problem in (10) is non-convex (due to the constraint) but can still be optimally solved (cf. [8]). Directly applying the known generic results will however require us to solve for the root of a non-linear function (via some iterative method such as Newton-Raphson). In the following proposition, by exploiting the particular structure of (10), specifically the fact that the Grammian in the constraint is rank one, we show that an optimal solution can be directly obtained in closed-form via simple analytical expressions. For brevity, we will focus on $i \geq 2$ and assume $\mathbf{D} \succ \mathbf{0}$ with $|\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}| > 0$. The remaining cases can be handled similarly and indeed are simpler.

Proposition 4. *An optimal solution of (10) is given by*

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{D}^{-1}\mathbf{q}, & \text{If } |\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}| \geq 1 \\ (\mathbf{D} - \hat{\lambda}\mathbf{v}\mathbf{v}^\dagger)^{-1}\mathbf{q}, \text{where } \hat{\lambda} = \frac{1-|\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}|}{\mathbf{v}^\dagger\mathbf{D}^{-1}\mathbf{v}}, & \text{Else}\end{cases}$$

Proof. We will prove this result by showing that strong duality holds for (10). First note that the unconstrained solution to (10), $\hat{\mathbf{x}} = \mathbf{D}^{-1}\mathbf{q}$, is also feasible (and hence optimal) whenever $|\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}| \geq 1$. Suppose now that $|\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}| < 1$. Let λ denote the non-negative Lagrangian parameter associated with the sole constraint and form the Lagrangian $L(\mathbf{x}, \lambda) = \lambda + \mathbf{x}^\dagger(\mathbf{D} - \lambda\mathbf{v}\mathbf{v}^\dagger)\mathbf{x} - \mathbf{q}^\dagger\mathbf{x} - \mathbf{x}^\dagger\mathbf{q}$. To obtain the dual function we need to minimize $L(\mathbf{x}, \lambda)$ over all $\mathbf{x}$. It can be shown that the dual function, denoted by $g(\lambda)$, is finite only if $\mathbf{D} - \lambda\mathbf{v}\mathbf{v}^\dagger \succ \mathbf{0}$. Under the latter condition we obtain that the minimizing $\mathbf{x}$, denoted by $\hat{\mathbf{x}}$,[4] as

$$\hat{\mathbf{x}} = (\mathbf{D} - \lambda\mathbf{v}\mathbf{v}^\dagger)^{-1}\mathbf{q} \qquad (11)$$

from which we get $g(\lambda) = \lambda - \mathbf{q}^\dagger(\mathbf{D} - \lambda\mathbf{v}\mathbf{v}^\dagger)^{-1}\mathbf{q}$. We now need to maximize the dual function over all $\lambda \geq 0$. This can be accomplished by defining $\tilde{\mathbf{q}} = \mathbf{D}^{-1/2}\mathbf{q}, \tilde{\mathbf{v}} = \mathbf{D}^{-1/2}\mathbf{v}$ and using rank-1 inverse update rule to write the dual function as

$$\begin{aligned} g(\lambda) &= \lambda - \tilde{\mathbf{q}}^\dagger(\mathbf{I} - \lambda\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger)^{-1}\tilde{\mathbf{q}} \\ &= \lambda - \tilde{\mathbf{q}}^\dagger(\mathbf{I} + \lambda\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger/(1-\lambda||\tilde{\mathbf{v}}||^2))\tilde{\mathbf{q}} \qquad (12)\end{aligned}$$

[4]We do not explicitly indicate the dependence of $\hat{\mathbf{x}}$ on λ.

We can now determine the unique non-negative λ maximizing the dual function as $\hat{\lambda} = \frac{1-|\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}}|}{||\tilde{\mathbf{v}}||^2} = \frac{1-|\mathbf{q}^\dagger\mathbf{D}^{-1}\mathbf{v}|}{\mathbf{v}^\dagger\mathbf{D}^{-1}\mathbf{v}}$. The corresponding $\hat{\mathbf{x}}$ (evaluated using (11) at $\lambda = \hat{\lambda}$) satisfies

$$\begin{aligned}\hat{\mathbf{x}}^\dagger\mathbf{v}\mathbf{v}^\dagger\hat{\mathbf{x}} &= \mathbf{q}^\dagger(\mathbf{D} - \hat{\lambda}\mathbf{v}\mathbf{v}^\dagger)^{-1}\mathbf{v}\mathbf{v}^\dagger(\mathbf{D} - \hat{\lambda}\mathbf{v}\mathbf{v}^\dagger)^{-1}\mathbf{q} \\ &= \tilde{\mathbf{q}}^\dagger(\mathbf{I} - \hat{\lambda}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger)^{-1}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger(\mathbf{I} - \hat{\lambda}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger)^{-1}\tilde{\mathbf{q}} \\ &= \left|\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}} + \hat{\lambda}\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}}||\tilde{\mathbf{v}}||^2/|\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}}|\right|^2 = \left|\frac{\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}}}{|\tilde{\mathbf{q}}^\dagger\tilde{\mathbf{v}}|}\right|^2 = 1\end{aligned}$$

Thus, $\hat{\mathbf{x}}$ is feasible for (10) and since complementary slackness also holds we see that strong duality holds, i.e.,

$$g(\hat{\lambda}) = \hat{\mathbf{x}}^\dagger\mathbf{D}\hat{\mathbf{x}} - \mathbf{q}^\dagger\hat{\mathbf{x}} - \hat{\mathbf{x}}^\dagger\mathbf{q}$$

which shows that $\hat{\mathbf{x}}$ is also optimal. □

The next subproblem of (8) that we consider is one where any arbitrary $\gamma \in \mathbb{C} : |\gamma| = 1$ has been specified and is held fixed. Thus, we have the following subproblem:

$$\begin{aligned}&\min_{\alpha,\beta\in\mathbb{C}} \left\{w_1|\beta|^2 + w_2|\alpha|^2 + w_3(|\beta - r'_{i,i+1}|^2 + |\alpha - s'_{i,i+1}|^2)\right\} \\ &\qquad \text{s.t. } \left|\Delta + (r'_{i,i})^\dagger\beta + (s'_{i,i})^\dagger\alpha\right|^2 \geq 1,\end{aligned}$$

where $\Delta = \gamma\sum_{j=1}^{i-1}(r'_{j,i})^\dagger\beta_j + \gamma\sum_{j=1}^{i-1}(s'_{j,i})^\dagger\alpha_j$ is now a constant. Then, upon defining $\tilde{\mathbf{x}} = [\tilde{\alpha}, \tilde{\beta}]^T$, where $\tilde{\alpha} = \sqrt{w_2+w_3}\alpha$, $\tilde{\beta} = \sqrt{w_1+w_3}\beta$ along with $\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1, \tilde{\theta}_2]^T$ where $\tilde{\theta}_1 = \frac{w_3 s'_{i,i+1}}{\sqrt{w_2+w_3}}$ and $\tilde{\theta}_2 = \frac{w_3 r'_{i,i+1}}{\sqrt{w_1+w_3}}$, we can show that this subproblem is equivalent to

$$\min_{\tilde{\mathbf{x}}\in\mathbb{C}^{2\times1}} \left\{\|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\theta}}\|^2\right\} \text{ s.t. } |\Delta^\dagger + \tilde{\mathbf{x}}^\dagger\tilde{\mathbf{v}}|^2 \geq 1. \qquad (13)$$

where $\tilde{\mathbf{v}} = [(s'_{i,i})/\sqrt{w_2+w_3},\ (r'_{i,i})/\sqrt{w_1+w_3}]^T$. The following proposition provides an optimal solution to (13).

Proposition 5. *An optimal solution of (13),* $\hat{\tilde{\mathbf{x}}}$ *, is given by* $\hat{\tilde{\mathbf{x}}} = \mathbf{V}\breve{\mathbf{x}}$*, where* $\mathbf{V}$ *is a* 2×2 *unitary matrix obtained via the Eigen-decomposition* $\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\dagger$*. Further, defining* $\breve{\boldsymbol{\theta}} = [\breve{\theta}_1, \breve{\theta}_2]^T = \mathbf{V}^\dagger\tilde{\boldsymbol{\theta}}$*, we can specify elements of* $\breve{\mathbf{x}} = [\breve{x}_1, \breve{x}_2]^T$ *as* $\breve{x}_2 = \breve{\theta}_2$ *and*

$$\breve{x}_1 = \begin{cases}\breve{\theta}_1 \ \text{If } |\Delta^\dagger + \|\tilde{\mathbf{v}}\|\breve{\theta}_1^\dagger| \geq 1 \\ \frac{\hat{\lambda}\Delta\|\tilde{\mathbf{v}}\| + \breve{\theta}_1}{1-\|\tilde{\mathbf{v}}\|^2\hat{\lambda}}, \text{where } \hat{\lambda} = \frac{1-|\Delta^\dagger + \|\tilde{\mathbf{v}}\|\breve{\theta}_1^\dagger|}{\|\tilde{\mathbf{v}}\|^2} \text{ Else}\end{cases} \qquad (14)$$

Proof. Adopting the notation defined in the proposition, we note the key point that $\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\dagger$ is rank one so $\mathbf{V}^\dagger\tilde{\mathbf{v}} = [||\mathbf{v}||, 0]^T$. As a result upon performing a coordinate rotation using $\mathbf{V}^\dagger$ we obtain that an optimal solution in rotated coordinates, $\breve{\mathbf{x}}$, must satisfy $\breve{x}_2 = \breve{\theta}_2$, since the constraint can be seen to only depend on $\breve{x}_1$. Further, the constrained optimization to obtain the remaining complex variable $\breve{x}_1$ is the following one

$$\min_{x\in\mathbb{C}} \left\{|x - \breve{\theta}_1|^2\right\} \text{ s.t. } |\Delta^\dagger + x^\dagger||\tilde{\mathbf{v}}|||^2 \geq 1. \qquad (15)$$

(15) can be solved by first checking whether the unconstrained solution $x = \breve{\theta}_1$ is feasible, which is done in the first condition of (14). Otherwise, we know that the optimal solution must satisfy the constraint with equality and can be recovered after some careful algebra. □

In order to solve (8) we adopt the following two step procedure. In the first step, we solve (9). If the obtained optimal solution to (9) yields a $\hat{\gamma} : |\hat{\gamma}| \geq 1$, then clearly we have obtained the optimal solution to (8) as well. Otherwise, we move to step two. Here, if $|\hat{\gamma}|$ is above a threshold[5] then we solve (13) using $\gamma = \frac{\hat{\gamma}}{|\hat{\gamma}|}$. Finally, if $|\hat{\gamma}|$ is below that threshold, we instead solve (13) after initializing it with $\gamma = 1$. We found this procedure to be quite competitive compared to one where a more exhaustive search is done in step two by repeatedly solving (13) over several different phase values of γ.

V. Alternate Reconstruction

In this section we adapt a popular least-squares based alternative. If we directly apply the modeling used with (unquantized) analog observations (cf. [4]) to our case, we would model each feedback as

$$\sqrt{\gamma_i}\mathbf{g}_i = \mathbf{B}_i^\dagger \mathbf{h} + \boldsymbol{\eta}_i, \;\; i = 1, \cdots, M, \tag{16}$$

where $\boldsymbol{\eta}_i$ is (spatially white) additive noise. Here instead of this naive approach, we assume that each feedback can be modeled as follows.

$$\exp(j\theta_i)\sqrt{\gamma_i}\mathbf{g}_i = \mathbf{B}_i^\dagger \mathbf{h} + \boldsymbol{\eta}_i, \;\; i = 1, \cdots, M, \tag{17}$$

where $\{\exp(j\theta_i)\}_{i=1}^M$ are unknown phase-terms. This permits us to pose an alternate reconstruction formulation that we describe next. Towards this end, let us define $\mathbf{g}(\boldsymbol{\theta}) = [\exp(j\theta_1)\sqrt{\gamma_1}\mathbf{g}_1^T, \cdots, \exp(j\theta_M)\sqrt{\gamma_M}\mathbf{g}_M^T]^T$ as the observation vector, which is defined for any choice of $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_M]^T$, along with $\mathbf{B} = [\mathbf{B}_1, \cdots, \mathbf{B}_M]$. Then, in order to utilize the subspace information at hand let us enforce that $\mathbf{h} = \mathbf{U}_{\rm sub}\mathbf{z}$ for some $\mathbf{z} \in \mathbb{C}^{S\times 1}$. Notice here we are imposing that the reconstructed channel must lie in the subspace so if the reliability of the subspace information is in doubt, we should increase $\mathcal{R}(\mathbf{U}_{\rm sub})$ by appending columns to $\mathbf{U}_{\rm sub}$. Consider the least squares problem

$$\min_{\mathbf{z}\in\mathbb{C}^{S\times 1}, \boldsymbol{\theta}\in[0,2\pi]^M} \{\|\mathbf{g}(\boldsymbol{\theta}) - \mathbf{B}^\dagger \mathbf{U}_{\rm sub}\mathbf{z}\|^2\} \tag{18}$$

Note that in (18) we are choosing the best possible phase ambiguity with respect to minimizing the squared error norm. Keeping $\boldsymbol{\theta}$ fixed, we can solve for $\mathbf{z}$ to obtain an optimal $\mathbf{z}$, denoted by $\hat{\mathbf{z}}(\boldsymbol{\theta})$, as $\hat{\mathbf{z}}(\boldsymbol{\theta}) = (\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+\mathbf{g}(\boldsymbol{\theta})$, where $(\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+$ is the pseudo-inverse of $\mathbf{B}^\dagger\mathbf{U}_{\rm sub}$. Substituting this optimal solution (which itself depends on the choice of $\boldsymbol{\theta}$) into (18) we obtain

$$\min_{\boldsymbol{\theta}\in[0,2\pi]^M} \{\|(\mathbf{I} - \mathbf{B}^\dagger\mathbf{U}_{\rm sub}(\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+)\mathbf{g}(\boldsymbol{\theta})\|^2\} \tag{19}$$

Further defining $\mathbf{G} = {\rm BlkDiag}\{\sqrt{\gamma_1}\mathbf{g}_1, \cdots, \sqrt{\gamma_M}\mathbf{g}_M\}$ and noting that $\mathbf{I} - \mathbf{B}^\dagger\mathbf{U}_{\rm sub}(\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+$ is a projection matrix, we can express (19) as

$$\min_{\boldsymbol{\theta}\in[0,2\pi]^M} \{\exp(j\boldsymbol{\theta})^\dagger \mathbf{G}^\dagger(\mathbf{I} - \mathbf{A}\mathbf{A}^+)\mathbf{G}\exp(j\boldsymbol{\theta})\} \tag{20}$$

where we have used $\mathbf{A} = \mathbf{B}^\dagger\mathbf{U}_{\rm sub}$ and $\exp(j\boldsymbol{\theta}) = [\exp(j\theta_1), \cdots, \exp(j\theta_M)]^T$. The problem in (20) can be readily solved for $M = 2$ and after some effort for $M = 3$ as well. However, for general M it is known to be an NP-hard problem which requires us to devise sub-optimal albeit efficient techniques. Notice that if we relaxed (20) and allowed optimizing over arbitrary vectors under an ℓ_2 norm-constraint, we would get the solution to be any (scaled) Eigen-vector corresponding to the minimal Eigenvalue of $\mathbf{G}^\dagger(\mathbf{I} - \mathbf{A}\mathbf{A}^+)\mathbf{G}$. We could then simply pick the phase-terms of all elements in that vector to obtain a sub-optimal solution to the original problem in (20). Here, we have modified our K-best approach for the objective in (20), which we found outperforms the approach based on the minimal Eigen-value. Once a (sub-)optimal solution of (20), say $\hat{\boldsymbol{\theta}}$, is recovered we can determine our reconstruction as $\mathbf{U}_{\rm sub}(\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+\mathbf{g}(\hat{\boldsymbol{\theta}})$. Also note that if we choose to employ the naive model in (16) we will obtain the reconstruction to be $\mathbf{U}_{\rm sub}(\mathbf{B}^\dagger\mathbf{U}_{\rm sub})^+\mathbf{g}(\mathbf{0})$ (i.e., we are supposing no ambiguity and fixing the unknown phase terms to be all zeros).

VI. Simulation Results

We consider the downlink of a single-cell in which the BS (equipped with $N_t = 32$ transmit antennas) communicates with 10 single receive antenna users. We adopt the popular single scattering ring channel model and simulate sum rate performance averaged over 30 drops. Each drop entails generating a fresh set of user locations followed by large-scale fading coefficients. Keeping the coefficients so generated fixed, within each drop sum-rate is averaged over thousand fast fading realizations. The average sum-rates corresponding to different drops are then averaged together to obtain the overall sum-rate. The key simulation aspects are discussed next.

• *Prebeam Matrix Design:* We suppose that a collection of pre-beam matrices, $\{\tilde{\mathbf{B}}_\ell\}$, is given out of which $M = 3$ matrices, $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$, are assigned to the user of interest. Each pre-beam matrix in the collection is 32×8 and comprises of a group of 8 columns from a (possibly over-sampled) Fourier matrix with 32 rows. We consider two cases. The first one is referred to here as the **orthogonal design**. In this design the first four matrices in the collection, $\{\tilde{\mathbf{B}}_\ell\}_{\ell=1}^4$ correspond to a partition of the 32×32 uniformly sampled (hence unitary DFT) Fourier matrix into 4 groups (each comprising of adjacent columns) so that $\tilde{\mathbf{B}}_\ell$ has columns with indices $\{8(\ell-1)+1, \cdots, 8\ell\}$ for $\ell = 1, \cdots, 4$. Each one of the remaining matrices in the collection, contains four columns from any two out of these four groups. The second design is referred to as the **non-orthogonal design**. Here we consider a 32×64 over-sampled Fourier matrix with oversampling factor 2. Then, the first eight pre-beam matrices in the collection, $\{\tilde{\mathbf{B}}_\ell\}_{\ell=1}^8$ correspond to a partition of the aforementioned 32×64 matrix into 8 groups (each comprising of adjacent columns) so that $\tilde{\mathbf{B}}_\ell$ has columns with indices $\{8(\ell-1)+1, \cdots, 8\ell\}$ for $\ell = 1, \cdots, 8$. Each of the remaining matrices in the collection, contains four columns from any two out of these eight groups. Notice that the non-orthogonal design entails a higher pilot overhead. Further, with either design upon pre-beamforming the pilot signals, each user estimates an 8×1 pre-beamformed channel vector and thereby employs the standardized 8-port quantization codebook to obtain its PMI vector report.

• *Prebeam Group Assignment:* The BS obtains estimates of $\{E[\|\tilde{\mathbf{B}}_\ell^\dagger\mathbf{h}\|^2]\}_{\ell=1}^L$, where $L = 4$ ($L = 8$) for orthogonal (non-orthogonal) design. We note that these quantities need to be obtained over a coarser time-scale and only those above a configurable threshold are of interest. The BS then assigns two groups corresponding to the two largest among these L computed values. The third assigned prebeam matrix is formed by concatenating four columns of the assigned matrix corresponding to the maximal computed value and four

[5]We used 0.01 in our simulations.

TABLE I
ORTHOGONAL PREBEAM DESIGN: QUANTIZED FEEDBACK

Reconstruction Scheme	Sum Rate (b/s/Hz)	Percentage Gain (over baseline)
(P1) with optimal	14.1381	32.84%
(P1) with K-best	14.1273	32.74%
Alternate Reconstruction w/o phase opt.	9.6678	-10.09%
Alternate Reconstruction w. phase opt.	11.9999	12.75%
Baseline	10.6429	0%

TABLE II
NON-ORTHOGONAL PREBEAM DESIGN: QUANTIZED FEEDBACK

Reconstruction Scheme	Sum Rate (b/s/Hz)	Percentage Gain (over baseline)
(P1) with K-best	20.0048	35.06%
Alternate Reconstruction w/o phase opt.	10.4539	-29.45%
Alternate Reconstruction w. phase opt.	14.5731	-1.65%
Baseline	14.8115	0%

columns of the second assigned one with the second largest value.

• *MU-MIMO Scheduling:* In each slot the BS obtains a reconstruction of each user's channel and then selects a subset of users that it decides to serve in that slot. The user selection we use is a practical one which assumes each reconstructed channel to be accurate and proceeds to employ the greedy zero-forcing method [15] that has been found to be effective.

• *Baseline:* The Baseline method we compare against entails assiging one pre-beam matrix, $\mathbf{B}_1$, corresponding to the maximal value among $\{E[\|\tilde{\mathbf{B}}_\ell^\dagger \mathbf{h}\|^2]\}_{\ell=1}^L$, to each user. The reconstructed channel is obtained by solving (P1) where now $M = 1$, in the absence of any auxiliary estimate (so $w_3 = 0$) but with or without subspace information. If reliable subspace information is not avaialble, we set $w_1 = w_2 = 1$ to determine the baseline solution as $\frac{\sqrt{\gamma_1}}{\|\mathbf{B}_1\mathbf{g}_1\|^2}\mathbf{B}_1\mathbf{g}_1$. On the other hand, with reliable subspace information we can enforce $\mathbf{h} = \mathbf{P}_{\rm sub}\mathbf{h}$ (by considering $w_2 >> w_1$) and then solve (P1) to obtain the baseline solution as $\frac{\sqrt{\gamma_1}}{\|\mathbf{P}_{\rm sub}\mathbf{B}_1\mathbf{g}_1\|^2}\mathbf{P}_{\rm sub}\mathbf{B}_1\mathbf{g}_1$.

A. Only Quantized Information

We first introduce our results in which only multiple quantized feedback are employed for reconstruction without utilizing subspace information. We solve (P1) where $M = 3$ after setting $w_1 = w_2 = 1$ and $w_3 = 0$. We use our proposed K-best reconstruction described in Section IV-A with $K = 3$. To benchmark we also plot the performance of the described baseline as well as the alternate reconstruction scheme of Section V. For the latter scheme we consider both phase optimization as well as no phase optimization which ignores the phase ambiguity problem. In Table I we present the sum rates of K-best and alternate reconstruction schemes for orthogonal pre-beam design. In addition, recalling Remark 1 in Section III that (P1) is optimally solvable in this case, we follow the methodology of Section III to obtain the optimal reconstruction and plot its resulting sum-rate. *From Table I we see that our proposed K-best reconstruction is almost the same in performance as the scheme that optimally solves (P1).* Further, our framework offers substantial gains over the baseline and the alternate reconstruction schemes. In particular, the popular alternate reconstruction scheme that ignores phase ambiguity is completely unsuited to realistic quantized feedback. In Table II we present our results for non-orthogonal pre-beam design. In this case we only solve (P1) via the K-best scheme since there is no efficient method to obtain its optimal solution. The substantial througput improvement provided by our scheme is evident.

TABLE III
ORTHOGONAL PREBEAM DESIGN: QUANTIZED FEEDBACK PLUS SUBSPACE

Reconstruction Scheme	Sum Rate (b/s/Hz)	Percentage Gain (over baseline)
(P1) with K-best	20.8534	15.21%
Alternate Reconstruction w. phase opt.	19.2483	6.34%
Baseline	18.001	0%

B. Quantized Information plus Subspace

We now consider combining the intantaneous quantized feedback and subspace information. Utilizing the available $\mathbf{U}_{\rm sub}$ we solve (P1) after setting $w_1 = 1$ & $w_2 = 600$ with $w_3 = 0$. We evaluate the reconstruction obtained upon solving (P1) with K-best scheme and present our results in Table III for the orthogonal pre-beam design. Notice that the gains are good even though the baseline itself incorporates knowledge of subspace and has significantly improved. Even larger gains were obtained with non-orthogonal pre-beam design. In either case the alternate reconstruction scheme (with phase optimization) yielded strictly inferior performance compared to our proposed one based on (P1).

Simulations with $M = 2$ pre-beam matrices per-user were also obtained. These results (which are skipped due to space constraints) also reinforced the superior performance of our reconstruction scheme. We also noticed over multiple different scenarios that a small value of the number of survivors parameter, K, is enough to capture most of the available gains.

VII. CONCLUSIONS

We proposed a new formulation for channel reconstruction that combines fine time-scale quantized feedback and coarse time-scale subspace information. We derived an efficient algorithm to solve the resulting non-convex quadratic programming via a K-best methodology and demonstrated its efficacy.

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