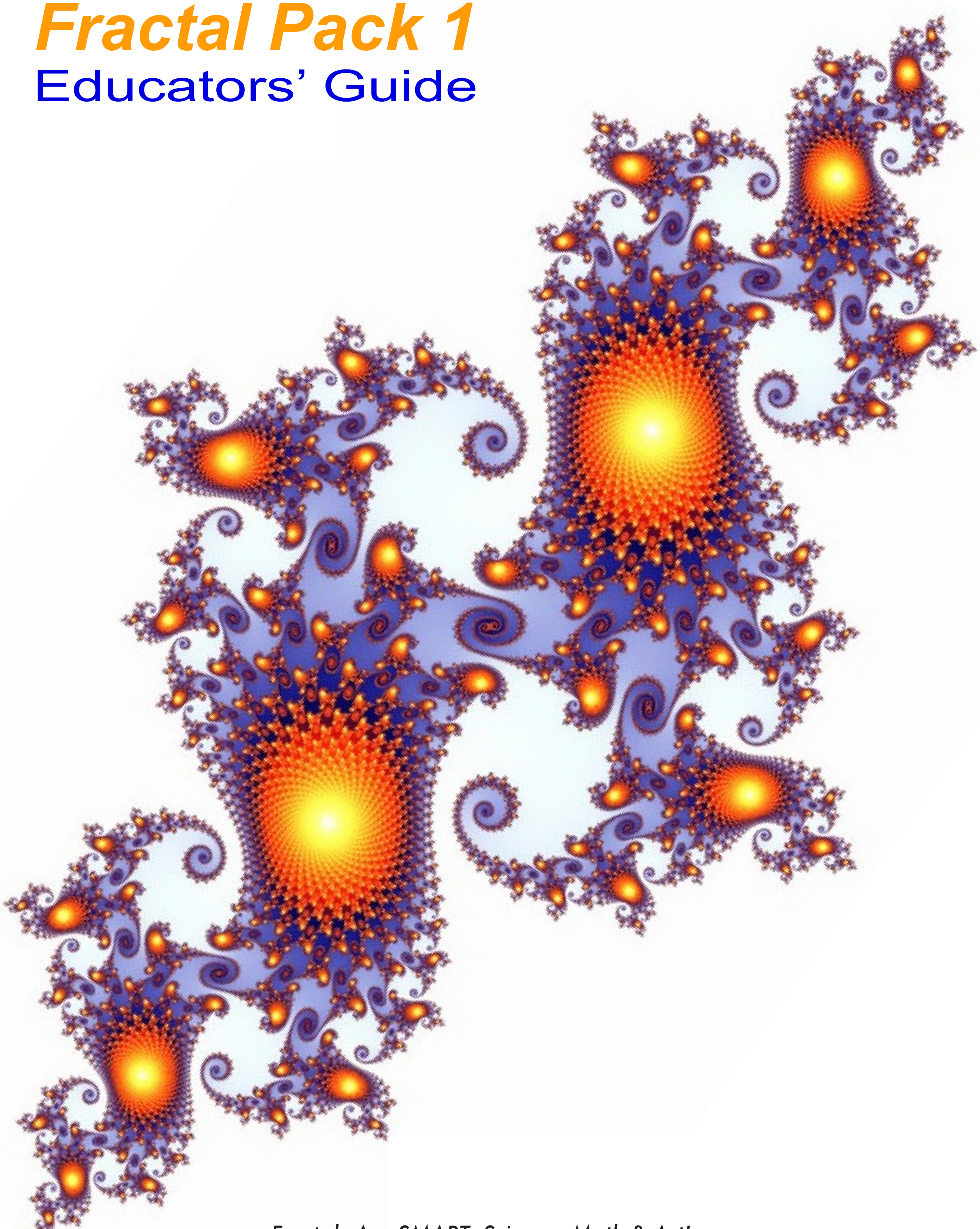
# Fractal Pack 1
## Educators' Guide



*Fractals Are SMART: Science, Math & Art!*
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# Contents

# INTRODUCTION:

## WHAT IS A FRACTAL?

A fractal is a never ending pattern that repeats itself at different scales. This property is called "Self-Similarity."

Fractals are extremely complex, sometimes infinitely complex - meaning you can zoom in and find the same shapes forever.

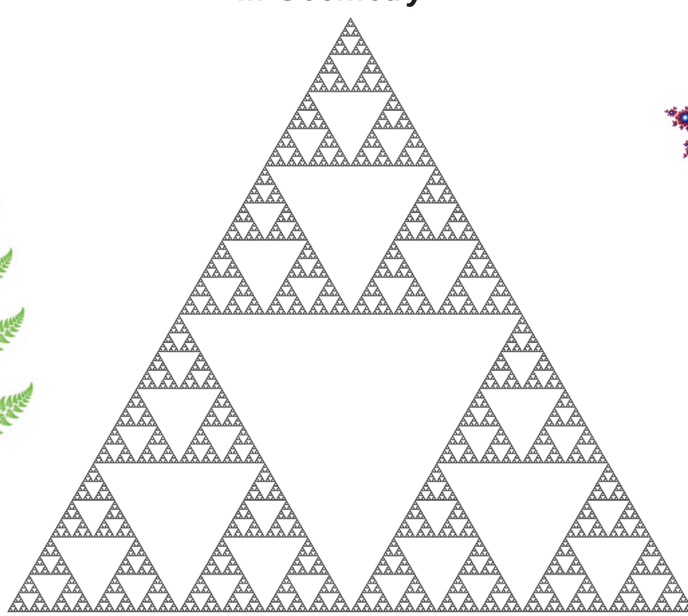
Amazingly, fractals are extremely simple to make.

A fractal is made by repeating a simple process again and again.

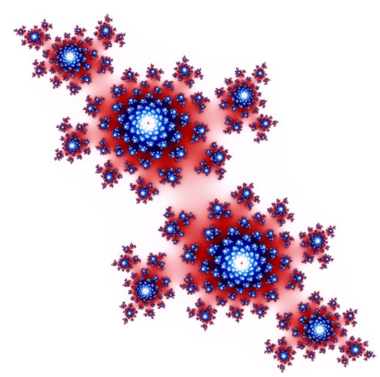
## WHERE DO WE FIND FRACTALS?

***In Nature***

***In Geometry***



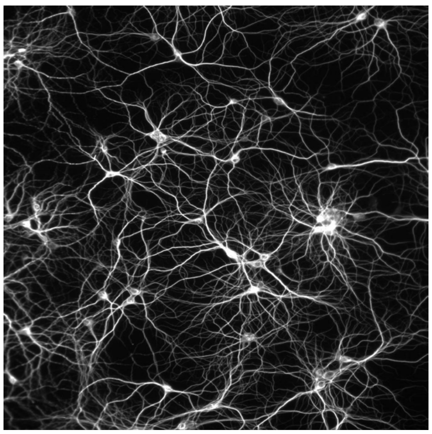
***In Algebra***



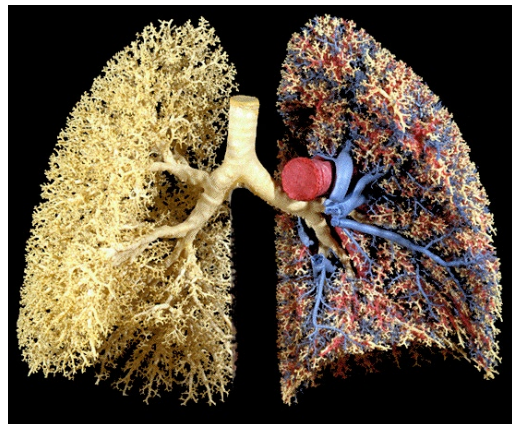
# NATURAL FRACTALS

## BRANCHING

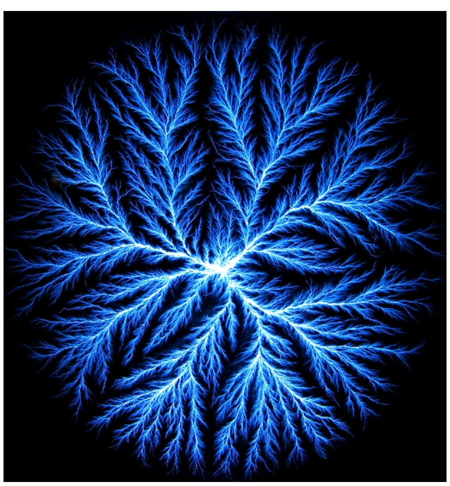
Fractals are found all over nature, spanning a huge range of scales. We find the same patterns again and again, from the tiny branching of our blood vessels and neurons to the branching of trees, lightning bolts, and river networks. Regardless of scale, these patterns are all formed by repeating a simple branching process.
*A fractal is a picture that tells the story of the process that created it.*



Neurons from the human cortex. The branching of our brain cells creates the incredibly complex network that is responsible for all we perceive, imagine, remember. Scale = 100 microns = $10^{-4}$ m.

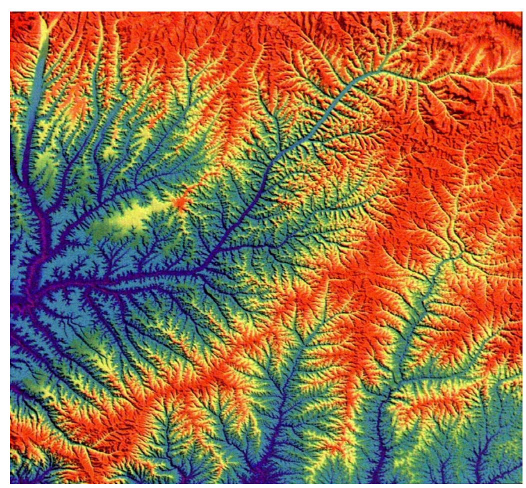


Our lungs are branching fractals with a surface area ~100 $m^2$. The similarity to a tree is significant, as lungs and trees both use their large surface areas to exchange oxygen and $CO_2$. Scale = 30 cm = $3*10^{-1}$ m.



Lichtenberg "lightning", formed by rapidly discharging electrons in lucite. Scale = 10 cm = $10^{-1}$ m.

Oak tree, formed by a sprout branching, and then each of the branches branching again, etc. Scale = 30 m = $3*10^{1}$ m.



River network in China, formed by erosion from repeated rainfall flowing downhill for millions of years. Scale = 300 km = $3*10^{5}$ m.

# NATURAL FRACTALS

## SPIRALS

The spiral is another extremely common fractal in nature, found over a huge range of scales. Biological spirals are found in the plant and animal kingdoms, and non-living spirals are found in the turbulent swirling of fluids and in the pattern of star formation in galaxies.

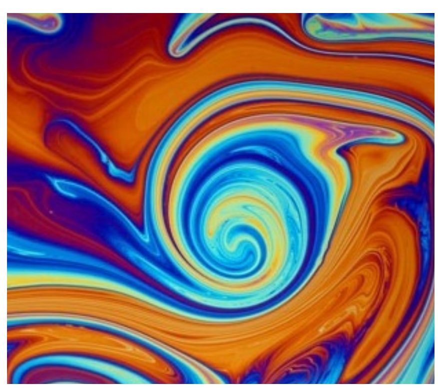
All fractals are formed by simple repetition, and combining expansion and rotation is enough to generate the ubiquitous spiral.

A fossilized ammonite from 300 million years ago. A simple, primitive organism, it built its spiral shell by adding pieces that grow and twist at a constant rate. Scale = 1 m.

A hurricane is a self-organizing spiral in the atmosphere, driven by the evaporation and condensation of sea water. Scale = 500 km = $5*10^5$ m.

A spiral galaxy is the largest natural spiral comprising hundreds of billions of stars. Scale = 100,000 ly = $\sim 10^{20}$ m.

The plant kingdom is full of spirals. An agave cactus forms its spiral by growing new pieces rotated by a fixed angle. Many other plants form spirals in this way, including sunflowers, pinecones, etc. Scale = 50 cm = $5*10^{-1}$ m.



The turbulent motion of fluids creates spirals in systems ranging from a soap film to the oceans, atmosphere and the surface of jupiter. Scale = 5 mm = $5*10^{-3}$ m.

A fiddlehead fern is a self-similar plant that forms as a spiral of spirals of spirals. Scale = 5 cm = $5*10^{-2}$ m.

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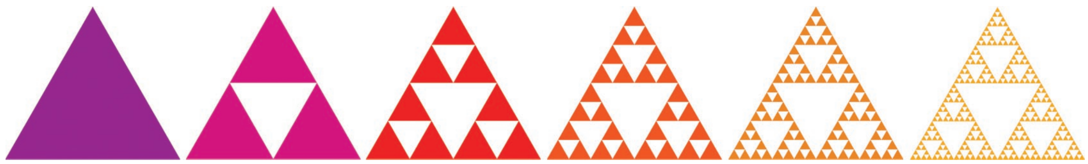
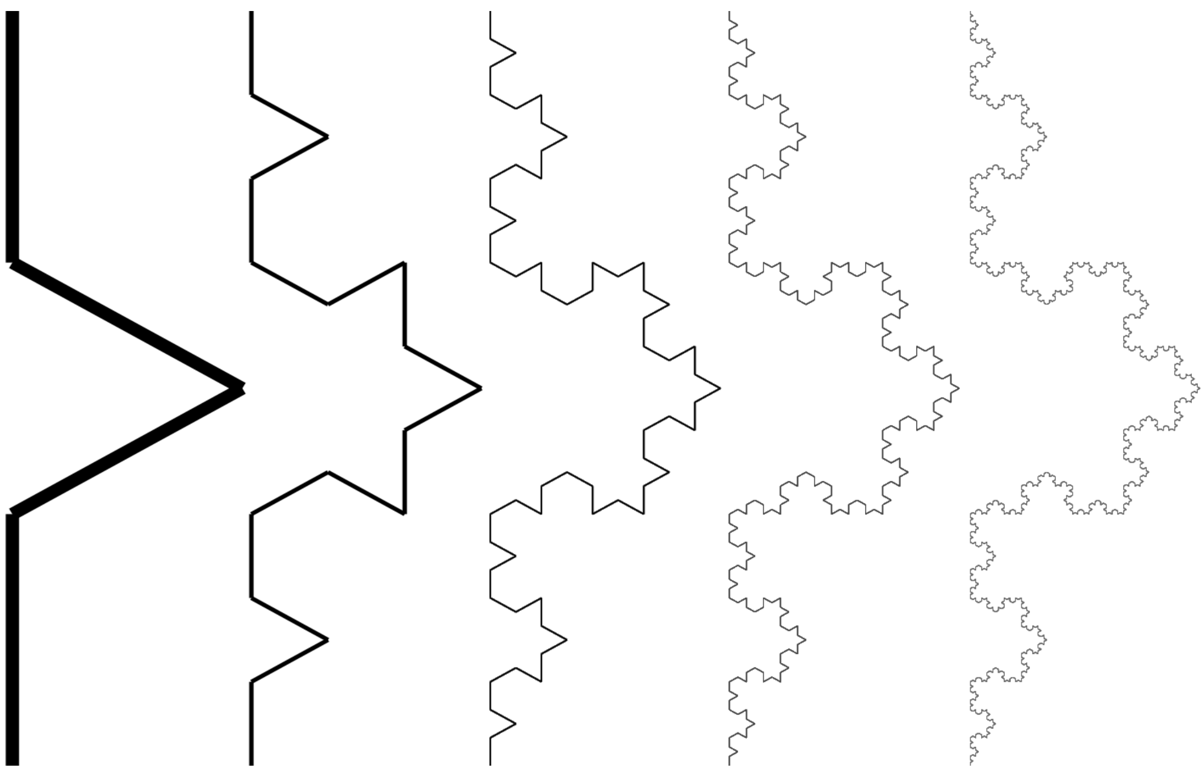
# GEOMETRIC FRACTALS

Purely geometric fractals can be made by repeating a simple process.



The Sierpinski Triangle is made by repeatedly removing the middle triangle from the prior generation. The number of colored triangles increases by a factor of 3 each step, 1,3,9,27,81,243,729, etc.

See the Fractivity on page 15 to learn to teach elementary school students how to draw and assemble Sierpinski Triangles.



The Koch Curve is made by repeatedly replacing each segment of a generator shape with a smaller copy of the generator. At each step, or iteration, the total length of the curve gets longer, eventually approaching infinity. Much like a coastline, the length of the curve increases the more closely you measure it.

# ALGEBRAIC FRACTALS

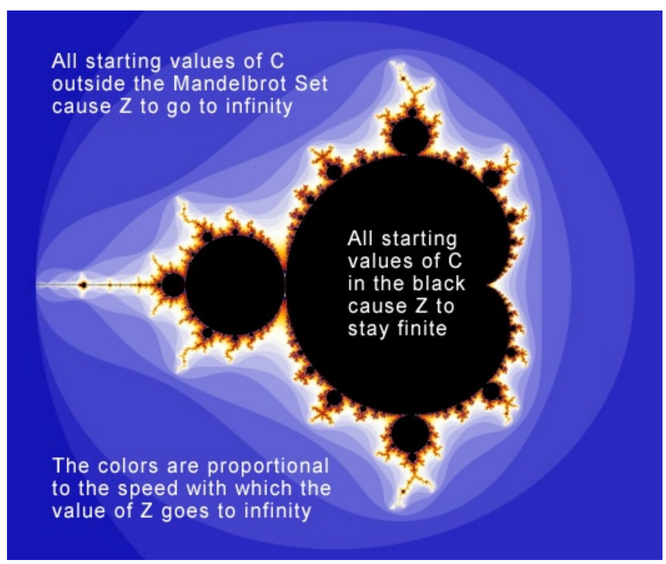
We can also create fractals by repeatedly calculating a simple equation over and over. Because the equations must be calculated thousands or millions of times, we need computers to explore them. Not coincidentally, the Mandelbrot Set was discovered in 1980, shortly after the invention of the personal computer.

## HOW DOES THE MANDELBROT SET WORK?

We start by plugging a value for the variable ‘C’ into the simple equation below. Each complex number is actually a point in a 2-dimensional plane. The equation gives an answer, ‘$Z_{new}$’ . We plug this back into the equation, as ‘$Z_{old}$’ and calculate it again.  We are interested in what happens for different starting values of ‘C’.

Generally, when you square a number, it gets bigger, and then if you square the answer, it gets bigger still. Eventually, it goes to infinity. This is the fate of most starting values of ‘C’. However, some values of ‘C’ do not get bigger, but instead get smaller, or alternate between a set of fixed values. These are the points inside the Mandelbrot Set, which we color black. Outside the Set, all the values of ‘C’ cause the equation to go to infinity, and the colors are proportional to the speed at which they expand.

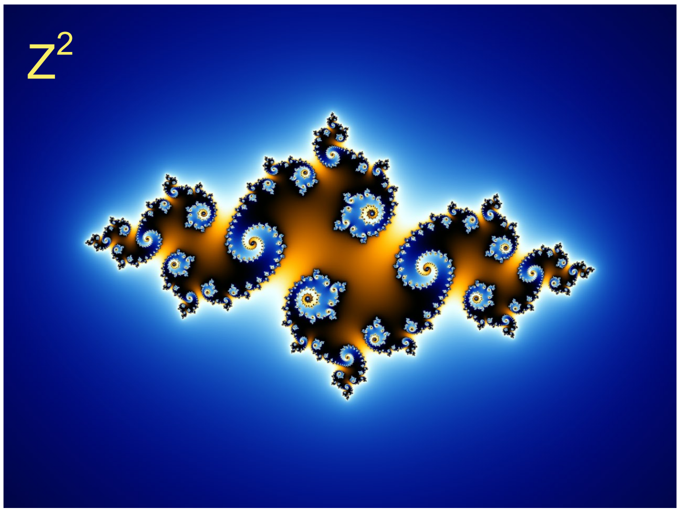
$$Z_{new} = Z_{old}^2 + C$$

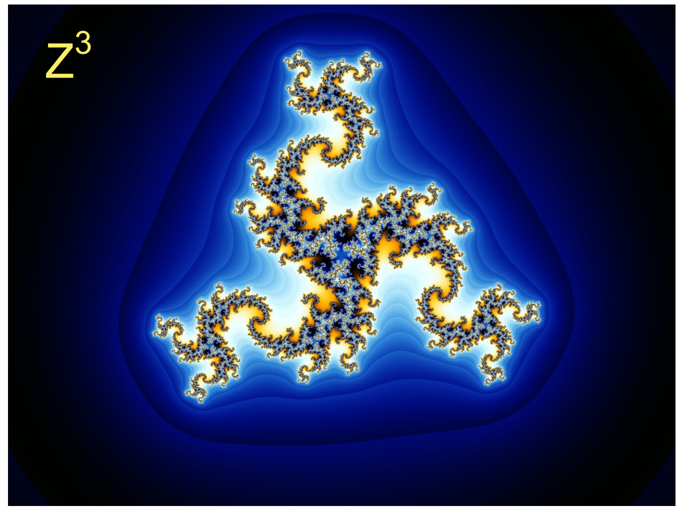


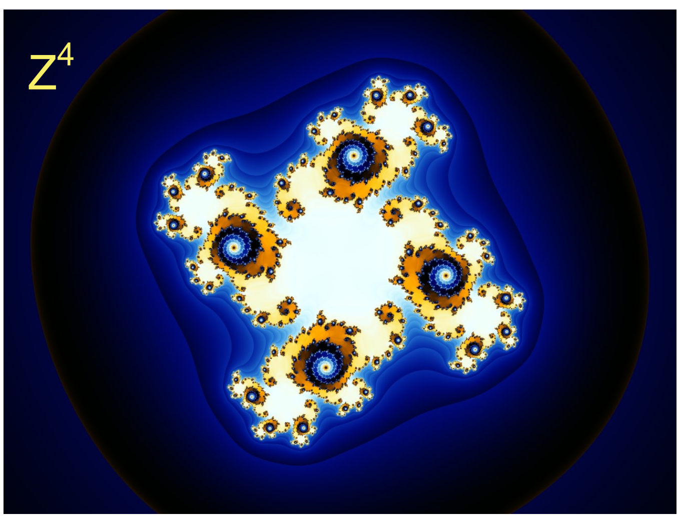

The interesting places in ths fractal are all on the edge. We can zoom in forever, and never find a clear edge. The deeper we explore, the longer the numbers become, and the slower the calculations are. Deep fractal exploration takes patience!

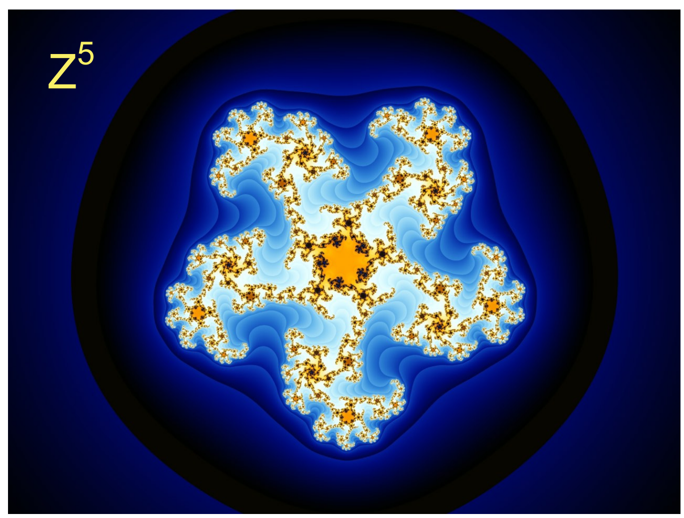
# PATTERNS & SYMMETRY

The great value of fractals for education is that they make abstract math visual. When people see the intricate and beautiful patterns produced by equations, they lose their fear and instead become curious.



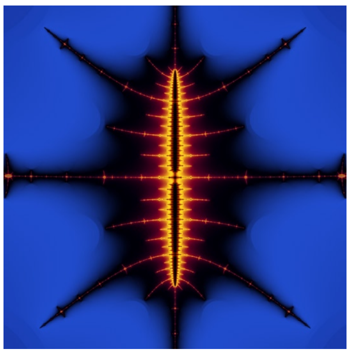
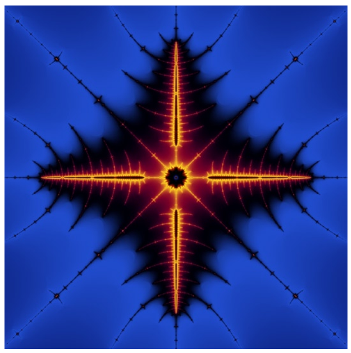
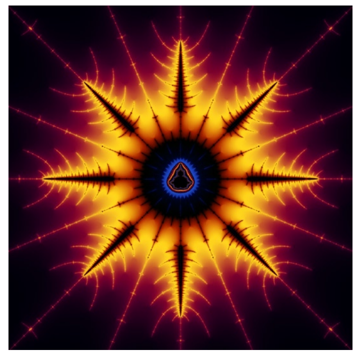



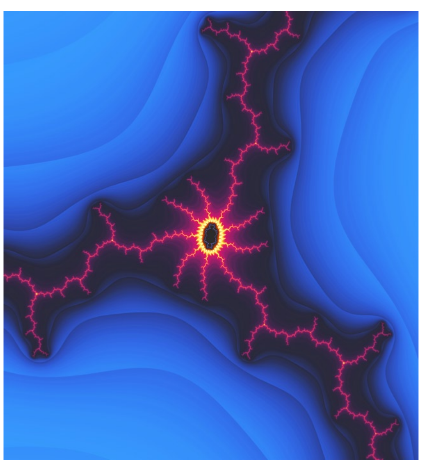
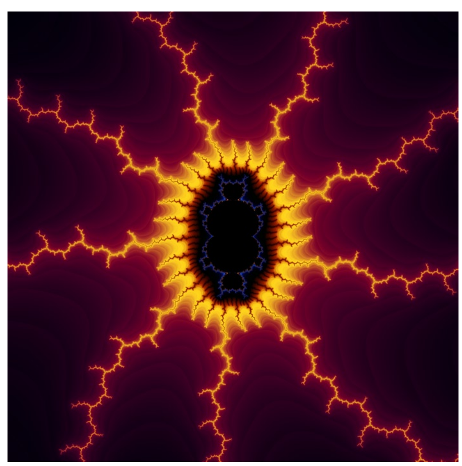

Exploring fractals is fun, and we can play with the equations to see what happens. The 4 images above are algebraic fractals known as Julia Sets. The first image in the upper left comes from the same equation as the Mandelbrot Set, $Z = Z^2 + C$. When we raise the exponent to $Z^3$ (i.e. Z*Z*Z), the Julia Set takes on a 3-fold symmetry, and so on. *The degree of symmetry always corresponds to the degree of the exponent.*

# PATTERNS & SYMMETRY

Just as we find branching fractals in nature, we also find branching within algebraic fractals like the Mandelbrot Set. Known as “Bifurcation”, branching in these fractals is a never-ending process. The four images below are successive zooms into a detail of the $Z = Z^2 + C$ Mandelbrot Set. Two-fold symmetry branches and becomes 4-fold, which doubles into 8-fold, and then 16-fold. The branching process continues forever, and the number of arms at any level is always a power of 2.


If we explore other algebraic fractals, we find similar patterns and progressions. The two images below are details from the $Z = Z^3 + C$ Mandelbrot Set. Since the equation involves Z *cubed*, the arms now branch in 3-fold symmetry. Each of the 3 arms branches into 3 more arms, becoming 9-fold symmetry. This then trifurcates into 27 arms, 81, 243, etc. where the number of arms is always a power of 3.

Again, the educational value of fractals is that they make the behavior of equations visible. Zooming into fractals, math ceases to be intimidating, and instead becomes entrancing.

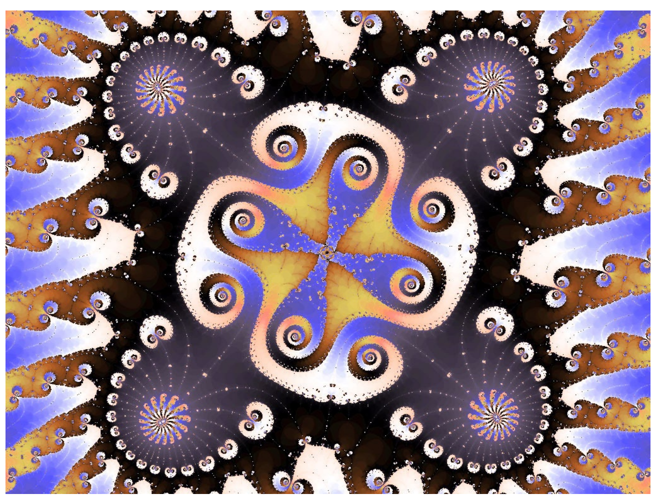
# IDEAS OF SCALE

## HOW BIG (OR SMALL) ARE FRACTALS?

Mathematical fractals are infinitely complex. This means we can zoom into them forever, and more detail keeps emerging. To describe the scale of fractals, we must use scientific notation:

| | | |
|---|---|---|
| Thousand | 1,000 | $10^3$ |
| Million | 1,000,000 | $10^6$ |
| Billion | 1,000,000,000 | $10^9$ |
| Trillion | 1,000,000,000,000 | $10^{12}$ |
| Quadrillion | 1,000,000,000,000,000 | $10^{15}$ |

Because of the limits of computer processors, all the fulldome fractal zooms stop at a magnification of $10^{16}$. Of course the fractals keep going, but it becomes much slower to compute deeper than that. $10^{16}$ (or ten quadrillion) is incredibly deep. To put it in perspective, the diameter of an atom is about $10^{-10}$ meters, so as we zoom six orders of magnitude smaller, we're looking at things a million times smaller than an atom!
Or, to look at it another way, as we zoom into the fractals, the original object keeps growing. How big does it get when we have zoomed in $10^{16}$ times? The orbit of the dwarf planet Pluto is about $10^{12}$ meters in diameter. If we start zooming in a 10 meter dome, then the original image grows to a size larger than our entire solar system - 100,000 times larger!



All of these zooms are just scratching the surface of the infinitely complex. Some fractals, like the Mandelbrot Set, become even more intricate and beautiful the deeper we explore. The image above exists at a depth of $10^{176}$ magnification!

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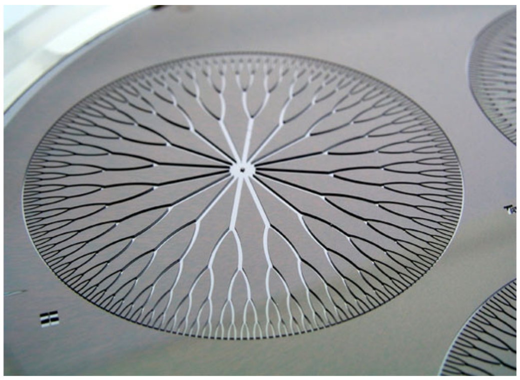
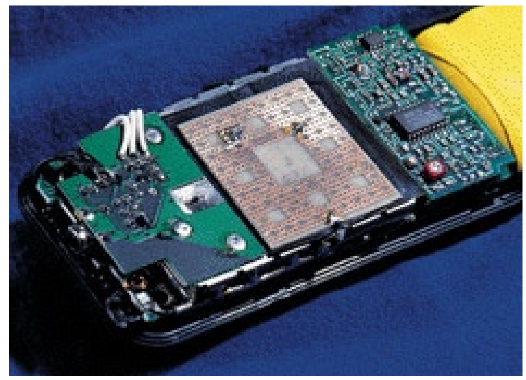
# FRACTAL APPLICATIONS

A commonly asked question is: What are fractals useful for?

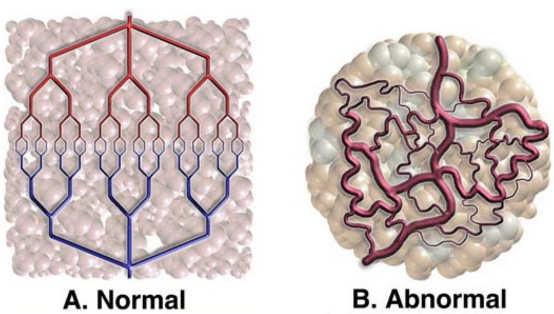
Nature has used fractal designs for at least hundreds of millions of years. Only recently have human engineers begun copying natural fractals for inspiration to build successful devices. Below are just a few examples of fractals being used in engineering and medicine.



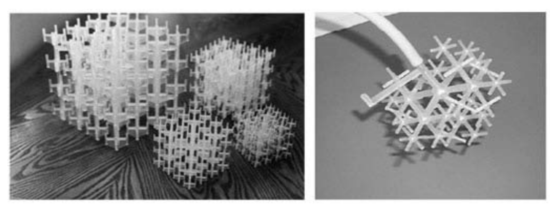
A computer chip cooling circuit etched in a fractal branching pattern. Developed by researchers at Oregon State University, the device channels liquid nitrogen across the surface to keep the chip cool.



Fractal antennas developed by Fractenna in the US and Fractus in Europe are making their way into cellphones and other devices. Because of their fractal shapes, these antennas can be very compact while receiving radio signals across a range of frequencies.




Researchers at Harvard Medical School and elsewhere are using fractal analysis to assess the health of blood vessels in cancerous tumors. Fractal analysis of CT scans can also quantify the health of lungs suffering from emphysema or other pulmonary illnesses.

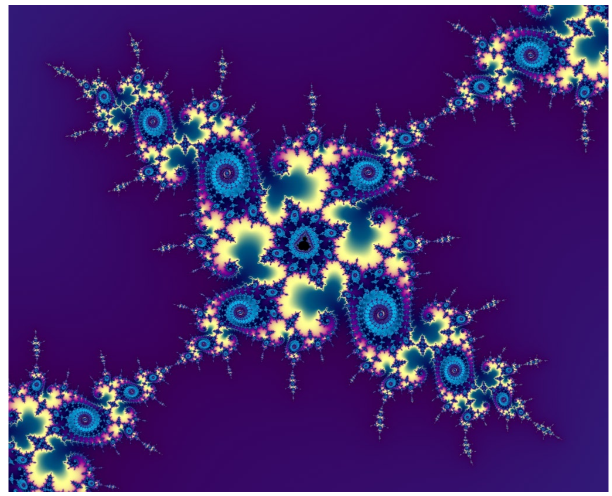


Amalgamated Research Inc (ARI) creates space-filling fractal devices for high precision fluid mixing. Used in many industries, these devices allow fluids such as epoxy resins to be carefully and precisely blended without the need for turbulent stirring.

# FULLDOME ANIMATIONS

## FRACTAL PACK 1

### CRYSTALOON

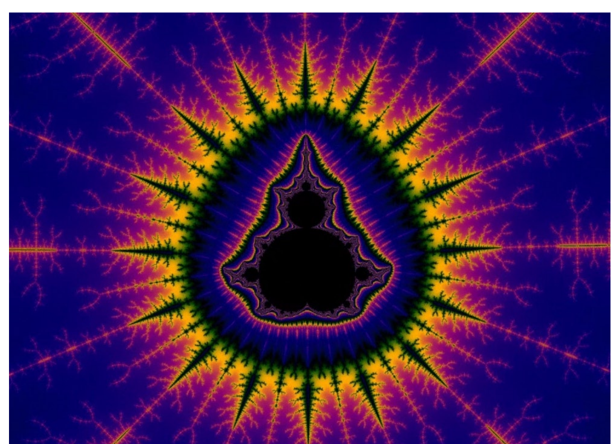


This fractal animation explores the Mandelbrot Set, the archetypal algebraic fractal. It comes from the very simplest possible non-linear equation:

$$Z_{new} = Z_{old}^{2} + C$$

We journey into a region known as the “Crystal Canyon”, due to the crystalline nature of the decorations. Zooming into one of the jewels along the edge, the journey progresses into a wheel of mesh-like crystals. The zoom continues into a sequence of self-similar units before veering into the connection point between units. At this point the patterns begin to bifurcate, turning from 2-fold to 4-fold symmetry, then 8, 16, 32, etc, all surrounding a miniature replica of... the entire Mandelbrot Set. The zoom culminates at a magnification of $10^{16}$, or a million times smaller than the diameter of a hydrogen atom.

### GALANGA

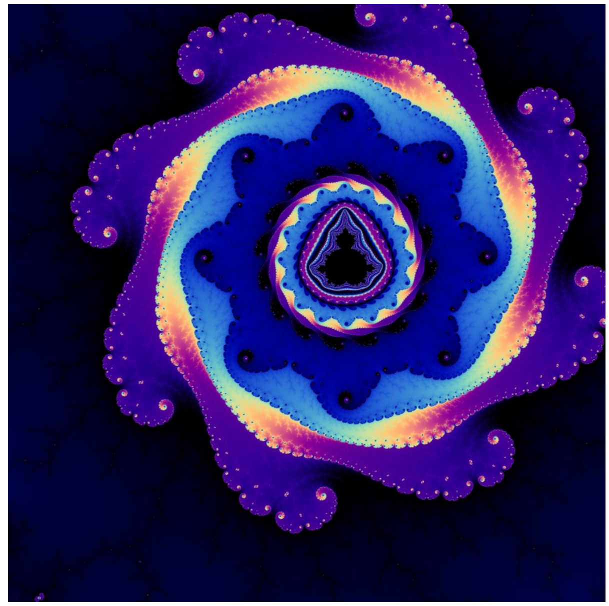


This zoom explores the “Snowflake Village,” another region within the Mandelbrot Set. One of the astonishing things about this fractal is that it is *not* perfectly self-similar. Different areas have vastly different structural styles, and the entire Set contains an infinite variety of shapes and patterns.

Another amazing discovery about the Mandelbrot Set is that this purely abstract entity contains structures that closely resemble the fractals of nature. In Galanga, we find objects reminiscent of snowflakes, flowers, pine trees, bacteria, chromosomes, insects, etc - all created from the extremely simple equation above.

## Pleoria

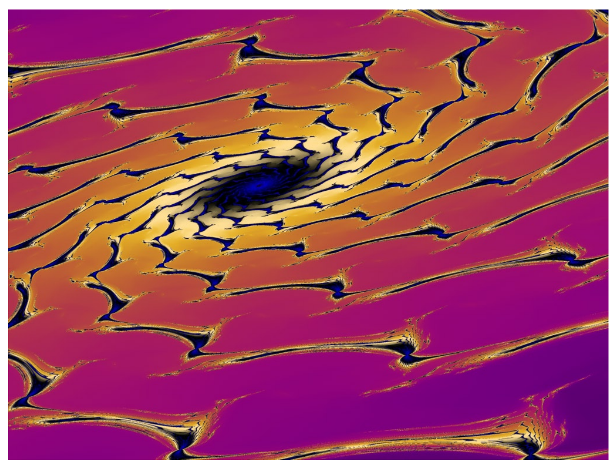
In this animation, we are using the same equation, $Z = Z^2 + C$ but we plot the inverse of the values, which causes the whole Set to turn dramatically inside out. This zoom explores the details around the edges of the miniature embedded Mandelbrot replicas, which give rise to structures we refer to as virtual Julia Sets. We encounter structures reminiscent of lightning bolts, brains and flowers.



## MORPHALINGUS

$$Z_{n+1} = Z_n^{\,2} + C + \frac{abs(oz-C)}{abs(Z_n - C)}$$

In this animation, we start with the same general formula for the Mandelbrot Set, but with an additional complex term added in. This results in a similar fractal, but with some very different characteristics. The zoom explores several different areas of the fractal, each of which has its own distinct musical accompaniment.



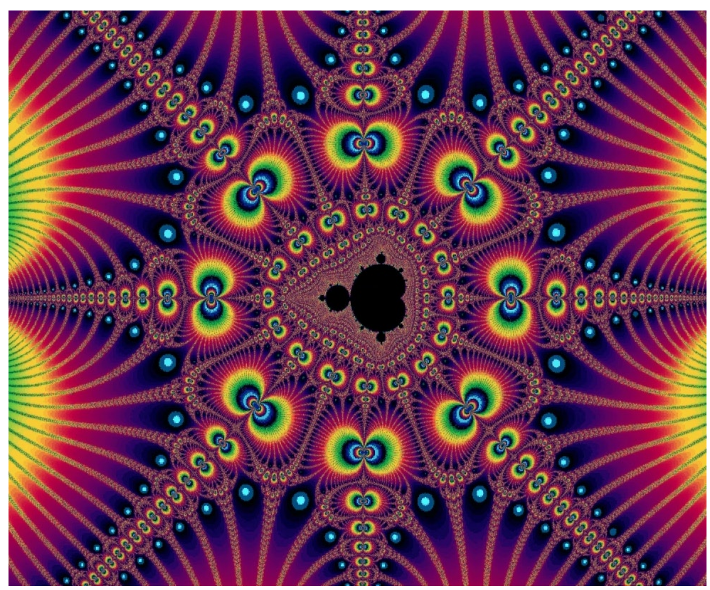
## FEATHERINO

In this animation, we are exploring the behavior of a very different equation:

$$Z_{new} = Z_{old} - a\frac{(Z_{old}^{4} - 1)}{4Z_{old}^{3}}$$

This equation was actually created over 300 years ago by Isaac Newton, and it describes the process known as Newton's Method for solving a polynomial equation. Of course, without a computer, Newton could not see how beautiful his equation really is. The fractal created by this equation is perfectly self-similar, and it is infinitely large as well as infinitely small.

## PEACOCK



This fractal comes from yet another equation:

$$Z_{new} = Z_{old} - \frac{(Z_{old}^{2} - 1)}{a(Z_{old}^{2} + 1)}$$

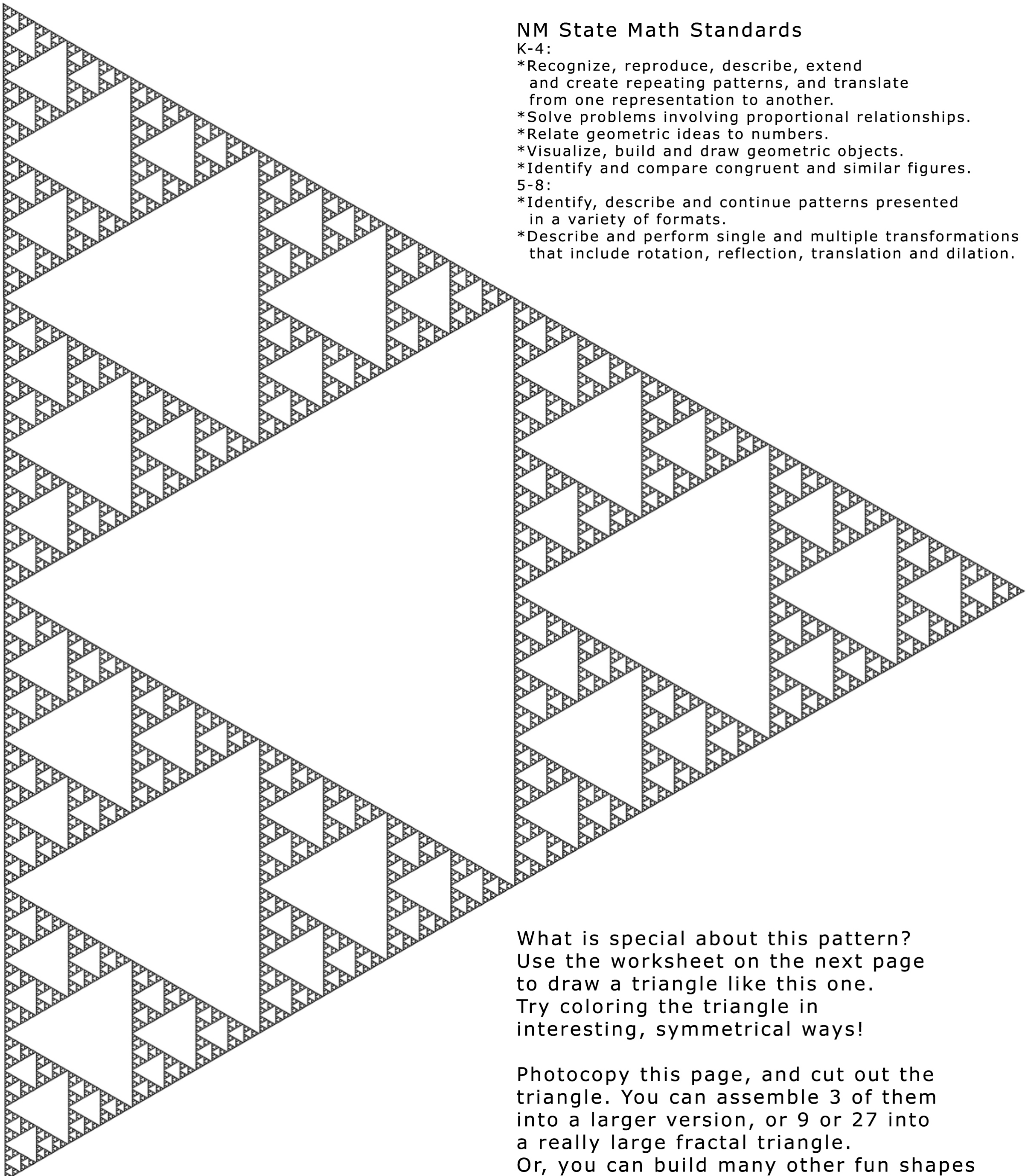
While this fractal equation is related to the Newton fractal above, it behaves very differently. It is not perfectly self-similar, but instead it gains in complexity as we zoom in. Further, embedded within this fractal, we find tiny copies of the Mandelbrot Set!

## GEOMETRIC FRACTALS

In this animation, we explore fractals that are made using simple geometric repetition of shapes rather than equations. We can watch trees branch and grow in 3-dimensions, ferns unfurl, and a host of abstract fractal shapes come into being, swirl about and dissolve into clouds of fractal dust.

This animation lets us see how simple it really is to grow fractals, and it helps us understand how the incredible complexity of natural forms all around us comes about by simple repetition.

# Sierpinski Triangle Fractal



NM State Math Standards
K-4:
*Recognize, reproduce, describe, extend and create repeating patterns, and translate from one representation to another.
*Solve problems involving proportional relationships.
*Relate geometric ideas to numbers.
*Visualize, build and draw geometric objects.
*Identify and compare congruent and similar figures.
5-8:
*Identify, describe and continue patterns presented in a variety of formats.
*Describe and perform single and multiple transformations that include rotation, reflection, translation and dilation.

What is special about this pattern?
Use the worksheet on the next page to draw a triangle like this one.
Try coloring the triangle in interesting, symmetrical ways!

Photocopy this page, and cut out the triangle. You can assemble 3 of them into a larger version, or 9 or 27 into a really large fractal triangle.
Or, you can build many other fun shapes with the triangles

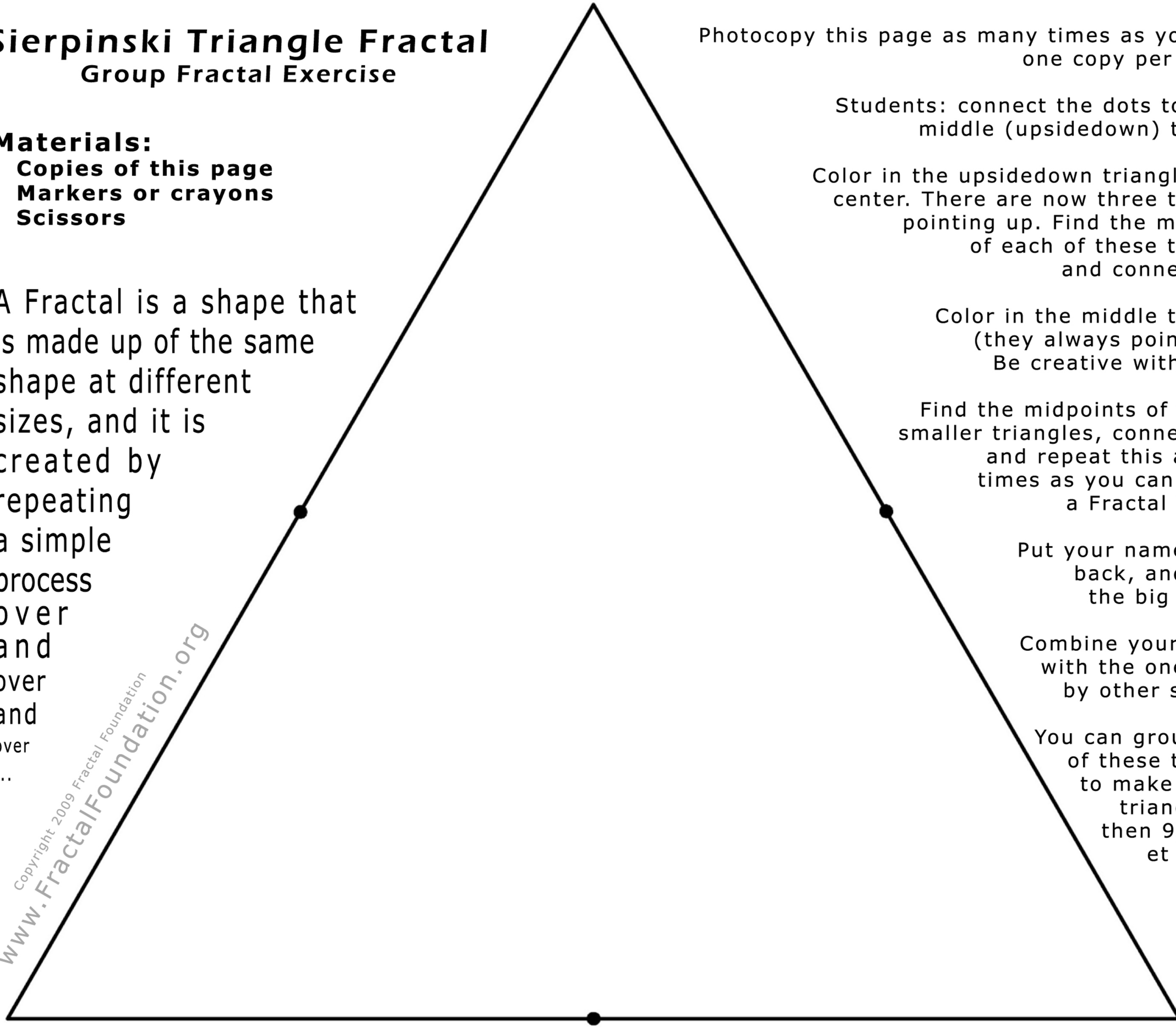
# Sierpinski Triangle Fractal

## Group Fractal Exercise

**Materials:**

**Copies of this page**
**Markers or crayons**
**Scissors**

A Fractal is a shape that is made up of the same shape at different sizes, and it is created by repeating a simple process over and over and over ...



Photocopy this page as many times as you need, one copy per student.

Students: connect the dots to form a middle (upsidedown) triangle.

Color in the upsidedown triangle in the center. There are now three triangles pointing up. Find the midpoints of each of these triangles, and connect them.

Color in the middle triangles (they always point down). Be creative with colors!

Find the midpoints of the new smaller triangles, connect them, and repeat this as many times as you can to form a Fractal Triangle.

Put your name on the back, and cut out the big triangle.

Combine your triangle with the ones made by other students.

You can group three of these together to make a bigger triangle, and then 9, 27, 81, et cetera...

# Fractivity: Explore Fractals with XaoS

First, download and install the XaoS program (either Mac or Windows version) from:
http://fractalfoundation.org/resources/fractal-software/

When you run the program, it opens with an image of the Mandelbrot set.
To navigate: just point the mouse and click! On a PC, the left button zooms in and the right zooms out. On a Mac, use ctrl-click to zoom out. To pan the image around, use both buttons together, or shift-click on the Mac.

Set the defaults: From the 'Filters' menu, enable Palette Emulator. From the "Calculation" menu select Iterations, and raise it to 2000. From the 'File' menu, select Save Configuration so you don't have to make these changes again.
Color palettes are randomly generated, and can be changed with the "P" key. To cycle the colors, use "Y". There are many filters and effects to explore from the menus.

XaoS can create many different fractal types, which can be accessed by using the number keys:
Keys 1 to 5 are Mandelbrot sets with various powers. The "normal" X^2 Mandelbrot set is on key 1. (Hitting "1" is a good way to reset yourself if you get lost!)
Key 6 is a Newton fractal, exponent 3, illustrating Newton's method for finding roots to 3'd order polynomial equations.
Key 7 is the Newton fractal for exponent 4.
Key 8,9, and 0 are Barnsley fractals.
Key A - N are several other fascinating fractal formulas

Julia Sets: Every point in the Mandelbrot set (and several of the other fractals) corresponds to a unique Julia set. To explore the relationship between the Mandelbrot and Julia fractals, press "J" to enter fast-Julia mode. When you find a Julia set you like, switch over to it by pressing "M".

To save a fractal, use "File->Save Image" to save the picture for use in other prgrams. Use "File->Save" to save the actual parameters of the file, which will allow you to return to the fractal in XaoS and keep exploring it further.

Finally - use the Help file and explore the excellent tutorials! Though written by Jan Hubicka - the initial programmer - originally in Czech, they are very useful both to learn how to use the program as well as to learn about the fractals. Enjoy!

## SCIENCE AND MATH EDUCATIONAL STANDARDS

From the National Council of Teachers of Mathematics (NCTM):

Recognize geometric shapes and structures in the environment and specify their location. (NCTM Geometry grades K-2)

Recognize and apply slides, flips, and turns; recognize and create shapes that have symmetry. (NCTM Geometry grades K-2)

Investigate, describe, and reason about the results of subdividing, combining, and transforming shapes. (NCTM Geometry grades 3-5)

Identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs. (NCTM Geometry grades 3-5)

Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, architecture, and everyday life.
(NCTM Geometry grades 6-8; 9-12)

Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations. (Grades 6-8 NCTM Expectations for Algebra Knowledge)

Use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts. (Grades 9-12 NCTM Expectations for Algebra Knowledge)

From the National Science Education Standard; National Academy of Sciences:

UNIFYING CONCEPTS AND PROCESSES STANDARD:
As a result of activities in grades K-12, all students should develop understanding and abilities aligned with the following concepts and processes:

- Systems, order, and organization
- Evidence, models, and explanation
- Constancy, change, and measurement
- Evolution and equilibrium
- Form and function

Structure and function in living systems; Diversity and adaptations of organisms; Interdependence of organisms.