Triangles with Integer Sides and Sharing Barrels
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Barrel sharing problems have been common recreational problems since the Middle Ages. The most common version has three persons wanting to share 7 full, 7 half-full and 7 empty barrels so that each gets the same amount of contents and the same number of barrels. I consider the general problem with N of each type of barrel. The number of solutions is seen to be the same as the number of triangles with integer sides and perimeter N. These triangles were studied in [**7**] and [**4**] by use of intricate summations. Their work is exposited and extended in [**6**]. Here I give a geometric approach using triangular coordinates which is easier to understand and brings out several further properties, including the connection between the number of incongruent triangles and the partitions into at most three parts. At the end I study more general barrel sharing problems.

Sharing Barrels

Suppose we have N barrels of each type: full, half-full and empty. Let f_i, h_i, e_i be the number of these that the i-th person receives, $i = 1,2,3$. These are clearly nonnegative integers and we shall assume this from now on. Then we have a fair sharing if and only if the following conditions hold.

$$f_i + h_i + e_i = N, \text{ for } i = 1,2,3.$$
$$f_i + h_i / 2 = N / 2, \text{ for } i = 1,2,3.$$
$$\sum_i f_i = \sum_i h_i = \sum_i e_i = N. \tag{1}$$

A little observation and manipulation shows that (1) implies $e_i = f_i$ and $h_i = N - 2f_i$, and hence that (1) is solved by knowing the f_i subject to:

$$\sum_i f_i = N;$$
$$f_i \le N / 2, \text{for } i = 1,2,3. \tag{2}$$

Integral Triangles

It is well known and easily seen that three nonnegative lengths x, y, z can form a triangle if and only if the three triangle inequalities hold:

$$x + y \ge z, \ y + z \ge x, \ z + x \ge y. \tag{3}$$

If we set $x + y + z = p$, then (3) is equivalent to:

$$x \le p/2, \ y \le p/2, \ z \le p/2 \tag{4}$$

(The triangle is nondegenerate if and only if the inequalities are all strict.) Hence the solutions for sharing N barrels of each type are just the integral lengths that form a triangle of perimeter N.

Triangular Coordinates

Consider a triangle of sides x, y, z, and perimeter p. Since $x + y + z = p$, we can view (x, y, z) as a point on the plane $x + y + z = p$, in the triangle cut off by the planes $x = 0$, $y = 0$, $z = 0$. This gives the standard representation of (x, y, z) in triangular coordinates as shown in Figure 1 for the case $p = 5$. (Ignore the broken lines in Figure 1b for the moment.) Letting the spacing between lines be our unit of distance, the point (x, y, z) is located x units from the right edge, y units from the left edge and z units from the bottom edge of the triangle. It is a classic property of the equilateral triangle that the sum of the perpendicular distances from an interior point (x, y, z) to the sides, i.e., $x + y + z$, is a constant, namely the altitude.
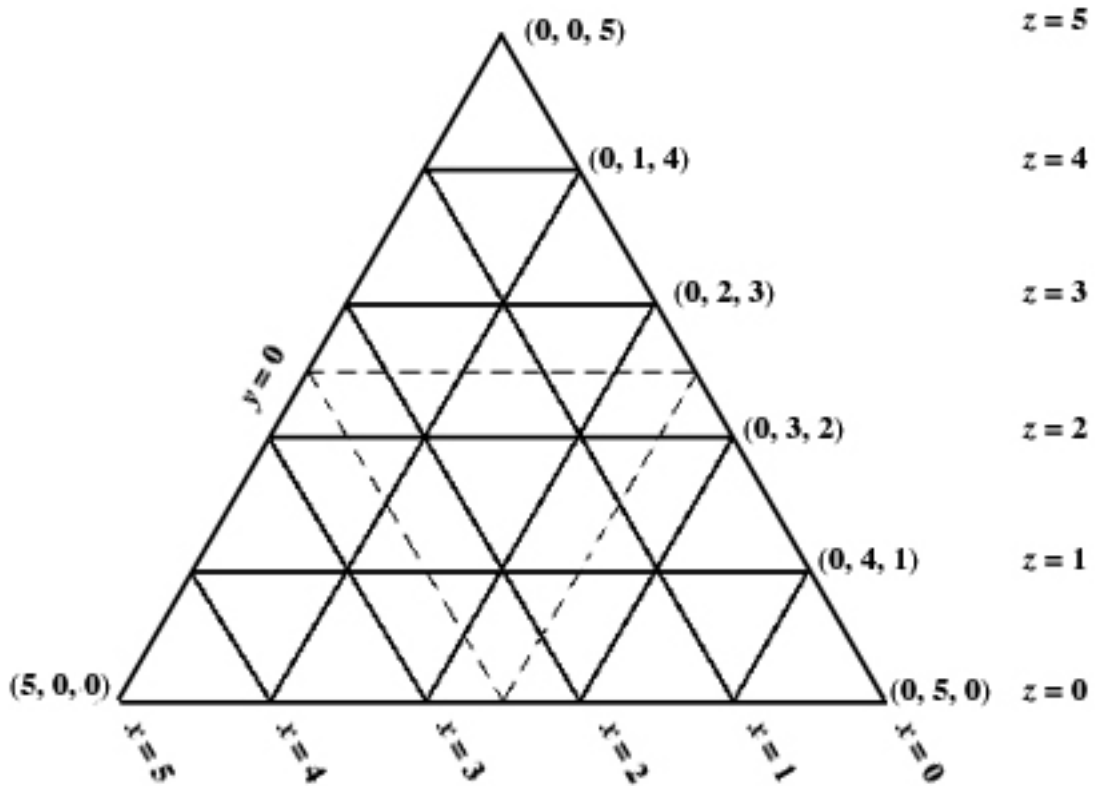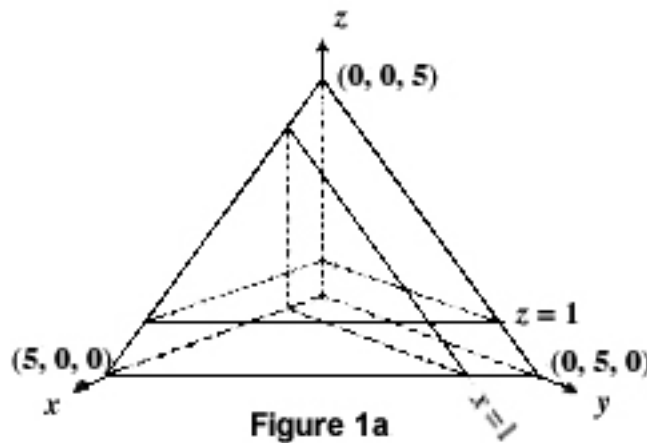
Figure 1a

Figure 1b

Triangular coordinates for $p = 5$

If we consider integral values of x, y, z with an integral sum p, we see that these points (x, y, z) form a triangular array having $p + 1$ points along an edge. We denote such an array as **TA**$(p + 1)$. **TA**$(p + 1)$ clearly has $1 + 2 + L + (p+1) = (p + 1)(p + 2)/2 = T(p + 1)$ points, where $T(p)$ denotes the pth triangular number.

The points along the edges of **TA**$(p + 1)$ correspond to at least one of x, y, z being 0, so the interior points correspond to all lengths being positive. These thus form a triangular array **TA**$(p$ - $2)$ with $T(p$ - $2)$ points. Readers will find it useful to draw diagrams as they read on.

The Number of Integral Triangles

In our triangular coordinates, we see that (x, y, z) corresponds to an integral triangle of perimeter p if and only if it is an integer point in **TA**$(p + 1)$ that lies inside the central region cut off by condition (4): $x \leq p/2, y \leq p/2, z \leq p/2,$, as indicated by the broken lines in Figure 1b for $p = 5$.

Let $T_1(p)$ be the number of integral triangles of perimeter p and let $T_2(p)$ be the number that are nondegenerate.

If p is odd (as in Figure 1b), let $p = 2q + 1$. Then the central region cut off by our conditions (4) is a triangle with base on the line $z = q$. This line contains $p + 1 - q = q + 2$ points, but the cut-off region omits the two end points, so our region is a **TA**(q), which contains $T(q)$ points. (Alternatively, take $T(2q + 2) - 3T(q + 1)$ to obtain $T(q)$.) All of these points correspond to nondegenerate triangles, so we have shown that $T_1(p) = T_2(p) = T(q)$. These are precisely the solutions of our barrel sharing problem for p barrels of each type.

If p is even, let $p = 2q$. Then the central region cut off by condition (4) is a triangle whose base is the whole line $z = q$, hence it is a **TA**$(q + 1)$ and we have $T_1(p) = T(q + 1)$. This is the number of solutions of our barrel sharing problem, since we do not restrict ourselves to nondegenerate solutions. But our central region certainly does contain degenerate triangles. We can remove all of these by excluding the lines $z = q, y = q, x = q$. This leaves a central region which is a **TA**$(q$ - $2)$, so $T_2(p) = T(q - 2)$. (As before, these results can be obtained by subtracting from $T(p + 1)$.)

Note that both $p = 2q$ - 2 and $p = 2q + 1$ give the same central region **TA**(q) of integer points corresponding to triangles of perimeter p, while both $p = 2q + 1$ and $p = 2q + 4$ give the same central region **TA**(q) of integer points corresponding to nondegenerate triangles of perimeter p. The latter half of the last sentence is the geometric basis of Theorem 3 in [7]. From these observations, we see that $T_1(p) = T_2(p + 3)$. This is also easily seen since adding one to each length gives a one-to-one correspondence between the triangles being counted.

The Number of Incongruent Integral Triangles

In enumerating the solutions of the barrel sharing problems, we do not really care which person gets which share, since each share is fair. If x, y, z is a fair distribution of full barrels, then we consider this as equivalent to y, x, z, etc. I.e.,all six permutations of x, y, z are considered as equivalent solutions.

Viewing x, y, z as sides of a triangle, there are six ways in which it can be congruent to another triangle. That is, one triangle is congruent to another if and only if the sides of one are a permutation of the sides of the other. These correspond to the six permutations of x, y, z and to the six symmetries of our triangular region.

So to count the number of inequivalent solutions of the barrel sharing problem or to count the number of incongruent integral triangles, we need to count the points of our central triangular region that are inequivalent under the symmetries of the triangle.

Let $T_3(p)$ be the number of incongruent integral triangles of perimeter p and let $T_4(p)$ be the number of those that are nondegenerate. Let $N(q)$ be the number of inequivalent points in **TA**(q). Figure 2 shows the inequivalent points for q = 4,5. Then $T_3(2q-2) = T_3(2q+1) = N(q)$ and $T_4(2q+1) = T_4(2q+4) = N(q)$. Again, there is a shift of three between the general case and the nondegenerate case, i.e., $T_3(p) = T_4(p+3)$.
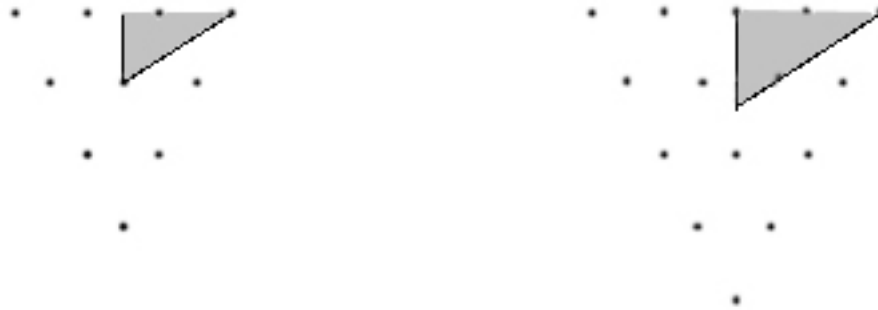
Figure 2

Inequivalent points for $q = 4, 5$

Theorem 1 $N(q+3) = N(q) + [(q+4)/2]$.

Proof. The array **TA**(q + 3) *is* obtained by bordering **TA**(q). The new inequivalent points are those in the border and they comprise half of a bordering edge. Such an edge has q + 3 points and we must count the midpoint when q + 3 is odd, giving $[(q+4)/2]$ new inequivalent points.

Corollary 1.1. *The sequence (N(q)) is determined by the recurrence in Theorem 1 and the initial conditions: N(1) = N(2) = 1, N(3) = 2. These values can be extended backward, consistently with the Theorem, to N(0) = N(- 1) = N(- 2) = N(- 3) = N(- 4) = 0.*

Corollary 1.2. $N(q + 6) = N(q) + q + 5.$

Repeated use of Corollary 1.2 gives us the following.

Corollary 1.3. *Let $q - 1 = 6k + r$, with $0 < r < 6$.*
If $r = 0$, then $N(q) = 6T(k) + I = 3k(k + 1) + 1$.
If $r \neq 0$, then $N(q) = 6T(k) + r(k + 1) = (3k + r)(k + 1)$.

This corollary holds for $q \geq -4$ and can be extended backward.

Corollaries 1 and 3 contain Theorems 1 and 2 of [7], but seem much simpler to me.

Table I

p = 2q or 2q + 1

p	q	$T(p)$	$N(p)$	$T_1(p)$	$T_2(p)$	$T_3(p)$	$T_4(p)$
0	0	0	0	1	0	1	0
1	0	1	1	0	0	0	0
2	1	3	1	3	0	1	0
3	1	6	2	1	1	1	1
4	2	10	3	6	0	2	0
5	2	15	4	3	3	1	1
6	3	21	5	10	1	3	1
7	3	28	7	6	6	2	2
8	4	36	8	15	3	4	1
9	4	45	10	10	10	3	3
10	5	55	12	21	6	5	2
11	5	66	14	15	15	4	4
12	6	78	16	28	10	7	3
13	6	91	19	21	21	5	5
14	7	105	21	36	15	8	4
15	7	120	24	28	28	7	7
16	8	136	27	45	21	10	5
17	8	153	30	36	36	8	8
18	9	171	33	55	28	12	7
19	9	190	37	45	45	10	10
20	10	210	40	66	36	14	8

Relation to Partitions

Looking up the sequence $N(q)$ in Sloane's invaluable handbook [**10**], one finds that it is the same as the number of ways that $q - 1$ can be partitioned into at most three parts. To see this, view **TA**(q) as the points (x, y, z) such that $x + y + z = q - 1$. Then taking just the inequivalent points is precisely the same as taking the partitions of $q - 1$ into at most three parts. Let $P_d(n)$ denote the number of partitions of n into at most d parts, so we have $N(n + 1) = P_3(n)$. Then Theorem 1 is a form of the known result that $P_3(n + 3) = P_3(n) + P_2(n + 3)$. This says that a partition of $n + 3$ either has *3* positive parts, and hence arises from a partition counted by $P_3(n)$ by adding 1 to each part, or has a zero part, and hence arises from a partition counted by $P_2(n + 3)$ by adding an extra part of 0. We see also that the number of partitions of $n + 3$ into exactly three parts (i.e., with no zero parts) is just $P_3(n)$.

We have seen that $T_3(2n - 2) = N(n)$ and that the latter is equal to $P_3(n - 1)$. We can see this another way as follows. $T_3(2n - 2)$ counts those triples x_1, x_2, x_3 such that $\sum_i x_i = 2n - 2$, with $0 \le x_i \le n - 1$. Letting $y_i = n - 1 - x_i$, we have that $\sum_i y_i = n - 1$, with $0 \le y_i \le n - 1$. Hence the triple y_1, y_2, y_3 is a partition of *n - 1*.

(The pretty correspondence between x_i and y_i has occurred to several people. It is in my unpublished *1982* paper on integral triangles and was also found by both N. J. Fine and P. Pacitti [6, pp. 45-46].)

In the context of barrel sharing, when $N = 2n - 2$, then the x_i are the f_i of Section 1 and so $y_i = N/2 - f_i = h_i/2$. This shows that, for even *N,* the sharing of barrels is determined by sharing the *N/2* pairs of half-full barrels in any way.

Similar arguments apply for the odd case and for the nondegenerate cases. For sharing $N = 2n + 1$ barrels of each type, each person must receive an odd number of half-full barrels. Thus the sharing is determined by giving each person one half-full barrel and then distributing the remaining $(N - 3)/2 = n - 1$ pairs of half-full barrels in any way. Thus $T_3(2n + 1) = T_3(2n - 2) = P_3(n - 1)$.

In [4] (and [6]), it is shown that the number of partitions of n into three positive parts, i.e., $P_3(n - 3)$ is $\{n^2/12\}$, where $\{x\}$ is the nearest integer to x, and hence that

$T_4(n) = \{n^2/12\} - \lfloor n/4 \rfloor \cdot \lfloor (n + 2/4) \rfloor$. I leave it to the reader to ponder the connection between this and my results: $T_4(2q + 1) = T_4(2q + 4) = N(q) = P_3(q - 1)$, Theorem 1 and its corollaries.

Historical Comments and Other Versions

The earliest examples of barrel sharing problems that I know of are in the ninth century collection attributed to Alcuin [3]. His problem 12 is our standard problem with 10 barrels of each type. Problem 51 is a variant - there are four barrels containing 10, 20,30,40 measures of wine and they are to be equally divided among four sons. Alcuin says only that the first two sons should take the 10 and 40 while the other two sons take the 20 and 30. Clearly some shifting of contents is required if each son is to get 25 measures of wine.

In the thirteenth century, Abbot Albert [1] gives the problem of dividing nine barrels containing $1, 2, к, 9$ measures among three persons.

In Bachet [5], we find examples where there are different numbers of barrels of the three types and an example where the barrels must be divided among four persons. (Ahrens [2] says that some of this material was added by the nineteenth century editor - I haven't seen earlier editions of [5] to verify this.)

If we have F full barrels, H half-full barrels and E empty barrels, then condition (1) becomes the following.

$$f_i + h_i + e_i = (F + H + E)/3, \text{ for } i = 1,2,3$$
$$f_i + h_i/2 = (F + H/2)/3, \text{ for } i = 1,2,3$$
$$\sum_i f_i = F, \sum_i h_i = H, \sum_i e_i = E. \tag{5}$$

When is there an integral solution? The existence of an integral solution imposes certain constraints on *F, H, E,* namely that $2F + H$ and $F + H + E$ must be divisible by 3. These are easily seen to be equivalent to: $F \equiv H \equiv E(\mathrm{mod}\,3)$. However, we already know that $F = H = E = 1$ has no solution, but looking closer gives the following.

Theorem 2. *There is a fair sharing of F full; H half-full and E empty barrels among three people if and only if*

$$F \equiv H \equiv E \pmod 3, \text{ and } H \neq 1$$

Proof. This is a special case of Theorem 3 below.

   Initially I thought that the number of solutions of (5) could be found since a solution of (5) would be given by knowing the f_i subject to:

$$\sum_i f_i - F;$$
$$f_i \leq (F + H/2)/3, \text{ for } i = 1,2,3. \tag{6}$$

However, one must also have $0 \leq f_i \leq F$ and $f_i \leq (F + H + E)/3$, and further, that $0 \leq h_i \leq H, h_i \leq (2F + H)/3, h_i \leq (F + H + E)/3$ and $0 \leq e_i \leq E, e_i \leq (F + H + E)/3$. These 11 sets of inequalities give a rather complex set of conditions on the f_i and the same holds if we try to express solutions in terms of the h_i or e_i.
   If we wish to share N barrels of each type among k persons, then condition (7) holds.

$$f_i + h_i + e_i = 3N/k, \text{ for } i = 1,2,\text{К },k.$$
$$2f_i + h_i = 3N/k, \text{ for } i = 1,2,\text{К },k.$$
$$\sum_i f_i = \sum_i h_i = \sum_i e_i = N. \tag{7}$$

Again, a solution is determined by knowing the f_i, now subject to simple conditions similar to (2):

$$\sum_i f_i = N;$$
$$f_i \leq 3N/2k, \text{ for } i = 1,2,\text{К },k \tag{8}$$

Geometrically, this leads to simplicial coordinates in $k - 1$ dimensions, but the problem is no longer the same as finding k integral lengths which form a k-gon of perimeter N, for which the conditions are:

$$\sum_i f_i = N;$$
$$f_i \leq N/2, \text{ for } i = 1,2,\text{К },k \tag{9}$$

It is possible to generalize and extend the previous ideas to find the number of inequivalent solutions of (9), but it is not very illuminating and does not give the simple connection with partitions that occur for $k = 3$. Further, this is not the number of incongruent integral k-gons of perimeter N, since, e.g., this considers a, b, c, d as the same as b, a, c, d and since a quadrilateral with sides a, b, c, d has infinitely many incongruent shapes.

Obviously, one can combine both of Bachet's ideas and try to divide F, H, E among four or k persons. Ozanam [9] gives a confused version of this — he seems to start with $F = H = E = 8$, divided among four people, but gives a solution *for $F = E = 6$, $H = 12$*, though he seems to distinguish 6 half-full barrels from 6 half-empty barrels. Some trial and error leads to the following.

Theorem 3. *There is a fair sharing of F full, H half-full and E empty barrels among k people if and only if:*

(a) $F \equiv E \pmod{k}$;
(b) $H \equiv -2F \pmod{k}$;
(c) *if $(2F + H) / k$ is odd, then $H \geq k$.*   (10)

Proof. The conditions for a fair sharing are:

(a) $f_i + h_i + e_i = (F + H + E)/k$, for $i = 1, 2, \text{К}, k$;
(b) $2f_i + h_i = (2F + H)/k$, for $i = 1, 2, \text{К}, k$;
(c) $\sum_i f_i = F, \sum_i h_i = H, \sum_i e_i = E.$ (11)

From (11 -a & b), we get $f_i - e_i = (F - E)/k$ for each i, so that (10-a & b) must hold if there is a solution. If $(2F + H)/k$ is odd, then (11-b) shows that h_i is odd, hence $h_i \geq 1$, for each i. Hence $H \geq k$ and the "only if" part of the theorem is proven.
   Suppose that condition (10) holds. Let $F \equiv f \pmod{k}$, with $0 \leq f < k$. If $f = 0$, then we have $F \equiv H \equiv E \equiv 0 \pmod{k}$ and there is an easy solution. Suppose now that $f > 0$. Distribute 1, 0, 1 (i.e., 1 full, 0 half-full and 1 empty barrel) to f people and 0, 2, 0 to the remaining k - f people. This leaves F - f, H - $2(k - f)$, $E - f$ barrels. We have $F - f \equiv E - f \equiv 0$ and $H - 2(k - f) \equiv H + 2f \equiv 0 \pmod{k}$, so these remaining barrels can be easily shared. So we will have a fair sharing, provided only that $H \geq 2(k - f)$, which we rewrite as $(2f + H)/k \geq 2$. Since $f > 0$, we have that $(2f + H) / k > 0$. If $(2f + H)/k = 1,$, then also $(2F + H) / k$ is odd and (10-c) says that $H \geq k$, which gives $(2f + H)/k > 1..$ Hence $(2f + H)/k \geq 2$ and our distribution can indeed be carried out to give a fair sharing.

   Note that for $k = 3$, we have $-2 \equiv 1 \pmod{k}$, so that condition (10) simplifies to give the conditions in Theorem 2.
   Kraitchik [8] has varied the problem still further by having 9 barrels of each of the following five types: full, 3/4 full, 1/2 full, 1/4 full and empty, to be divided among 5 people!

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