GCIs Make Reasoning in Fuzzy DL with the Product T-norm Undecidable

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1 Introduction

Fuzzy variants of Description Logics (DLs) were introduced in order to deal with applications where not all concepts can be defined in a precise way. A great variety of fuzzy DLs have been investigated in the literature [12,8]. In fact, compared to crisp DLs, fuzzy DLs offer an additional degree of freedom when defining their expressiveness: in addition to deciding which concept constructors (like conjunction, disjunction, existential restriction) and which TBox formalism (like no TBox, acyclic TBox, general concept inclusions) to use, one must also decide how to interpret the concept constructors by appropriate functions on the domain of fuzzy values $[0, 1]$. For example, conjunction can be interpreted by different t-norms (such as Gödel, Łukasiewicz, and product) and there are also different options for how to interpret negation (such as involutive negation and residual negation). In addition, one can either consider all models or only so-called witnessed models [10] when defining the semantics of fuzzy DLs.

Decidability of fuzzy DLs is often shown by adapting the tableau-based algorithms for the corresponding crisp DL to the fuzzy case. This was first done for the case of DLs without general concept inclusion axioms (GCIs) [19,17,14,6], but then also extended to GCIs [16,15,18,4,5]. Usually, these tableau algorithms reason w.r.t. witnessed models.[1] It should be noted, however, that in the presence of GCIs there are different ways of extending the notion of witnessed models from [10], depending on whether the witnessed property is required to apply also to GCIs (in which case we talk about strongly witnessed models) or not (in which case we talk about witnessed models).

The paper [4] considers the case of reasoning w.r.t. fuzzy GCIs in the setting of a logic with product t-norm and involutive negation. More precisely, the tableau algorithm introduced in that paper is supposed to check whether an ontology consisting of fuzzy GCIs and fuzzy ABox assertions expressed in this DL has a strongly witnessed model or not.[2] Actually, the proof of correctness of this algorithm given in [4] implies that, whenever such an ontology has a strongly witnessed model, then it has a finite model. However, it was recently shown in [2]

[1] In fact, witnessed models were introduced in [10] to correct the proof of correctness for the tableau algorithm presented in [19].

[2] Note that the authors of [4] actually use the term "witnessed models" for what we call "strongly witnessed models."

that this is not the case in the presence of general concept inclusion axioms, i.e., there is an ontology written in this logic that has a strongly witnessed model, but does not have a finite model. Of course, this does not automatically imply that the algorithm itself is wrong. In fact, if one applies the algorithm from [4] to the ontology used in [2] to demonstrate the failure of the finite model property, then one obtains the correct answer, and in [2] the authors actually conjecture that the algorithm is still correct. However, incorrectness of the algorithm has now independently been shown in [3] and in [1]. Thus, one can ask whether the fuzzy DL considered in [4] is actually decidable. Though this question is not answered in [1], the paper gives strong indications that the answer might in fact be "no." More precisely, [1] contains a proof of undecidability for a variant of the fuzzy DL considered in [4], which (i) additionally allows for strict GCIs, i.e., GCIs whose fuzzy value is required to be *strictly greater* than a given rational number; and (ii) where the notion of strongly witnessed models used in [4] is replaced by the weaker notion of witnessed models.

In this paper, we consider a different fuzzy DL with product t-norm, where disjunction and involutive negation are replaced by the constructor implication, which is interpreted as the residuum. In this logic, residual negation can be expressed, but neither involutive negation nor disjunction. It was introduced in [10], where decidability of reasoning w.r.t. witnessed models was shown for the case without GCIs. In [7], an analogous decidability result was shown for the case of reasoning w.r.t. so-called quasi-witnessed models. Following [7], we call this logic $*$-$\mathcal{ALE}$. In the present paper we show that adding GCIs makes reasoning in $*$-$\mathcal{ALE}$ undecidable w.r.t. several variants of the notion of witnessed models (including witnessed, quasi-witnessed, and strongly witnessed models).

2 Preliminaries

In this section, we introduce the logic $*$-$\mathcal{ALE}$ and some of the properties that will be useful throughout the paper.

The syntax of this logic is slightly different from standard description logics, as it allows for an implication constructor, and no negation or disjunction. $*$-$\mathcal{ALE}$ *concepts* are built through the syntactic rule

$$C ::= A \mid \bot \mid \top \mid C_1 \sqcap C_2 \mid C_1 \to C_2 \mid \exists r.C \mid \forall r.C$$

where A is a *concept name* and r is a *role name*.

A $*$-$\mathcal{ALE}$ *ABox* is a finite set of *assertion axioms* of the form $\langle a : C \rhd q\rangle$ or $\langle (a, b) : r \rhd q\rangle$, where C is a $*$-$\mathcal{ALE}$ concept, $r \in \mathcal{N_R}$, q is a rational number in $[0, 1]$, a, b are *individual names* and $\rhd$ is either $\geq$ or $=$. A $*$-$\mathcal{ALE}$ *TBox* is a finite set of *concept inclusion axioms* of the form $\langle C \sqsubseteq D \geq q\rangle$, where C, D are $*$-$\mathcal{ALE}$ concepts and q is a rational number in $[0, 1]$. A $*$-$\mathcal{ALE}$ *ontology* is a tuple $(\mathcal{A}, \mathcal{T})$, where $\mathcal{A}$ is a $*$-$\mathcal{ALE}$ ABox and $\mathcal{T}$ a $*$-$\mathcal{ALE}$ TBox. In the following we will often drop the prefix $*$-$\mathcal{ALE}$, and speak simply of e.g. *TBoxes* and *ontologies*.

The semantics of this logic extends the classical DL semantics by interpreting concepts and roles as fuzzy sets over an interpretation domain. Given a non-empty domain Δ, a *fuzzy set* is a function $F : \Delta \to [0, 1]$, with the intuition that an element $\delta \in \Delta$ belongs to F with *degree* $F(\delta)$. Here, we focus on the *product t-norm* semantics, where logical constructors are interpreted using the product t-norm $\otimes$ and its *residuum* $\Rightarrow$ defined, for every $\alpha, \beta \in [0,1]$, as follows:

$$\alpha \otimes \beta := \alpha \cdot \beta,$$
$$\alpha \Rightarrow \beta := \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta/\alpha & \text{otherwise.} \end{cases}$$

The semantics of $*$-$\mathcal{ALE}$ is based on interpretations. An *interpretation* is a tuple $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty set, called the *domain*, and the function $\cdot^\mathcal{I}$ maps each individual name a to an element of $\Delta^\mathcal{I}$, each concept name A to a function $A^\mathcal{I} : \Delta^\mathcal{I} \to [0,1]$ and each role name r to a function $r^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$. The interpretation function is extended to arbitrary $*$-$\mathcal{ALE}$ concepts as follows. For every $\delta \in \Delta^\mathcal{I}$,

$$\begin{aligned}
\top^\mathcal{I}(\delta) &= 1, \\
\bot^\mathcal{I}(\delta) &= 0, \\
(C_1 \sqcap C_2)^\mathcal{I}(\delta) &= C_1^\mathcal{I}(\delta) \otimes C_2^\mathcal{I}(\delta) \\
(C_1 \to C_2)^\mathcal{I}(\delta) &= C_1^\mathcal{I}(\delta) \Rightarrow C_2^\mathcal{I}(\delta) \\
(\exists r.C)^\mathcal{I}(\delta) &= \sup_{\gamma \in \Delta^\mathcal{I}} r^\mathcal{I}(\delta, \gamma) \otimes C^\mathcal{I}(\gamma) \\
(\forall r.C)^\mathcal{I}(\delta) &= \inf_{\gamma \in \Delta^\mathcal{I}} r^\mathcal{I}(\delta, \gamma) \Rightarrow C^\mathcal{I}(\gamma).
\end{aligned}$$

The interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ satisfies the assertional axiom $\langle a : C \rhd q \rangle$ iff $C^\mathcal{I}(a^\mathcal{I}) \rhd q$, it satisfies $\langle (a,b) : r \rhd q \rangle$ iff $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \rhd q$ and it satisfies the concept inclusion $\langle C \sqsubseteq D \geq q \rangle$ iff $\inf_{\delta \in \Delta^\mathcal{I}} (C^\mathcal{I}(\delta) \Rightarrow D^\mathcal{I}(\delta)) \geq q$. This interpretation is called a *model* of the ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

In fuzzy DLs, reasoning is often restricted to witnessed models [10]. An interpretation $\mathcal{I}$ is called *witnessed* if it satisfies the following two conditions:

(wit1) for every $\delta \in \Delta^\mathcal{I}$, role r and concept C there exists $\gamma \in \Delta^\mathcal{I}$ such that $(\exists r.C)^\mathcal{I}(\delta) = r^\mathcal{I}(\delta, \gamma) \otimes C^\mathcal{I}(\gamma)$, and

(wit2) for every $\delta \in \Delta^\mathcal{I}$, role r and concept C there exists $\gamma \in \Delta^\mathcal{I}$ such that $(\forall r.C)^\mathcal{I}(\delta) = r^\mathcal{I}(\delta, \gamma) \Rightarrow C^\mathcal{I}(\gamma)$.

This model is called *weakly witnessed* if it satisfies **(wit1)** and *quasi-witnessed* if it satisfies **(wit1)** and the condition

(wit2') for every $\delta \in \Delta^\mathcal{I}$, role r and concept C, either $(\forall r.C)^\mathcal{I} = 0$ or there exists $\gamma \in \Delta^\mathcal{I}$ such that $(\forall r.C)^\mathcal{I}(\delta) = r^\mathcal{I}(\delta, \gamma) \Rightarrow C^\mathcal{I}(\gamma)$.

In the presence of GCIs, witnessed interpretations are sometimes further restricted [6,2,8] to satisfy

(wit3) for every two concepts C, D, there is a γ such that

$$\inf_{\eta \in \Delta^\mathcal{I}} (C^\mathcal{I}(\eta) \Rightarrow D^\mathcal{I}(\eta)) = C^\mathcal{I}(\gamma) \Rightarrow D^\mathcal{I}(\gamma).$$

Witnessed interpretations that satisfy this third restriction **(wit3)** are called *strongly witnessed* interpretations.

We say that an ontology $\mathcal{O}$ is *consistent* (resp. *weakly witnessed consistent*, *quasi-witnessed consistent*, *witnessed consistent*, *strongly witnessed consistent*) if it has a model (resp. a weakly witnessed model, a quasi-witnessed model, a witnessed model, a strongly witnessed model). Obviously, strongly witnessed consistency implies witnessed consistency, which implies quasi-witnessed consistency, which itself implies weakly witnessed consistency. The converse implications, however, need not hold; for instance, a quasi-witnessed consistent $*$-$\mathcal{ALE}$ ontology that has no witnessed models can be derived from the example in [7].

We now describe some properties of t-norms and axioms that will be useful for the rest of the paper. For every $\alpha, \beta \in [0,1]$ it holds that $\alpha \Rightarrow \beta = 1$ iff $\alpha \leq \beta$. Thus, given two concepts C, D, the axiom $\langle C \sqsubseteq D \geq 1 \rangle$ expresses that $C^{\mathcal{I}}(\delta) \leq D^{\mathcal{I}}(\delta)$ for all $\delta \in \Delta^{\mathcal{I}}$. Additionally, $1 \Rightarrow \beta = \beta$ and $0 \Rightarrow \beta = 1$ for all $\beta \in [0,1]$, and $\alpha \Rightarrow 0 = 0$ for all $\alpha \in (0,1]$.

In the following, we will use the expression $\langle C \stackrel{r}{\rightsquigarrow} D \rangle$ to abbreviate the axioms $\langle C \sqsubseteq \forall r.D \geq 1 \rangle, \langle \exists r.D \sqsubseteq C \geq 1 \rangle$. To understand this abbreviation, consider an interpretation $\mathcal{I}$ satisfying $\langle C \stackrel{r}{\rightsquigarrow} D \rangle$ and let $\delta, \gamma \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(\delta, \gamma) = 1$. From the first axiom it follows that

$$\begin{aligned} C^{\mathcal{I}}(\delta) \leq (\forall r.D)^{\mathcal{I}}(\delta) &= \inf_{\eta \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(\delta, \eta) \Rightarrow D^{\mathcal{I}}(\eta) \\ &\leq r^{\mathcal{I}}(\delta, \gamma) \Rightarrow D^{\mathcal{I}}(\gamma) = 1 \Rightarrow D^{\mathcal{I}}(\gamma) = D^{\mathcal{I}}(\gamma). \end{aligned}$$

From the second axiom it follows that

$$\begin{aligned} C^{\mathcal{I}}(\delta) \geq (\exists r.D)^{\mathcal{I}}(\delta) &= \sup_{\eta \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(\delta, \eta) \cdot D^{\mathcal{I}}(\eta) \\ &\geq r^{\mathcal{I}}(\delta, \gamma) \cdot D^{\mathcal{I}}(\gamma) = D^{\mathcal{I}}(\gamma), \end{aligned}$$

and hence, both axioms together imply that $C^{\mathcal{I}}(\delta) = D^{\mathcal{I}}(\gamma)$. In other words, $\langle C \stackrel{r}{\rightsquigarrow} D \rangle$ expresses that the value of $C^{\mathcal{I}}(\delta)$ is propagated to the valuation of the concept D on all r successors with degree 1 of δ. Conversely, given an interpretation $\mathcal{I}$ such that $r^{\mathcal{I}}(\delta, \gamma) \in \{0,1\}$ for all $\delta, \gamma \in \Delta^{\mathcal{I}}$, if $r^{\mathcal{I}}(\delta, \gamma) = 1$ implies $C^{\mathcal{I}}(\delta) = D^{\mathcal{I}}(\gamma)$, then $\mathcal{I}$ is a model of $\langle C \stackrel{r}{\rightsquigarrow} D \rangle$.

For a concept C, and a natural number $n \geq 1$, the expression C^n will denote the concatenation of C with itself n times; that is, $C^n := \bigsqcap_{j=1}^{n} C$. The semantics of $\sqcap$ yields $(C^n)^{\mathcal{I}}(\delta) = (C^{\mathcal{I}}(\delta))^n$, for every model $\mathcal{I}$ and $\delta \in \Delta^{\mathcal{I}}$.

We will show that consistency of $*$-$\mathcal{ALE}$ ontologies w.r.t. the different variants of witnessed models introduced above is undecidable. We will show this using a reduction from the Post correspondence problem, which is well-known to be undecidable [13].

Definition 1 (PCP). *Let* $(v_1, w_1), \ldots, (v_m, w_m)$ *be a finite list of pairs of words over an alphabet* $\Sigma = \{1, \ldots, s\}, s > 1$. *The* Post correspondence problem *(PCP)* *asks whether there is a non-empty sequence* $i_1, i_2, \ldots, i_k$*,* $1 \leq i_j \leq m$ *such that* $v_{i_1} v_{i_2} \cdots v_{i_k} = w_{i_1} w_{i_2} \cdots w_{i_k}$*. If such a sequence exists, then the word* $i_1 i_2 \cdots i_k$ *is called a* solution *of the problem.*

We assume w.l.o.g. that there is no pair v_i, w_i where both words are empty. For a word $\mu = i_1 i_2 \cdots i_k \in \{1, \ldots, m\}^*$, we will denote as v_μ and w_μ the words $v_{i_1} v_{i_2} \cdots v_{i_k}$ and $w_{i_1} w_{i_2} \cdots w_{i_k}$, respectively.

The alphabet Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s+1$ in which 0 never occurs. Using this intuition, we will express the empty word as the number 0.

In the following reductions, we will encode the word w in Σ^* using the number $2^{-w} \in [0, 1]$. We will construct an ontology whose models encode the search for a solution. The interpretation of two designated concept names A and B at a node will correspond to the words v_μ, w_μ, respectively, for $\mu \in \{1, \ldots, m\}^*$.

3 Undecidability w.r.t. Witnessed Models

We will show undecidability of consistency w.r.t. witnessed models by constructing, for a given instance $\mathcal{P} = ((v_1, w_1), \ldots, (v_m, w_m))$ of the PCP, an ontology $\mathcal{O}_\mathcal{P}$ such that for every witnessed model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ and every $\mu \in \{1, \ldots, m\}^*$ there is an element $\delta_\mu \in \Delta^\mathcal{I}$ with $A^\mathcal{I}(\delta_\mu) = 2^{-v_\mu}$ and $B^\mathcal{I}(\delta_\mu) = 2^{-w_\mu}$. Additionally, we will show that this ontology has a witnessed model whose domain has only these elements. Then, $\mathcal{P}$ has a solution iff for every witnessed model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ there exist a $\delta \in \Delta^\mathcal{I}$ such that $A^\mathcal{I}(\delta) = B^\mathcal{I}(\delta)$.

Let $\delta \in \Delta^\mathcal{I}$ encode the words $v, w \in \Sigma^*$; i.e., $A^\mathcal{I}(\delta) = 2^{-v}$ and $B^\mathcal{I}(\delta) = 2^{-w}$, and let $i, 1 \leq i \leq m$. Assume additionally that we have concept names V_i, W_i with $V_i^\mathcal{I}(\delta) = 2^{-v_i}$ and $W_i^\mathcal{I}(\delta) = 2^{-w_i}$. We want to ensure the existence of a node γ that encodes the concatenation of the words v, w with the i-th pair from $\mathcal{P}$; i.e. vv_i and ww_i. This is done through the TBox

$$\mathcal{T}_\mathcal{P}^i := \{\langle \top \sqsubseteq \exists r_i.\top \geq 1\rangle, \langle (V_i \sqcap A^{(s+1)^{|v_i|}}) \overset{r_i}{\rightsquigarrow} A\rangle, \langle (W_i \sqcap B^{(s+1)^{|w_i|}}) \overset{r_i}{\rightsquigarrow} B\rangle\}.$$

Recall that we are viewing words in Σ^* as natural numbers in base $s+1$. Thus, the concatenation of two words u, u' corresponds to the operation $u \cdot (s+1)^{|u'|} + u'$. We then have that

$$(V_i \sqcap A^{(s+1)^{|v_i|}})^\mathcal{I}(\delta) = V_i^\mathcal{I}(\delta) \cdot (A^\mathcal{I}(\delta))^{(s+1)^{|v_i|}} = 2^{-vv_i}.$$

If $\mathcal{I}$ is a witnessed model of $\mathcal{T}_\mathcal{P}^i$, then from the first axiom it follows that $(\exists r_i.\top)^\mathcal{I}(\delta) = 1$, and according to **(wit1)**, there must exist a $\gamma \in \Delta^\mathcal{I}$ with $r^\mathcal{I}(\delta, \gamma) = 1$. The last two axioms then ensure that $A^\mathcal{I}(\gamma) = 2^{-vv_i}$ and $B^\mathcal{I}(\gamma) = 2^{-ww_i}$; thus, the concept names A and B encode, at node γ, the words vv_i and ww_i, as desired. If we want to use this construction to recursively construct all the pairs of concatenated words defined by $\mathcal{P}$, we need to ensure also that $V_j^\mathcal{I}(\gamma) = 2^{-v_j}$, $W_j^\mathcal{I}(\gamma) = 2^{-w_j}$ hold for every $j, 1 \leq j \leq m$. This can be done through the axioms

$$\mathcal{T}_\mathcal{P}^0 := \{\langle V_j \overset{r_i}{\rightsquigarrow} V_j\rangle, \langle W_j \overset{r_i}{\rightsquigarrow} W_j\rangle \mid 1 \leq i, j \leq m\}.$$

It only remains to ensure that there is a node δ_ε where $A^{\mathcal{I}}(\delta_\varepsilon) = B^{\mathcal{I}}(\delta_\varepsilon) = 1 = 2^0$ (that is, where A and B encode the empty word) and $V_j^{\mathcal{I}}(\delta_\varepsilon) = 2^{-v_j}$, $W_j^{\mathcal{I}}(\delta_\varepsilon) = 2^{-w_j}$ hold for every $j, 1 \leq i \leq m$. This condition is easily enforced through the ABox[3]

$$\mathcal{A}_{\mathcal{P}}^0 := \{\langle a : A = 1\rangle, \langle a : B = 1\rangle\} \cup \{\left\langle a : V_i = 2^{-v_i}\right\rangle, \left\langle a : W_i = 2^{-w_i}\right\rangle \mid 1 \leq i \leq m\}.$$

Finally, we include a concept name H that must be interpreted as 0.5 in every domain element. This is enforced by the following axioms:

$$\mathcal{A}_0 := \{\langle a : H = 0.5\rangle\},$$
$$\mathcal{T}_0 := \{\langle H \overset{r_i}{\rightsquigarrow} H\rangle \mid 1 \leq i \leq m\}.$$

This concept name will later be used to detect whether $\mathcal{P}$ has a solution (see Theorem 3).

Let now $\mathcal{O}_{\mathcal{P}} := (\mathcal{A}_{\mathcal{P}}, \mathcal{T}_{\mathcal{P}})$ where $\mathcal{A}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}}^0 \cup \mathcal{A}_0$ and $\mathcal{T}_{\mathcal{P}} := \mathcal{T}_0 \cup \bigcup_{i=0}^{m} \mathcal{T}_{\mathcal{P}}^i$. We define the interpretation $\mathcal{I}_{\mathcal{P}} := (\Delta^{\mathcal{I}_{\mathcal{P}}}, \cdot^{\mathcal{I}_{\mathcal{P}}})$ as follows:

- $\Delta^{\mathcal{I}_{\mathcal{P}}} = \{1, \ldots, m\}^*$,
- $a^{\mathcal{I}_{\mathcal{P}}} = \varepsilon$,

for every $\mu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$,

- $A^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-v_\mu},\ B^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-w_\mu},\ H^{\mathcal{I}_{\mathcal{P}}}(\mu) = 0.5$,

and for all $j, 1 \leq j \leq m$

- $V_j^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-v_j},\ W_j^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-w_j}$, and
- $r_j^{\mathcal{I}_{\mathcal{P}}}(\mu, \mu j) = 1$ and $r_j^{\mathcal{I}_{\mathcal{P}}}(\mu, \mu') = 0$ if $\mu' \neq \mu j$.

It is easy to see that $\mathcal{I}_{\mathcal{P}}$ is in fact a witnessed model of $\mathcal{O}_{\mathcal{P}}$, since every node has exactly one r_i successor with degree greater than 0, for every $i, 1 \leq i \leq m$. More interesting, however, is that for every witnessed model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, there is an homomorphism from $\mathcal{I}_{\mathcal{P}}$ to $\mathcal{I}$ as described in the following lemma.

Lemma 2. *Let $\mathcal{I}$ be a witnessed model of $\mathcal{O}_{\mathcal{P}}$. Then there exists a function $f : \Delta^{\mathcal{I}_{\mathcal{P}}} \to \Delta^{\mathcal{I}}$ such that, for every $\mu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$, $C^{\mathcal{I}_{\mathcal{P}}}(\mu) = C^{\mathcal{I}}(f(\mu))$ holds for every concept name C and $r_i^{\mathcal{I}}(f(\mu), f(\mu i)) = 1$ holds for every $i, 1 \leq i \leq m$.*

Proof. The function f is built inductively on the length of μ. First, as $\mathcal{I}$ is a model of $\mathcal{A}_{\mathcal{P}}$, there must be a $\delta \in \Delta^{\mathcal{I}}$ such that $a^{\mathcal{I}} = \delta$. Notice that $\mathcal{A}_{\mathcal{P}}$ fixes the interpretation of all concept names on δ and hence $f(\varepsilon) = \delta$ satisfies the condition of the lemma.

[3] Notice that equality is necessary for this construction; since there is no negation constructor, it is not possible to express $\langle a : X = q\rangle$ with $q < 1$ using only axioms of the form $\langle a : Y \geq q'\rangle$.

Let now μ be such that $f(\mu)$ has already been defined. By induction, we can assume that $A^{\mathcal{I}}(f(\mu)) = 2^{-v_\mu}, B^{\mathcal{I}}(f(\mu)) = 2^{-w_\mu}, H^{\mathcal{I}}(f(\mu)) = 0.5$, and for every j, $1 \leq j \leq m$, $V_j^{\mathcal{I}}(f(\mu)) = 2^{-v_j}, W_j^{\mathcal{I}}(f(\mu)) = 2^{-w_j}$. Since $\mathcal{I}$ is a witnessed model of $\langle \top \sqsubseteq \exists r_i.\top \geq 1\rangle$, for all $i, 1 \leq i \leq m$ there exists a $\gamma \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(f(\mu), \gamma) = 1$, and as $\mathcal{I}$ satisfies all the axioms of the form $\langle C \overset{r}{\rightsquigarrow} D\rangle \in \mathcal{T}_{\mathcal{P}}$, it follows that

$$A^{\mathcal{I}}(\gamma) = 2^{-v_\mu v_i} = 2^{-v_{\mu i}}, \quad B^{\mathcal{I}}(\gamma) = 2^{-w_\mu w_i} = 2^{-w_{\mu i}},$$

$H^{\mathcal{I}}(\gamma) = 0.5$ and for all $j, 1 \leq j \leq m$, $V_j^{\mathcal{I}}(\gamma) = 2^{-v_j}, W_j^{\mathcal{I}}(\gamma) = 2^{-w_j}$. Setting $f(\mu i) = \gamma$ thus satisfies the required property. □

From this lemma it follows that, if the PCP $\mathcal{P}$ has a solution μ for some $\mu \in \{1, \ldots, m\}^+$, then every witnessed model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$ contains a node $\delta = f(\mu)$ such that $A^{\mathcal{I}}(\delta) = B^{\mathcal{I}}(\delta)$; that is, where A and B encode the same word. Conversely, if every witnessed model contains such a node, then in particular $\mathcal{I}_{\mathcal{P}}$ does, and thus $\mathcal{P}$ has a solution. The question is now how to detect whether a node with this characteristics exists in every model. We will extend $\mathcal{O}_{\mathcal{P}}$ with axioms that further restrict $\mathcal{I}_{\mathcal{P}}$ to satisfy $A^{\mathcal{I}_{\mathcal{P}}}(\mu) \neq B^{\mathcal{I}_{\mathcal{P}}}(\mu)$ for every $\mu \in \{1, \ldots, m\}^+$. This will ensure that the extended ontology will have a model iff $\mathcal{P}$ has no solution.

Suppose for now that, for some $\mu \in \{1, \ldots, m\}^*$, it holds that

$$2^{-v_\mu} = A^{\mathcal{I}_{\mathcal{P}}}(\mu) > B^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-w_\mu}.$$

We then have that $v_\mu < w_\mu$ and hence $w_\mu - v_\mu \geq 1$. It thus follows that

$$(A \to B)^{\mathcal{I}_{\mathcal{P}}}(\mu) = 2^{-w_\mu}/2^{-v_\mu} = 2^{-(w_\mu - v_\mu)} \leq 2^{-1} = 0.5$$

and thus $((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu) \leq 0.5$. Likewise, if $A^{\mathcal{I}_{\mathcal{P}}}(\mu) < B^{\mathcal{I}_{\mathcal{P}}}(\mu)$, we also get $((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu) \leq 0.5$. Additionally, if $A^{\mathcal{I}_{\mathcal{P}}}(\mu) = B^{\mathcal{I}_{\mathcal{P}}}(\mu)$, then it is easy to verify that $((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu) = 1$. From all this it follows that, for every $\mu \in \{1, \ldots, m\}^*$,

$$A^{\mathcal{I}_{\mathcal{P}}}(\mu) \neq B^{\mathcal{I}_{\mathcal{P}}}(\mu) \quad \text{iff} \quad ((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu) \leq 0.5. \tag{1}$$

Thus, the instance $\mathcal{P}$ has no solution iff for every $\mu \in \{1, \ldots, m\}^+$ it holds that $((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu) \leq 0.5$.

We define now the ontology $\mathcal{O}'_{\mathcal{P}} := (\mathcal{A}_{\mathcal{P}}, \mathcal{T}'_{\mathcal{P}})$ where

$$\mathcal{T}'_{\mathcal{P}} := \mathcal{T}_{\mathcal{P}} \cup \{\langle \top \sqsubseteq \forall r_i.(((A \to B) \sqcap (B \to A)) \to H) \geq 1\rangle \mid 1 \leq i \leq m\}.$$

Theorem 3. *The instance $\mathcal{P}$ of the PCP has a solution iff the ontology $\mathcal{O}'_{\mathcal{P}}$ is not witnessed consistent.*

Proof. Assume first that $\mathcal{P}$ has a solution $\mu = i_1 \cdots i_k$ and let $u = v_\mu = w_\mu$ and $\mu' = i_1 i_2 \cdots i_{k-1} \in \{1, \ldots, m\}^*$. Suppose there is a witnessed model $\mathcal{I}$ of $\mathcal{O}'_{\mathcal{P}}$. Since $\mathcal{O}_{\mathcal{P}} \subseteq \mathcal{O}'_{\mathcal{P}}$, $\mathcal{I}$ must also be a model of $\mathcal{O}_{\mathcal{P}}$. From Lemma 2 it then follows that there are nodes $\delta, \delta' \in \Delta^{\mathcal{I}}$ such that $A^{\mathcal{I}}(\delta) = A^{\mathcal{I}_{\mathcal{P}}}(\mu) = B^{\mathcal{I}_{\mathcal{P}}}(\mu) = B^{\mathcal{I}}(\delta)$ and $r_{i_k}^{\mathcal{I}}(\delta', \delta) = 1$. Then, $((A \to B) \sqcap (B \to A))^{\mathcal{I}}(\delta) = 1$ and hence

$$(((A \to B) \sqcap (B \to A)) \to H)^{\mathcal{I}}(\delta) = 1 \Rightarrow 0.5 = 0.5.$$

This then means that $(\forall r_{i_k}.(((A \to B) \sqcap (B \to A)) \to H))^{\mathcal{I}}(\delta') \leq 0.5$, violating one of the axioms in $\mathcal{T}'_{\mathcal{P}}$. Hence $\mathcal{I}$ is cannot be a model of $\mathcal{O}'_{\mathcal{P}}$.

For the converse, assume that $\mathcal{O}'_{\mathcal{P}}$ is not witnessed consistent. Then $\mathcal{I}_{\mathcal{P}}$ is not a model of $\mathcal{O}'_{\mathcal{P}}$. Since it is a model of $\mathcal{O}_{\mathcal{P}}$, there must exist an $i, 1 \leq i \leq m$ such that $\mathcal{I}_{\mathcal{P}}$ violates the axiom $\langle \top \sqsubseteq \forall r_i.(((A \to B) \sqcap (B \to A)) \to H) \geq 1 \rangle$. This means that there is some $\mu \in \{1, \ldots, m\}^*$ such that

$$(\forall r_i.(((A \to B) \sqcap (B \to A)) \to H))^{\mathcal{I}_{\mathcal{P}}}(\mu) < 1.$$

Since $r_i^{\mathcal{I}_{\mathcal{P}}}(\mu, \mu') = 0$ for all $\mu' \neq \mu i$ and $r_i^{\mathcal{I}_{\mathcal{P}}}(\mu, \mu i) = 1$, this implies that $(((A \to B) \sqcap (B \to A)) \to H)^{\mathcal{I}_{\mathcal{P}}}(\mu i) < 1$, i.e. $((A \to B) \sqcap (B \to A))^{\mathcal{I}_{\mathcal{P}}}(\mu i) > 0.5$. From (1) it follows that $A^{\mathcal{I}_{\mathcal{P}}}(\mu i) = B^{\mathcal{I}_{\mathcal{P}}}(\mu i)$ and hence μi is a solution of $\mathcal{P}$. □

Corollary 4. *Witnessed consistency of* $*$-$\mathcal{ALE}$ *ontologies is undecidable.*

Notice that in the proofs of Lemma 2 and Theorem 3, the second condition of the definition of witnessed models was never used. Moreover, the witnessed interpretation $\mathcal{I}_{\mathcal{P}}$ is obviously also weakly witnessed. We thus have the following corollary.

Corollary 5. *Weakly witnessed consistency and quasi-witnessed consistency of* $*$-$\mathcal{ALE}$ *ontologies are undecidable.*

4 Undecidability w.r.t. Strongly Witnessed Models

Unfortunately, the model $\mathcal{I}_{\mathcal{P}}$ constructed in the previous section is not a strongly witnessed model of $\mathcal{O}_{\mathcal{P}}$ since, for instance, $\inf_{\eta \in \Delta^{\mathcal{I}_{\mathcal{P}}}} (\top^{\mathcal{I}_{\mathcal{P}}}(\eta) \Rightarrow A^{\mathcal{I}_{\mathcal{P}}}(\eta)) = 0$, but there is no $\delta \in \Delta^{\mathcal{I}_{\mathcal{P}}}$ with $A^{\mathcal{I}_{\mathcal{P}}}(\delta) = 0$. Thus, the construction of $\mathcal{O}_{\mathcal{P}}$ does not yield an undecidability result for strongly witnessed consistency in $*$-$\mathcal{ALE}$.

Thus, we need a new reduction that proves undecidability of strongly witnessed consistency. This reduction will follow a similar idea to the one used in the previous section, in which models describe a search for a solution of the PCP $\mathcal{P}$. However, rather than building the whole search tree, models will describe only individual branches of this tree. The condition **(wit3)** will be used to ensure that at some point in this branch a solution is found.

Before describing the reduction in detail, we recall a property of t-norms. From a t-norm $\otimes$ and residuum $\Rightarrow$, one can express the minimum and maximum operators as follows [9]:

- $\min(\alpha, \beta) = \alpha \otimes (\alpha \Rightarrow \beta)$,
- $\max(\alpha, \beta) = \min(((\alpha \Rightarrow \beta) \Rightarrow \beta), ((\beta \Rightarrow \alpha) \Rightarrow \alpha))$.

We can thus introduce w.l.o.g. the $*$-$\mathcal{ALE}$ concept constructor max with the obvious semantics. We will use this constructor to simulate the non-deterministic choices in the search tree as described next.

Given an instance $\mathcal{P} = ((v_1, w_1), \ldots, (v_m, w_m))$ of the PCP, we define the ABox $\mathcal{A}^0_\mathcal{P}$ and the TBox $\mathcal{T}^0_\mathcal{P}$ as in the previous section, and for every $i, 1 \leq i \leq m$ we construct the TBox

$$\mathcal{T}^{s}{}^{i}_{\mathcal{P}} := \{\langle C_i \sqsubseteq \exists r_i.\top \geq 1\rangle, \langle V_i \sqcap A^{(s+1)^{|v_i|}} \overset{r_i}{\rightsquigarrow} A\rangle, \langle W_i \sqcap B^{(s+1)^{|w_i|}} \overset{r_i}{\rightsquigarrow} B\rangle\}.$$

The only difference between the TBoxes $\mathcal{T}^i_\mathcal{P}$ and $\mathcal{T}^{s}{}^{i}_{\mathcal{P}}$ is in the first axiom. Intuitively, the concept names C_i encode the choice of the branch in the tree to be expanded. If $C_i^\mathcal{I}(\delta) = 1$, there will be an r_i successor with degree 1, and the i-th branch of the tree will be explored. For this intuition to work, we need to ensure that at least one of the C_is is interpreted as 1 in every node. On the other hand, we can stop expanding the tree once a solution has been found. Using this intuition, we define the ontology $\mathcal{O}^s_\mathcal{P} := (\mathcal{A}^s_\mathcal{P}, \mathcal{T}^s_\mathcal{P})$ where

$$\begin{aligned}
\mathcal{A}^s_\mathcal{P} &:= \mathcal{A}^0_\mathcal{P} \cup \{a : \max(C_1, \ldots, C_m) = 1\},\\
\mathcal{T}^s_\mathcal{P} &:= \mathcal{T}^0_\mathcal{P} \cup \bigcup_{i=1}^{m} \mathcal{T}^{s}{}^{i}_{\mathcal{P}} \cup \{\langle (A \sqcap B) \to \bot \sqsubseteq \bot \geq 1\rangle\} \cup\\
&\qquad \{\langle \top \sqsubseteq \forall r_i. \max((A \to B) \sqcap (B \to A), C_1, \ldots, C_m) \geq 1\rangle \mid 1 \leq i \leq m\}.
\end{aligned}$$

Theorem 6. *The instance $\mathcal{P}$ of the PCP has a solution iff the ontology $\mathcal{O}^s_\mathcal{P}$ is strongly witnessed consistent.*

Proof. Let $\nu = i_1 i_2 \cdots i_k$ be a solution of $\mathcal{P}$ and let $\mathsf{pre}(\nu)$ denote the set of all prefixes of ν. We build the finite interpretation $\mathcal{I}^s_\mathcal{P}$ as follows:

- $\Delta^{\mathcal{I}^s_\mathcal{P}} := \mathsf{pre}(\nu)$,
- $a^{\mathcal{I}^s_\mathcal{P}} = \varepsilon$,

for all $\mu \in \Delta^{\mathcal{I}^s_\mathcal{P}}$,

- $A^{\mathcal{I}^s_\mathcal{P}}(\mu) = 2^{-v_\mu}$, $B^{\mathcal{I}^s_\mathcal{P}}(\mu) = 2^{-w_\mu}$,

and for all $j, 1 \leq j \leq m$

- $V_j^{\mathcal{I}^s_\mathcal{P}}(\mu) = 2^{-v_j}$, $W_j^{\mathcal{I}^s_\mathcal{P}}(\mu) = 2^{-w_j}$,
- $C_j^{\mathcal{I}^s_\mathcal{P}}(\mu) = 1$ if $\mu j \in \mathsf{pre}(\nu)$ and $C_j^{\mathcal{I}^s_\mathcal{P}}(\mu) = 0$ otherwise, and
- $r_j^{\mathcal{I}^s_\mathcal{P}}(\mu, \mu j) = 1$ if $\mu j \in \mathsf{pre}(\nu)$ and $r_j^{\mathcal{I}^s_\mathcal{P}}(\mu, \mu') = 0$ if $\mu' \in \mathsf{pre}(\nu)$ and $\mu' \neq \mu j$.

We show now that $\mathcal{I}^s_\mathcal{P}$ is a model of $\mathcal{O}^s_\mathcal{P}$. Since $\mathcal{I}^s_\mathcal{P}$ is finite, it follows immediately that it is also strongly witnessed. Clearly $\mathcal{I}^s_\mathcal{P}$ satisfies all axioms in $\mathcal{A}^0_\mathcal{P}$; additionally, we have that $C_{i_1}^{\mathcal{I}^s_\mathcal{P}}(\varepsilon) = 1$ and thus, $\mathcal{I}^s_\mathcal{P}$ satisfies $\mathcal{A}^s_\mathcal{P}$. The axiom $\langle (A \sqcap B) \to \bot \sqsubseteq \bot \geq 1\rangle$ expresses that $(A \sqcap B)^{\mathcal{I}^s_\mathcal{P}}(\mu) \Rightarrow 0 = 0$, and hence $(A \sqcap B)^{\mathcal{I}^s_\mathcal{P}}(\mu) > 0$ for all $\mu \in \mathsf{pre}(\nu)$, which clearly holds. We now show that the rest of the axioms are also satisfied for every $\mu \in \mathsf{pre}(\nu)$. Let

$\mu \in \mathsf{pre}(\nu) \setminus \{\nu\}$. Then we know that there exists $i, 1 \leq i \leq m$ such that $C_i^{\mathcal{I}_\mathcal{P}^s}(\mu) = 1$ and $r_i^{\mathcal{I}_\mathcal{P}^s}(\mu, \mu i) = 1$; thus $\mathcal{I}_\mathcal{P}^s$ satisfies the axioms in ${\mathcal{T}^s}_\mathcal{P}^i$. Moreover, $C_j^{\mathcal{I}_\mathcal{P}^s}(\mu) = 0 = r_j^{\mathcal{I}_\mathcal{P}^s}(\mu, \mu')$ for all $j \neq i$ and all $\mu' \in \mathsf{pre}(\nu)$ which means that $\mathcal{I}_\mathcal{P}^s$ trivially satisfies all axioms in ${\mathcal{T}^s}_\mathcal{P}^j$.

If $\mu i = \nu$, then as ν is a solution $((A \to B) \sqcap (B \to A))^{\mathcal{I}_\mathcal{P}^s}(\mu i) = 1$; otherwise, there is a $j, 1 \leq j \leq m$ with $\mu i j \in \mathsf{pre}(\nu)$ and thus $C_j^{\mathcal{I}_\mathcal{P}^s}(\mu i) = 1$. This means that $\mathcal{I}_\mathcal{P}^s$ satisfies the last axioms in $\mathcal{T}_\mathcal{P}^s$. Finally, if $\mu = \nu$, then $r_i^{\mathcal{I}_\mathcal{P}^s}(\mu, \mu') = 0$ and $C_i(\mu) = 0$, for all $\mu' \in \mathsf{pre}(\nu), 1 \leq i \leq m$, and thus the axioms are all trivially satisfied.

For the converse, let $\mathcal{I}$ be a strongly witnessed model of $\mathcal{O}_\mathcal{P}^s$. Then, there must be an element $\delta_0 \in \Delta^\mathcal{I}$ with $a^\mathcal{I} = \delta_0$. Since $\mathcal{I}$ must satisfy all axioms in $\mathcal{A}_\mathcal{P}^s$, there is an $i_1, 1 \leq i_1 \leq m$ such that $C_{i_1}^\mathcal{I}(\delta_0) = 1$. Since it must satisfy the axioms in ${\mathcal{T}^s}_\mathcal{P}^{i_1}$, there must exist a $\delta_1 \in \Delta^\mathcal{I}$ with $r_{i_1}^\mathcal{I}(\delta_0, \delta_1) = 1$, $A^\mathcal{I}(\delta_1) = 2^{-v_{i_1}}$, and $B^\mathcal{I}(\delta_1) = 2^{-w_{i_1}}$. If $A^\mathcal{I}(\delta_1) = B^\mathcal{I}(\delta_1)$, then i_1 is a solution of $\mathcal{P}$. Otherwise, from the last set of axioms in $\mathcal{T}_\mathcal{P}^s$, there must exist an $i_2, 1 \leq i_2 \leq m$ with $C_{i_2}^\mathcal{I}(\delta_1) = 1$. We can then iterate this same process to generate a sequence $i_3, i_4, \ldots$ of indices and $\delta_2, \delta_3, \ldots \in \Delta^\mathcal{I}$ where $A^\mathcal{I}(\delta_k) = 2^{-v_{i_1} \cdots v_{i_k}}$, and $B^\mathcal{I}(\delta_k) = 2^{-w_{i_1} \cdots w_{i_k}}$.

If there is some k such that $A^\mathcal{I}(\delta_k) = B^\mathcal{I}(\delta_k)$, then $i_1 \cdots i_k$ is a solution of $\mathcal{P}$. Assume now that no such k exists. We then have an infinite sequence of indices $i_1, i_2, \ldots$ and since for every $i, 1 \leq i \leq m$ either $v_i \neq 0$ or $w_i \neq 0$, then at least one of the sequences $v_{i_1} \cdots v_{i_k}, w_{i_1} \cdots w_{i_k}$ diverges. Thus, for every natural number n there is a k such that either $v_{i_1} \cdots v_{i_k} > n$ or $w_{i_1} \cdots w_{i_k} > n$; equivalently, $(A \sqcap B)^\mathcal{I}(\delta_k) < 1/n$. This implies that

$$\inf_{\eta \in \Delta^\mathcal{I}} (\top^\mathcal{I}(\eta) \Rightarrow (A \sqcap B)^\mathcal{I}(\eta)) = 0$$

and since $\mathcal{I}$ is strongly witnessed, there must exist a $\gamma \in \Delta^\mathcal{I}$ with

$$0 = \top^\mathcal{I}(\gamma) \Rightarrow (A \sqcap B)^\mathcal{I}(\gamma) = (A \sqcap B)^\mathcal{I}(\gamma).$$

But from this it follows that $((A \sqcap B) \to \bot)^\mathcal{I}(\gamma) \Rightarrow 0 = 0$, contradicting the axiom $\langle (A \sqcap B) \to \bot \sqsubseteq \bot \geq 1 \rangle$ of $\mathcal{T}_\mathcal{P}^s$. Thus, $\mathcal{P}$ has a solution. □

Notice that, if $\mathcal{P}$ has no solution, then $\mathcal{O}_\mathcal{P}^s$ still has witnessed models, but no strongly witnessed models. It is also relevant to point out that $\mathcal{O}_\mathcal{P}^s$ has a strongly witnessed model iff it has a finite model. In fact, the condition of strongly witnessed was only used for ensuring finiteness of the model, and hence, that a solution is indeed found.

Corollary 7. *For $*$-$\mathcal{ALE}$ ontologies, strongly witnessed consistency and consistency w.r.t. finite models are undecidable.*

5 Conclusions

We have shown that consistency of $*$-$\mathcal{ALE}$ ontologies w.r.t. a wide variety of models, ranging from finite models to weakly witnessed models, is undecidable if the product t-norm semantics are used. Whether consistency in general, that is, without restricting the class of interpretations used, is also undecidable is still an open problem. In [11] it was shown that, if only *crisp* axioms are used, then consistency is equivalent to quasi-witnessed consistency. However, it is unclear how to extend this result to the fuzzy axioms used in this paper.

As future work we plan to study whether these undecidability results still hold if the disjunction and negation constructors are used in place of the implication considered in this paper. Additionally, we will study the decidability status of these logics if different t-norms are chosen for the semantics.

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