THE

MESSENGER OF MATHEMATICS.

EDITED BY

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VOL. LII.

[May 1922—April 1923].

Cambridge: BOWES & BOWES.

London: MACMILLAN & CO. Ltd.

Glasgow: MACLEHOSE, JACKSON & CO.

1923.

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MESSENGER OF MATHEMATICS.

FACTORISATION OF $\mathcal{N}$ & $\mathcal{N}' = (x^n \mp y^n) \div (x \mp y)$, &c.

[when $x - y = n$].

By *Lt.-Col. Allan Cunningham, R E*, Fellow of King's College, London.

[The Author is indebted to Mr. H. J. Woodall for suggestions and help in reading the Proofs].

1. *Introduction.* Using the abbreviation—

M.A.P.F. meaning *Max. Algebraic Prime Factor*..... ..... .(1).

and notation—

$\mathbf{N}_n$ = M.A.P.F. of $(x^n - y^n)$, $\mathbf{N}_n'$ = M.A.P.F. of $(x^n + y^n)$, [x, y unrestricted].(2*a*),

$\mathcal{N}_n$ = M.A.P.F. of $(x^n - y^n)$, $\mathcal{N}_n'$ = M.A.P.F. of $(x^n + y^n)$, [$x - y = n$] ....(2*b*),

Z_n = M.A.P.F. of $(x^n - y^n)$, Z_n' = M.A.P.F. of $(x^n + y^n)$, [$x - y = 1$] .. ..(2*c*),

H_n = M.A.P.F. of $(y^n - 1^n)$, H_n' = M.A.P.F. of $(y^n + 1^n)$, [$x = 1$]................(2*d*),

it is proposed in this Memoir to study the properties of the two quantities $\mathcal{N}_n$, $\mathcal{N}_n'$ defined in (2*b*) above with a special view to their factorisation.

The eight quantities above are all included under the generic title of "*n*-ans", whilst they will be distinguished as:

General n-ans;	*Sub-n-ans;*	*Sub-simple n-ans;*	*Simple n-ans.*
$\mathbf{N}_n$, $\mathbf{N}_n'$	$\mathcal{N}_n$, $\mathcal{N}_n'$	Z_n, Z_n'	H_n, H_n'

The subscript n indicates the *exponent* of the algebraic form; it will be *omitted* when not necessary to specify the exponent, so as to simplify the notation.

It will be seen that the six forms $\mathcal{N}_n$, $\mathcal{N}_n'$; Z_n, Z_n'; H_n, H_n' are only special forms of the General *n*-ans $\mathbf{N}_n$, $\mathbf{N}_n'$. The forms H_n, H_n' are the only ones that have been as yet much studied: the forms Z_n, Z_n' were studied in the present Author's previous* Paper on this subject. It will be shown that the forms $\mathcal{N}_n$, $\mathcal{N}_n'$ here treated of are closely related to the forms Z_n, Z_n' of that Paper (being in fact generalised forms thereof) and also to the forms H_n, H_n'. The treatment in this Paper closely follows that used in the previous paper.

The main divisions of the Subject are—

General, Art. 1–5.
Congruence Roots, Art. 6–16.
Chains & Aurifeuillians. Art. 17–21.
Simplest $\mathcal{N}_n$, Art 23.
Sub-Cubans, Art. 24–33.
Congruence Tables, C3–C15.
Factorisation Tables, F3–F15.

* "Factorisation of $\mathcal{N}$ & $\mathcal{N}' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = 1$], in the *Messenger of Mathematics*, vol. xlix, 1919, pp. 1—36.

1*a*. *Notation.* All symbols here denote *integers.*

p, a, b, c denote *odd primes*, [a $\neq$ b $\neq$ c]; I means an *integer*.

ω, Ω denote *odd* numbers; ϵ, e, E denote *even* numbers.

M.A.P.F.; **N**, **N'**, N, N', Z, Z', H, H' are explained in Art. I.

x, y; x', y' are styled *roots* of $N \equiv 0$, $N' \equiv 0$ (mod p & p^k).

z, w; z', w' are styled *roots* of $Z \equiv 0$, $Z' \equiv 0$ (mod p & p^k).

η, η' are styled *roots* of $H \equiv 0$, $H' \equiv 0$ (mod p & p^k).

The accented letters x', y', z', w', η' belong especially to N', Z', H'; but the accents will be omitted when there is no risk of mistakes.

$v = xy$ or $x'y'$ (a contraction for shortness' sake).

(x, y), $\{x', y'\}$ are used as abbreviations for N, N', exhibiting the elements x, y, $x' y'$.

m, M mean *Multiples of*

$\tau(n)$ denotes the *Totient* of n; $\tau(\text{a}) = \text{a}-1$, $\tau(\text{a}^2) = \text{a}(\text{a}-1)$, $\tau(\text{ab}) = \tau(\text{a}).\tau(\text{b})$.

A, **A'** denote an *Aurifeuillian* (Art. 19) or *Aut-Aurifeuillian* (Art. 21).

1*b*. *Working condition.* To avoid unnecessary obvious factors in **N**, **N'**, &c., it is assumed throughout that—

x and y, x' and y' have no common factors..............(3),

This involves that x, y and x', y' are all prime to n ....(3*a*).

2. *Simpler forms of* N_n, N'_n. The Sub-n-ans (2*h*) take the following simple forms for the simpler cases of $n = \text{a}$, a^2, ab, 2ω:—

1°. $n = \text{a}$; $N = (x^n - y^n) \div (x - y)$; $N' = (x^n + y^n) \div (x + y)$ ....(4*a*),

2°. $n = \text{a}^2$; $N = (x^n - y^n) \div (x^\text{a} - y^\text{a})$; $N' = (x^n + y^n) \div (x^\text{a} + y^\text{a})$.........(4*b*),

3°. $n = \text{ab}$; $N = \dfrac{(x^n - y^n)(x - y)}{(x^\text{a} - y^\text{a})(x^\text{b} - y^\text{b})}$; $N' = \dfrac{(x^n + y^n)(x + y)}{(x^\text{a} + y^\text{a})(x^\text{b} + y^\text{b})}$.........(4*c*),

4°. $n = 2\omega$; $N = $ M.A.P.F. of $(x^n - y^n) = $ M.A.P.F. of $(x^{\frac{1}{2}n} + y^{\frac{1}{2}n})$.........(4*d*).

The four Cases above are the only ones dealt with in this Memoir. To have treated more complicated cases (*e.g.* $n = \text{a}^3$, a^4, &c.; abc, &c.; $n = \epsilon$, &c.) would have needed a Memoir of great length.

3. *N & N' as functions of x, xy.* It will now be shown that N and N' can always be expressed as functions of n^2 and xy.

It is easily seen that N, N' are *symmetric functions* of x, y of *even* degree, which can be arranged as a sum of pairs of terms of form—

$$N \text{ \& } N' = \Sigma_r A_r (xy)^{\alpha_r} . (x^{\epsilon_r \beta_r} + y^{\epsilon_r \beta_r}) \quad \text{....(5)},$$

where $\epsilon_r = 2^{k_r}$, $\beta_r = \omega$ ....(5*a*),

a form which sufficiently exhibits the symmetry (in x, y).

And, it will suffice to show that—(under the condition $x - y = n$)—the quantity $(x^{e_r\beta_r} + y^{e_r\beta_r})$ is always expressible as a function of n^2, xy. This may be shown by taking $e_r = 2, 2^2, 2^3$, &c.,; $\beta_r = 1, 3, 5, 7$, &c., ..., in succession.

In what follows $xy = v$ is written (for shortness), and $x - y = n$ is substituted.

$$x^2 + y^2 = (x-y)^2 + 2xy = n^2 + 2v \quad \ldots\ldots (6a),$$

$$x^4 + y^4 = (x^2+y^2)^2 - 2(xy)^2 = n^4 + 4n^2v + 2v^2 \quad \ldots (6b),$$

$$x^8 + y^8 = (n^4 + 4n^2v + 2v^2)^2 - 2v^4 \quad \ldots (6c),$$

&c. = &c.

$$x^6 + y^6 = (x^3 - y^3)^2 + 2(xy)^3$$

$$= (x-y)^2.(x^2 + xy + y^2)^2 + 2(xy)^3$$

$$= n^2(n^2 + 3v)^2 + 2v^3 \quad \ldots (6d),$$

$$x^{10} + y^{10} = (x^2 + y^2)^5 - 5x^2y^2(x^6 + y^6) - 10x^4y^4(x^2 + y^2)$$

$$= n^{10} + 10n^8v + 35n^6v^2 + 50n^4v^3 + 25n^2v^4 + 2v^5 \quad \ldots (6e),$$

$$x^{12} + y^{12} = \{n^2(n^2 + 3v)^2 + 2v^3\}^2 - 2v^6 \quad \ldots (6f),$$

&c. = &c.

The mode in which these are successively formed suffices to show that

$(x^{e_r\beta_r} + y^{e_r\beta_r})$ is always a homogeneous function of n^2, xy. . .(7),

whence it follows from (5) that

N & N' are always expressible as homogeneous functions of n^2, xy (8)

3*a*. *Linear forms of* N, N'. It is known that—using M to denote "multiple of"—

$n = \text{a}, \text{a}^2, \text{a}^3$, &c., gives $\frac{1}{a}N_n = 1 + M(n)$ . . (9a),

$n = \text{ab}$ gives $N_n = 1 + M(n)$ . . . .(9b),

whilst $N_n' = 1 + M(n)$ *always* (9c).

3*b*. N & N' *as functions of* n, v *continued*. The following Table shows the quantities N, N' expressed as functions of n, and $v = xy$ for all *odd* values of $n \not> 15$.

n	N_n	N_n'
3	3^2+3v	3^2+v
5	$5^4+5v(5^2+v)$	$5^4+v(3.5^2+v)$
7	$7^6+7v(7^2+v)^2$	$7^6+v(7^2+v)(5.7^2+v)$
9	$9^6+3v(9^2+v)(3.9^2+v)$	$9^6+v(6.9^4+9.9^2v+v^2)$
11	$11^{10}+11v(11^2+v)(11^6+3.11^4v+4.11^2v^2+v^3)$	$11^{10}+v(9.11^8+28.11^6v+35.11^4v^2+15.11^2v^3+v^4)$
13	$13^{12}+13v(13^2+v)^2(13^6+3.13^4v+5.3^2v^2+v^3)$	$13^{12}+v(13^2+v)(13^8+34.13^6v+50.13^4v^2+20.13^2v^3+v^4)$
15	$15^8+v(15^2+v)(7.15^4+7.15^2v+v^2)$	$15^8+v(9.15^6+26.15^4v+24.15^2v^2+v^3)$

The following inferences seem evident from the Table (though not easily proved in a general manner). [Here $\tau(n)$ means "Totient of n"].

$$X_n = n^{\tau(n)} + v.\phi_1(n, v), \; X_n' = n^{\tau(n)} + v.\phi_2(n, v), \; always \ldots\ldots(11),$$

$$X_n = n^{\tau(n)} + v(n^2 + v).\phi(n, v), \text{ when } n > 3 \ldots\ldots(11a).$$

[The whole of the Results of Art. 3, 3*a*, 3*b* above merge into the similarly numbered Results of Art. 3, 3*a*, 3*b* of the previous Paper, by writing $x - y = n = 1$; and may also be derived from those Results by introducing powers of n^2 therein, so as to render them homogeneous in n^2, v].

4. *Quadratic Forms.* The numbers **N**, **N'** are known to be expressible in the following 2^{ic} forms (when n has no square factors)—

$$n = 4k+1 \text{ has } \mathbf{N}_n = X^2 - nY^2, \quad \mathbf{N}_n' = X'^2 - nY'^2 \ldots\ldots(12a),$$

$$\mathbf{N}_n = T^2 - nxyU^2, \quad \mathbf{N}_n' = T'^2 + nxyU'^2 \ldots\ldots(12b),$$

$$n = 4k-1 \text{ has } \mathbf{N}^n = X^2 + nY^2, \quad \mathbf{N}_n' = X'^2 + nY'^2 \ldots\ldots(12c),$$

$$\mathbf{N}_n = T^2 + nxyU^2, \quad \mathbf{N}_n' = T'^2 - nxyU'^2 \ldots\ldots(12d).$$

When $n = a^2, a^3$, &c., then a should be substituted for n in the above formulæ; and when n contains square factors (as well as other factors) certain modifications are necessary.

The forms for X, Y, T, U, &c., for X_n, X_n' are the same as for the general n-ans $\mathbf{N}_n$, $\mathbf{N}_n'$, so need not be detailed here. It will suffice to say that the 2^{ic} parts (X, Y, &c.) of X_n, X_n' are not generally expressible as functions of x, y.

5. *Factorisation by the Factor-Tables.* This factorisation of X, X' may be effected to a certain extent (up to the limit of $\frac{1}{n}.X$ & $X' \not> 10{,}017{,}000$) by the large* Factor-Tables: but this can be done only to the very limited extent shown below:

	$n =$	3	5	7	9	11	15
Limit of x	in X	5477	85	17	16	6	8
	in X'	3166	58	15	14	5	7

so that, to push it further, other means—(explained in next Article)—must be sought

6. *Congruence-Tables.* Solutions (x, y, x', y') of the Congruences

$$X_n \equiv 0, \text{ and } X_n' \equiv 0 \pmod{p \;\&\; p^\kappa} \ldots\ldots(13),$$

for all primes (p) capable of acting as divisors of the forms X_n & X_n' would evidently supply divisors of X_n & X_n'.

It will now be shown how to find solutions (x, y), (x', y')—or (as they are often called)—*Roots* of those congruences.

* *Factor-Tables for the first ten millions*, by D. N. Lehmer, Washington, 1909; these extend to 10017000.

And, since the differences are constant, viz. $x - y = x' - y' = n$, it will evidently suffice to record *one*—say x & x')—of each pair of these roots (x, y), (x', y') in Tables of Solutions. It will be shown that they are intimately connected with the roots of the associated Congruences

$$Z \equiv 0, \quad Z' \equiv 0; \quad H \equiv 0, \quad H' \equiv 0.$$

7*a*. *Special divisor* n, (or a). The Theory of Numbers shows that

n is a non-divisor of every N_n' ....(15*a*),

n is a divisor of every N_n, when—and only when—$n = \text{a}$ .....(15*b*),

a is a divisor of every N_n, when—and only when—$n = \text{a}^\kappa$......(15*c*),

n^2 and a^2 are non-divisors of all N_n and N_n' ....(15*d*),

As every N_n is divisible by either n or a (when $n = \text{a}, \text{a}^2, \text{a}^3$, &c.) it is convenient to deal in general (in those cases) in what follows with $\frac{1}{n}.N_n$ or $\frac{1}{\text{a}}N_n$, instead of with N_n itself.

7*b*. *Form of divisors.* Excluding the exceptional divisor (n or a)—see Art. 7*a*—it is known that—

Every divisor (p) of N & N' must be of form $p = 2mn + 1$ ....(16).

7*c*. *Number of roots* (x, x') *of Congruences.* It is known that

The number of incongruous roots (x, x') of each of the Congruences (13) is $= \tau(n)$, [n being *odd*]...(17),

where $\tau(n)$ means the Totient of n.

8. *Roots* (x, x') *from factorisations.* Every actually factorised number N, N' evidently supplies one root (x or x') for every prime (p) and prime-power (p^κ) contained in N or N': so that a few roots (x, x') of the Congruences (13) will be supplied by the factorisations found from the Factor-Tables (Art. 5); but this number is evidently very limited when $n > 3$ (see Art. 5).

9. *Connexion of roots* x, x' *with* η, η'. By a process precisely similar to that used in Art. 9 of the previous Paper (q, v), it may be shown that

$$x \equiv \frac{-n}{\eta - 1}, \quad v \equiv \frac{+n}{\eta' + 1}; \qquad x' \equiv \frac{-n}{\eta - 1}, \quad x' \equiv \frac{+n}{\eta + 1} \pmod{p \ \& \ p^\kappa} \ \ldots(18).$$

Now extensive Tables* exist of the roots η, η' modulo

* The Author has prepared extensive Tables of η, η' for *all* exponents $n \not> 15$ for all primes and prime-powers p & $p^\kappa \not> 100000$; most of these are in type, and printed off, in a series of seven volumes styled *Binomial Factorisations*. Mr T. G. Creak has prepared similar Tables for $n > 15$ to 40 for $p \not> 10000$, and in some cases up to $\not> p$ 10000, or even 100000.

p & p^κ. Thus the roots x, x' may be computed from the known roots η, η' by means of the Congruences (18) and this affords one of the readiest means of computing x, x'.

9a. *Connexion of roots x, x' with z, z'.*

$X_n \equiv 0$ gives $(y/x)^n \equiv +1 \pmod{p}$; $Z_n \equiv 0$ gives $(w/z)^n \equiv +1 \pmod{p}$.

Hence $y/x \equiv w/z \pmod{p}$;

Here $y = x - n$, and $w = z - 1$; whence $x \equiv nz \pmod{p}$............(18a).

Similarly it will be found that $x' \equiv nz' \pmod{p}$............(18b),

and similar results may be found for the modulus p^κ.

By these simple formulæ (18a, b) it is easy to compile Tables of solution (x, x') of the Congruences (13) from known* solutions (z, z') of $Z \equiv 0$, $Z' \equiv 0$.

From Results (18a, b) follow the important consequences that all relations between several roots z, z' are easily converted into *similar* relations between their corresponding roots x, x', and—in particular—

Homogeneous relations between several roots z, z' are converted into the similar relations between x, x' by simply changing z, z' into x, x'..................................(18c).

The Results 19a—of Art. 10—of the previous Paper will accordingly be converted into the similar theorems for the x, x' of the present, and simply stated below (without separate proof).

[To facilitate comparison the previous Results bear the *same numbering* as those from which they are derived in the previous Paper: noting that the x, x' of that Paper become the z, z' of the present Paper].

10. *Conjugate Roots* (x_r, x_s), (x_r', x_s'). The whole set of roots x, and similarly the set of roots x', may be grouped *in pairs* (x_r, x_s), (x_r', x_s') derived from (η_r, η_s), (η_r', η_s') where

$\eta_r . \eta_s \equiv +1$, and $\eta_r' . \eta_s' \equiv +1$ (mod p or p^κ)..................(19).

From Art. 10 of the previous Paper, it follows that in the present case—

$x_r + x_s = n$, or $p + n$, (or $p^\kappa + n$) ....(20a),

$x_r' + x_s' = n$, or $p + n$, (or $p^\kappa + n$) ....(20b).

These results show that it suffices to compute one-half of the complete set of $\tau(n)$ roots of each kind (x, x'); the other half being obtained by simple subtraction from (20a, b).

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* A set of such Tables were given in the previous Paper for all *odd* values of $n = 3, 5, 7, 9, 11, 15$ for all primes (p), and prime-powers $p^\kappa \not> 1000$: the x, x' of those Tables are changed into z, z' in the present Paper.

11. *Associate Roots.* Let x_r, x_r' be the roots obtained from the "associate pair" η_r, η_r', which are such that

$$\eta_r + \eta_r' \equiv 0, \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (21),$$

Then from Art. 11 of the previous Paper it follows that in the present case—

$$x_r + x_r' \equiv 2x_{2r}, \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (22a),$$

$$x_r \cdot x_r' \equiv n x_{2r}, \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (22b),$$

$$1/x_r + 1/x_r' \equiv 2/n, \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (22c).$$

The above three Results are true for all values of r (prime to n), and may be used as *succession-formulæ* for computing the complete sets of roots (x and x') of any prime (p), or prime-power (p^κ) from *one* given root x or x'.

Thus x_1', or x_1, may be computed from a given x_1, or x_1', by

$$x_1' \equiv \frac{nx_1}{2x_1 - n}, \quad x_1 \equiv \frac{nx_1'}{2x_1' - n}, \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (23),$$

and x_2 may be computed from

$$x_2 \equiv \tfrac{1}{n}(x_1 x_1'), \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (23a),$$

and the formula (22a)

$$x_2 = \tfrac{1}{2}(x_1 + x_1'), \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (23b),$$

may be used as a check on the work, and so on.

12. *Simple formula for* x' (n prime). Take

$$X' = \eta' + \eta'^3 + \eta'^5 + \ldots + \eta'^{n-2} \ldots\ldots (24),$$

Then from Art. 12 of the previous Paper, it follows that in the present case—

$$x' = X', \text{ (mod } p) \ldots\ldots (24a).$$

This formula is easy to use for computing x' (when n is prime), especially when n is small, in which case it takes the following simple forms—

$n =$	3	5	7	11	
$x' \equiv$	$3(\eta_1')$	$5(\eta_1' + \eta_3')$	$7(\eta_1' + \eta_3' + \eta_5')$	$11(\eta_1' + \eta_3' + \eta_5' + \eta_7' + \eta_9')$	(24b).

[But, to use these formulæ at all conveniently, it is necessary to have Tables of η' arranged* so as to show each root η_r', *along with the index* r, showing thereby its derivation from η_1'. Any root may then be taken for η_1', and the *corresponding* η_r can be pretty easily picked out].

There appears to be no simple formula for the roots x similar to the above for x'.

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* The Tables I. for each value of n in Reuschle's *Tafeln Complexer Primzahlen, &c.*, give the roots of the M.A.P.F. of $y^n - 1 \equiv 0$ (mod p) arranged in this way for most values of $n \nless 105$ for all suitable primes $p > 1000$. The present Author's Tables of η, η'—(quoted in footnote of Art. 9)—are arranged in the *numerical order* of the roots η, η'; so are not convenient for the purpose of this Article.

13. *Sum of Roots.* From Art. 13 of the previous Paper it follows that here

$$\Sigma(x) = \Sigma(x') = \tfrac{1}{2}n.\tau(n), \text{ [for } p \;\&\; p^\kappa] \;\ldots\ldots\ldots\ldots (25).$$

14. *Product of Roots.* From Art. 14 of the previous Paper it follows that here the Residues of the continued products of the whole set of roots x, and of the whole set x', are as shown below:

$$n = \text{a} \text{ gives } \Pi(x) \equiv n^{\tau(n)-1}; \quad \Pi(x') \equiv n^{\tau(n)} \pmod{p} \ldots\ldots (26a),$$

$$n = \text{a}^2 \text{ gives } \Pi(x) \equiv \text{a}^{2\tau(n)-1}; \quad \Pi(x') \equiv n^{\tau(n)} \pmod{p} \ldots\ldots (26b),$$

$$n = \text{ab gives } \Pi(x) \equiv n^{\tau(n)}; \quad \Pi(x') \equiv n^{\tau(n)} \ldots\ldots\ldots (26c).$$

[These results may be used as a Test of the accuracy of the Tables].

15. *Congruence-Tables.* The Tables* C_3, C_5, C_7, C_9, &c.—at end of this Memoir—give the complete set of roots (x, x') of both Congruences $X_n \equiv 0$, $X_n' \equiv 0$ (mod p & p^κ) for the odd values of $n = 3, 5, 7, 9, 11, 15$ for all primes and prime-powers (p & p^κ) proper to each index n, up to p & $p^\kappa \ngtr 1000$.

16. *Factorisation-Tables.* The Tables* F_5, F_7, F_9, F_{11}, F_{15}—(at end of the Text)—give the factorisation into prime factors—(as far as found possible with the means available)—of both X & X' up to following limits of x, x'—

$n =$	5,	7,	9,	11,	15;
x & $x' \ngtr$	187,	60,	74,	43,	49.

The aids to factorisation used were:—

1°. The *Congruence-Tables* quoted in Art. 15: these have enabled *all* divisors $\ngtr 1000$ to be cast out (none available for $n = 13, 17$).

2°. Certain *Numerical*† *MS Canons* (2^{ary}, 3^{ary}, 5^{ary}, ..., 11^{ary}), which give the residues (R, both ±) of x^r (mod p & p^κ)) for each of the Bases 2, 3, 5, ..., 11, up to p & $p^\kappa \ngtr 10000$; and up to the limit $r = 100$ for Base 2, and $r = 50$ for the other Bases.

These Canons have enabled all divisors $\ngtr 10000$ to be cast out when $x \ngtr 11$.

16a. Explanation of signs (., ;, :, ?, †, ‡) in the Tables F_3, ..., F_{15}.

1°. A semi-colon (;) on the extreme right shows that the factorisation (into prime factors) is *complete*.

2°. A full stop (.) on the extreme right shows the presence of other (unknown) factors (each > 1000).

* These Tables were computed partly by the Author, partly by an Assistant (the late Mr. R. F. Woodward) under the Author's superintendence. Every root x, x' was checked by one of the Rules in Art. 11.

† These Canons are still in MS. They await funds for publication! They were computed by an Assistant (Miss A. Woodward) under the Author's superintendence. The 2^{ary} and 10^{ary} Canons have been computed also by Mr. H. J. Woodall, A.R.C.Sc. The two copies have been collated.

3°. A colon (:) in the middle is used to separate two important *algebraic* factors, *e.g.* Chain-Factors, or Aurifeuillian Factors (Art. 17, 18).

3°*a* A bar ( ) in the middle is used to separate the two factors $(X - A)$ $(X + A)$ of the form $(X^2 - A^2)$;—(as in Art. 22).

4°. The signs (†, ‡) on the extreme right show that all factors $< 10^3, 10^4$ respectively have been cast out.

5°. A query (?) on the extreme right of a large factor $(> 10^7)$ shows that the composition of this factor is unknown (but it contains no factor $< 10^5$).

17. *Numerical Chains.* Let N_1, N_2, N_3, &c.,* be a series of *composite* numbers, each *formed in the same way* from a pair of elements (x, y), so that—

$$N_1 = f(x_1, y_1),\ N_2 = f(x_2, y_2),\ N_3 = f(x_3, y_3),\ \ldots,\ N_r = f(x_r, y_r) = L_r M_r \ldots (28),$$

when the functional operator (f) is the same throughout. When the factors (L, M) of every three successive numbers (N_{r-1}, N_r, N_{r+1}) are so connected that

$$M_{r-1} = L_r,\ M_r = L_{r+1} \text{ (for all values of } r) \ldots\ldots (29)$$

then the numbers (N_r) are said to be *in chain*†: the series is styled a *Chain-series*†, and the factors (L, M) are styled *Chain-Factors*†.

The salient properties of such Chains are—

$$\frac{N_2 N_4 N_6 \ldots N_{2r}}{N_1 N_2 N_3 \ldots N_r} = \frac{M_{2r}}{L_1} \ldots\ldots (30a),$$

$$\frac{N_1 N_3 N_5 \ldots N_{2r+1} N_{2r+1}}{N_2 N_4 N_6 \ldots N_{2r}} = L_1 M_{2r-1} \ldots\ldots (30b),$$

$$N_1 N_2 N_3 \ldots N_r = L_1 (L_2 L_3 \ldots L_r)^2 . M_r$$
$$= L_1 (M_1 M_2 \ldots M_{r-1})^2 . M_r \ldots\ldots (30c),$$

$$N_1 N_2 N_3 \ldots N_r = \square, \text{ if } L_1 M_r = \square \ldots\ldots (30d).$$

17*a*. *Arithmetical Nexus.* An interesting variety of this last occurs when $L_1 = M_r$; this gives

$$N_1 N_2 N_3 \ldots N_r = (L_1 L_2 L_3 \ldots L_r)^2 = (M_1 M_2 M_3 \ldots M_r)^2. \ldots (30e).$$

This variety is styled an arithmetical *Nexus*. If repeated to left and right, it forms a continuous *periodic* chain, thus the Nexus N_1, N_2, N_3 in which $L_1 = M_3$ gives

$$\ldots N_1, N_2, N_3, N_1, N_2, N_3, N_1, N_2, N_3, \ldots$$

and any three *consecutive* members of it form the *same* Nexus, thus

N_1, N_2, N_3; N_2, N_3, N_1; N_3, N_1, N_2 are the *same Nexus*.

Notation. In numerical work the Chain-Factors (L, M) of a member (N) of a Chain are separated by a colon (:), [thus $N = 91 = 7 : 13$]; so that this colon is really a special *sign of multiplication*].

* The N_1, N_2, N_3, &c., of this Article are not necessarily of the type of the N, N' of this Memoir, but are conditioned *only as here stated*.

† These terms were introduced by the present Author.

Examples of Chains. These may be formed by the simple Rule explained in Art. 32 of the previous Paper. Simple as the Method is (in principle), its application—when subject (as now) to the condition $x - y = n$—is practically limited to the cases of Sub-Cubans and Sub-Quintans, as the numbers (N_n) rise so rapidly as to be beyond the (present) powers of factorisation, which is an essential step in Chain-formation.

Examples of Sub-Cuban Chains will be found in Art. 30*b*.

Here follow a few Examples of Sub-Quintan Chains and Nexuses. The following notation is used for sake of brevity:

$$(x_r, y_r) = (x_r^5 - y_r^5) \div (x_r - y_r) = L_r . M_r; \quad \{x_r, y_r\} = (x_r^5 + y_r^5) \div (x_r + y_r) = L_r . M_r.$$

r	1	2	3	4	5
x_r, y_r	(3, 2)	(7, 2)	(19, 14)	{24, 19}	{31, 26}
L_r, M_r	1:11;	11:61;	61.31.41;	31.41:191;	191:61.61;

r	6	7	8	9
x_r, y_r	(68, 63)	{34, 29}	(9, 4)	{12, 7}
L_r, M_r	61.61:121.41;	121.41:211;	211:11;	11:31 41;

r	1	2	3	
x_r, y_r	(7, 2)	(19, 14)	{12, 7}	form a Nexus of 3 links.
L_r, M_r	11.61;	61:31.41;	31 41:11;	

Nos. 4, 5, 6, 7, 8, 9 of the first Series above also form a Nexus of 6 links.

18. *Aurifeuillians, Ant-Aurifeuillians* (A, A').

It is known that the general n-an N_n or N_n' may always be expressed in one of the impure 2^{ic} forms shown below—the determinant D of the form—($D = \pm\, n\,xy,\ \pm\, \mathrm{a}\,xy$, &c.)—depending on the form of n.

Form of n	N	N'	
$n = 4\alpha + 1 \mp \square$	$P^2 - nxyK^2$	$P'^2 + nxyK'^2$	............(31*a*),
$n = \mathrm{a}^2,\ \ \mathrm{a} = 4\alpha + 1$	$P^2 - \mathrm{a}xyK^2$	$P'^2 + \mathrm{a}xyK'^2$	............(31*b*),
$n = \mathrm{ab},\ \ \mathrm{a} = 4\alpha + 1$	$P^2 - \mathrm{a}xyK^2$	$P'^2 + \mathrm{a}xyK'^2$	............(31c),
$n = 4\alpha - 1$	$P'^2 + nxyK'^2$	$P^2 - nxyK^2$	............(31*a'*),
$n = \mathrm{a}^2,\ \ \mathrm{a} = 4\alpha - 1$	$P'^2 + \mathrm{a}xyK'^2$	$P^2 - \mathrm{a}xyK^2$	............(31*b'*),
$n = \mathrm{ab},\ \mathrm{a} = 4\alpha - 1$	$P'^2 + \mathrm{a}xyK'^2$	$P^2 - \mathrm{a}xyK^2$	............(31*c'*).

Here, introducing the condition

$$nxy = \square, \ = (n\tau v)^2, \text{ or } \mathrm{a}\,xy = \square = (\mathrm{a}\tau v)^2 \quad \ldots\ldots\ldots(32),$$

N and N' become either a *sum*, or a *difference of squares*, viz.

$$\mathrm{N} = P^2 - Q^2, \text{ or } P'^2 + Q'^2; \quad \mathrm{N}' = P'^2 + Q'^2, \text{ or } P^2 - Q^2 \quad \ldots\ldots(33),$$

where $\quad Q^2 = n\,xyK^2, \text{ or } \mathrm{a}\,xyK^2; \quad Q'^2 = n\,xyK'^2, \text{ or } \mathrm{a}\,xyK'^2 \quad \ldots\ldots(33a).$

When **N** or **N'** $= P^2 - Q^2$, it is styled an *Aurifeuillian**, denoted by **A**: and, when **N** or **N'** $= P'^2 + Q'^2$, it is styled an *Ant-Aurifeuillian**, denoted by **A'**; and the condition (32) producing **A** or **A'** is styled the *Aurifeuillian* condition.*

And, since x, y are supposed mutually prime (Art. 1*b*) this condition requires that x, y should be one of the forms—

$$x = \tau^2,\ n\upsilon^2,\ a\upsilon^2,\ b\upsilon^2,\ a\tau^2,\ b\tau^2 \ldots\ldots\ldots\ldots\ldots\ldots (34a),$$

$$y = n\upsilon^2,\ \tau^2,\ \tau^2,\ \tau^2,\ b\upsilon^2,\ a\upsilon^2 \ldots\ldots\ldots\ldots\ldots\ldots (34b),$$

but,—in the case of Sub-n-ans—x, y should both be prime to n, so that the conditions (32) *violate the "working condition"* (Art. 1*b*). This shows that

$\frac{1}{n}.N$ and N' cannot be *explicitly* either an **A** or an **A'**.........(35).

It may however happen that—with certain indices (n)—one or both of $\frac{1}{n}N_n$, N'_n may be expressible in some other way in form **N** or **N'** (with different x, y of course) to which the Aurifeuillian condition (32) is applicable. This happens markedly in the case of Sub-Cubans ($n = 3$), as will appear later (Art. 30*a*, 31, 32): it is not at present known to happen with other indices ($n > 3$).

19. *Aurifeuillians* (**A**). These are the more interesting of the two functions (**A**, **A'**) introduced in Art. 18; as $\frac{1}{n}N$, or N', being then (algebraically) expressible as a *difference of squares*, is hereby always (algebraically) factorisable into two factors, say L, M: thus—

$$\tfrac{1}{n}.N \text{ or } N' = \mathbf{A} = P^2 - Q^2 = L.M,\ [Q = n\tau\upsilon K, \text{ or } a\tau\upsilon K, \text{ \&c.}] \ldots\ldots (36),$$

$$L = P - Q, \quad M = P + Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36a).$$

The two factors, say L, M are styled the *Aurifeuillian-Factors* of **A**, and have the property

L and M are both expressible in the same *pure* 2^ic forms as their product **A** ....(37).

[In numerical work the Aurifeuillian Factors (L, M) of an Aurifeuillian (N) are separated by a colon (:)—(see Note at end of Art. 17)].

21. *Ant-Aurifeuillians* (**A'**). These are of interest chiefly as giving the algebraic expression of $\frac{1}{n}N$, or N', as a *sum of squares*—(which materially helps factorisation)—

$$\tfrac{1}{n}.N \text{ or } N' = P'^2 + Q'^2,\ [Q' = n\tau\upsilon.K', \text{ or } a\tau\upsilon.K', \text{ \&c.}] \ldots\ldots (39).$$

---

* These terms are due to the present Author: see his Memoir *On Aurifeuillians*, in *Proc. Lond. Math. Soc.*, vol. xxix, 1898.

The following general relation holds between P, P' and between Q, Q'.

If $P = \psi(x, y)$, and $Q = \psi(x, y)$; then

$$P' = \psi(x, -y), \text{ and } Q' = \psi(x, -y), \text{ and \textit{vice versâ}} \quad \ldots (40).$$

22. *Simplest factorisable* N_n. Take $x = \xi^2$, $y = \eta^2$; and—(for shortness)—

Let $(x, y)_n$, $\{x, y\}_n$ denote the M.A.P.F. of $(x^n - y^n)$ and $(x^n + y^n)$, in general.

$$\text{Then } \mathbf{N}_n = (x, y)_n = (\xi^2, \eta^2)_n = (\xi, \eta)_n \cdot \{\xi, \eta\}_n = L.M \quad \ldots\ldots \ldots (41).$$

Thus $\mathbf{N}_n$ is resolved (algebraically) into two factors (L, M).

The restriction $\xi^2 - \eta^2 = x - y = n$ of this Memoir permits of extremely few cases (*i.e.* possible values of ξ, η) for each value of n, depending in fact on the factorisation of n itself into *two* factors (say $n = 1.n$; $n = \text{a}.\text{b}$, &c.).

i. $n = 1.n$; has *only one* case, $\xi = \frac{1}{2}(n+1)$, $\eta = -\frac{1}{2}(n-1)$, giving $\xi - \eta = n$, $\xi + \eta = 1$.

L is of form X_n, and has the factor a when $n = \text{a}, \text{a}^2, \text{a}^3$, &c.; and is also of form H' (see Art. 1).

M is of form X_n', and is also of form H (see Art. 1).

ii. $n = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$ has also *one case*, $\xi = \lambda$, $\eta = \bar{1}$. giving $\xi - \eta = \lambda + 1$, $\xi + \eta = \lambda - 1$.

Here L is of form H', and M is of form H (see Art. 1).

iii. $n = \alpha\beta$, [α, β may be composite, but without common factor], has also *one* case;

$$\xi = \tfrac{1}{2}(\alpha + \beta), \quad \eta = -\tfrac{1}{2}(\alpha - \beta), \quad [\alpha > \beta], \text{ giving } \xi - \eta = \alpha, \ \xi + \eta = \beta.$$

The above include all the (at present) *completely* factorisable cases; as, when n is > 15, the factors L, M become generally too large for the present means of factorisation.

The Table below gives the complete factorisation of all the cases at present possible.

Case	n	ξ, η	x, y	a	L	M
i	3	2, $\bar{1}$	4, 1	3	3	7;
i	5	3, $\bar{2}$	9, 4	5	5.11	211;
i	7	4, $\bar{3}$	16, 9	7	7.379	14197;
i	9	5, 4	25,16	3	3.3907	19.1459;
i	11	6, 5	36,25	11	11.23.353.419	‡
i	13	7, 6	49,36	13	13. †	53.79.20021003 ? †
i	15	8, $\bar{7}$	64,49	—	61 231661	31.181.2011;
ii, iii	15	4, $\bar{1}$	16, 1	—	61.1321	151.331;
ii	21	5, 2	25, 4	—	43.1009.3613	127.2525293;
iii	35	6, $\bar{1}$	36, 1	—	71. *	631.701.2311.9241.585131;

* No divisor < 100000.

23. *Properties of Sub-Cubans* (N_{iii}, N_{iii}'). The following Articles are devoted to developing the properties of Sub-Cubans (N_{iii}, N_{iii}'). These are so numerous—compared with those of other indices ($n>3$)—as to require separate treatment. They are dealt with as follows:—

General properties, Art. 24, 25	Aurifeuillians, Art. 29—33
Factorisation, Art. 26, 27	Perfect powers, Art. 28—34.

24. *Cuban Identity.* Every General Cuban $\mathbf{N}_{iii}$, $\mathbf{N}_{iii}'$ can be expressed in *three* equivalent Cuban forms, one of $\mathbf{N}_{iii}$, two of $\mathbf{N}_{iii}'$, viz.

$$\mathbf{N}_{iii} = \frac{x^3-y^3}{x-y} = \frac{z^3+x^3}{z+x} = \frac{z^3+y^3}{z+y} = \mathbf{N}_{iii}', \text{ identically, } [z=x+y] \ldots\ldots (43).$$

As N_{iii} is always divisible by 3—(Art. 7*a*)—it is convenient to deal with $\frac{1}{3}.N_{iii}$ in future in place of N_{iii}.

Introducing now the condition $x-y=3$ of this Memoir, $\frac{1}{3}.N_{iii}$ and N_{iii}' become

$$\tfrac{1}{3}.N_{iii} = \frac{Y^3-1^3}{Y-1} = \frac{X^3+1^3}{X+1} = \frac{X^3+Y^3}{X+Y}$$

$$\text{where } X=x-1,\ Y=y+1,\ X-Y=1,\ [x-y=3] \ldots (43a).$$

$$N_{iii}' = \frac{y^3-3^3}{y-3} = \frac{x^3-3^3}{x+3} = \frac{x^3+y^3}{x+y},\ [x-y=3] \ldots\ldots\ldots\ldots (43b).$$

Hereby $\frac{1}{3}N_{iii}$ has also the form

$$\tfrac{1}{3}N_{iii} = H_{iii} = H_{iii}' \ldots\ldots\ldots\ldots (43c).$$

Properties latent in $\frac{1}{3}.N_{iii}$, N_{iii}' are in many cases obvious in one or other of these equivalent forms. This leads to many properties peculiar to Sub-Cubans, which obtain in no other Sub-n-ans.

24*a*. *Various formulæ* for $\frac{1}{3}N_{iii}$, N_{iii}'.

$$\tfrac{1}{3}N_{iii} = y^2+3y+3 = x^2-3x+3 = x^2-xy+y^2-6 \ldots\ldots (44a),$$

$$= Y^2+Y+1 = X^2-X+1 = X^2-XY+Y^2 \ldots\ldots (44a'),$$

$$= H_{iii} \qquad = H_{iii}' \ldots\ldots (44a''),$$

$$N_{iii}' = y^2+3y+9 = x^2-3x+9 = x^2-xy+y^2 \ldots\ldots (44b),$$

$$\tfrac{1}{3}N_{iii} = xy+3; \quad N_{iii}' = xy+9 \ldots\ldots (44c),$$

Hence $$N_{iii}' - \tfrac{1}{3}N_{iii} = 6, \text{ always} \ldots\ldots (44c').$$

24*b*. 2^ic^ *forms* (A, B) *of* N_{iii}, N_{iii}'. All Cubans can be expressed in the 2^ic^ form (A^2+3B^2), the precise form depending on whether x or y is *even*.

In the case of N_{iii}, N_{iii}' this takes the following forms—(noting that one of x, y must be *even*, since $x - y = n = \omega$)—

$$\tfrac{1}{3}N_{iii} = (\tfrac{1}{2}y)^2 + 3(\tfrac{1}{2}y+1)^2, \quad N_{iii}' = (\tfrac{1}{2}y+3)^2 + 3(\tfrac{1}{2}y)^2, \ [y=\epsilon] \ \ldots\ldots\ldots\ldots (45a),$$

$$= (\tfrac{1}{2}x)^2 + 3(\tfrac{1}{2}x-1)^2, \quad = (\tfrac{1}{2}x-3)^2 + 3(\tfrac{1}{2}x)^2, \ [x=\epsilon] \ \ldots\ldots\ldots\ldots (45b).$$

and it is here seen that

$$\tfrac{1}{3}N_{iii} \text{ has } B - A = 1, \quad N_{iii}' \text{ has } A - B = 3, \ [y=\epsilon] \ \ldots\ldots\ldots\ldots (46a),$$

$$\text{,,} \quad \text{,,} \quad A - B = 1, \quad \text{,,} \quad \text{,,} \quad B - A = 3, \ [x=\epsilon] \ \ldots\ldots\ldots\ldots (46b).$$

These suffice to distinguish Sub-Cubans N_{iii}, N_{iii}' *from each other*, and also *from all other Cubans* $\mathbf{N}_{iii}$ as follows—

Every $\mathbf{N} = A^2 + 3B^2$ with $A - B = 1$ is $= \frac{1}{3}N_{iii}$ .... (47a),

Every $\mathbf{N} = A^2 + 3B^2$ with $A - B = 3$ is $= N_{iii}'$ .... (47b),

No other $\mathbf{N} = A^2 + 3B^2$ can be either $\frac{1}{3}N_{iii}$ or N_{iii}' .... (47b).

24c. *Expression of a given number* (N) *as* $\frac{1}{3}N_{iii}$ *or* N_{iii}'. The Results in Art. 24b make it easy to test whether a *given* number is a Sub-Cuban, or not, and to find its elements A, B, x, y.

1°. If the given $N = \frac{1}{3}N_{iii}$, then $A^2 + 3B^2 = N$, and $A - B = \pm 1$.
Solving for B gives $B = \frac{1}{4}\{\sqrt{(4N-3)} \mp 1\}$, [*one* of the $\mp$ signs gives $B = I$].
And $A = B \pm 1$; [*one* of the $\pm$ signs satisfies $A^2 + 3B^2 = N$].
Lastly, $x = 2A$ when $A > B$, $y = 2A$ when $A < B$.

2°. If the given $N = N_{iii}'$, then $A^2 + 3B^2 = N$, and $A - B = \pm 3$.
Solving for B gives $B = \frac{1}{4}\{\sqrt{(4N-27)} \mp 3\}$; [only *one* of the $\mp$ signs gives $B = I$].
And $A = B \pm 3$; [*one* of the signs satisfies $A^2 + 3B^2 = N$]
Lastly, $x = 2B$ when $A < B$, $y = 2B$ when $A > B$.

Ex. Given $N = 400060003$; If $N = \frac{1}{3}N_{iii}$, then $B = \frac{1}{4}\{\sqrt{(4N-3)} \mp 1\} = 10001$;
$A = 10000$ or 10002; 10000 satisfies $A^2 + 3B^2 = N$.
Lastly, $A < B$ gives $y = 20000$, $x = 20003$.

25. *Equality of* $\frac{1}{3}N_{iii}$, N_{iii}'. Results (45a, b) shew that there are *only two* cases of $\frac{1}{3}N_{iii} = N_{iii}'$, given by $x = 1$, or $y = \pm 1$. viz.:

1°. $$\tfrac{1}{3}N_{iii} = \tfrac{1}{3}\cdot\frac{4^3 - 1^3}{4-1} = 7 = \frac{1^3 + \bar{2}^3}{1 + \bar{2}} = \frac{2^3 + \bar{1}^3}{2 + \bar{1}} = N_{iii}' \ \ldots\ldots\ldots (48a),$$

2°. $$\tfrac{1}{3}N_{iii} = \tfrac{1}{3}\cdot\frac{5^3 - 2^3}{5-2} = 13 = \frac{4^3 + 1^3}{4+1} = N_{iii}' \ \ldots\ldots\ldots (48b).$$

26. *Factorisation of* N_{iii} & N_{iii}'. Table F3a give the complete factorisation of $\frac{1}{3}N_{iii}$ and N_{iii}' up to the limit of x and $x' < 75$—(sufficient to show the sub-cuban properties)—with the values of x, y from which they arise, and the elements (A, B) of their 2^{ic} partition ($A^2 + 3B^2$).

The highest numbers $\frac{1}{3}N_{iii}$, N_{iii}' within the range of the large Factor Tables are given by $x = 3166$, viz.:

$$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} \cdot \frac{3166^3 - 3163^3}{3166 - 3163} = 2917 \cdot 3433;$$

$$N_{iii}' = \frac{3166^3 + 3163^3}{3166 + 3133} = 7 \cdot 73 \cdot 19397;$$

The form $\frac{1}{3}N_{iii}$,—owing to its variant form $\frac{1}{3}N_{iii} = (Y^3 - 1) \div (Y - 1)$—see (43*a*)—can be completely factorised up to the limit $x = 10002$ by aid of certain special Congruence-Tables of $(\eta^3 - 1) \div (\eta - 1) \equiv 0 \pmod{p}$,—(quoted below*)—and in very numerous cases up to much higher limits. Here follows an Example of each kind :—

1°. $$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} \cdot \frac{93979^3 - 93976^3}{93979 - 93976} = \frac{93977^3 - 1^3}{93977 - 1} = 7.12721.99181;$$

2°. $$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} \cdot \frac{792920^3 - 792917^3}{792920 - 792917} = \frac{792918^3 - 1^3}{792918 - 1} = 13\,99829.484459;$$

The form N_{iii}' may often be reduced within the power of the large Factor-Tables for values of $x' > 3166$ by aid of the Congruence Tables of $(x'^3 + y'^3) \div (x' + y') \equiv 0 \pmod{p}$ at end of this Memoir (when factors < 1000 exist.

Ex. $$N_{iii}' = \frac{99785^3 + 99782^3}{99785 + 99782} = \frac{99785^3 + 3^3}{99785 + 3} = 37 \cdot 43 \cdot 997 \cdot 6277;$$

27. *Simplest Factorisable* $\frac{1}{3}N_{iii}$. The form $\frac{1}{3}N_{iii}$ has one obviously factorisable Case :—

In formula (43*a*) take $Y = \eta^2$, giving $x = \eta^2 + 2$, $y = \eta^2 - 1$ ....(49)

Then—by (43*a*)—

$$\tfrac{1}{3}N_{iii} = \frac{\eta^6 - 1}{\eta^2 - 1} = \frac{\eta^3 - 1}{\eta - 1} \cdot \frac{\eta^3 + 1}{\eta + 1} = L.M, \text{ (suppose)} \quad \text{...........(50)}.$$

Here—(by 43)—L, M are seen to be

$$L = \tfrac{1}{3} \cdot \frac{(\eta + 1)^3 - (\eta - 2)^3}{(\eta - 1) - (\eta - 2)}, \quad M = \tfrac{1}{3} \cdot \frac{(\eta + 2)^3 - (\eta - 1)^3}{(\eta + 2) - (\eta - 1)} \quad \text{..........(50}a\text{)},$$

so that here L, M are *consecutive* members of the $\frac{1}{3}N_{iii}$ series.

Hence η must be of form $\eta = 3\rho$, and using suffixes of N to denote the value of the element x in $\frac{1}{3}N_{iii}$,

$$\tfrac{1}{3}N_{\eta^2+2} = \tfrac{1}{3} \cdot \frac{(\eta^2 + 2)^3 - (\eta^2 - 1)^3}{(\eta + 1) - (\eta^2 - 2)} = L.M = \tfrac{1}{3}N_{\eta+1}\tfrac{1}{3}N_{\eta+2} \quad \text{.....(50}b\text{)},$$

or, in words—

The product of every pair of consecutive $\frac{1}{3}N_{iii}$—(with $x = 3\rho + 1$, $3\rho + 2$)—is a member of the same series with $x = (3\rho)^2 + 2$ ....(50*c'*),

and the series of $\frac{1}{3}N_{9\rho^2+2}$ contains the whole of the members of the $\frac{1}{3}N_{iii}$ series in order, and without repetition ....(50*d*).

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* Tables of Least Roots (η, η') of $(\eta^3 \mp 1) \div (\eta \mp 1) \equiv 0$ (mod p & p^κ) extending to p & $p^\kappa \ngtr 10^4$ compiled by the present author, contained in vol i of his *Binomial Factorisations*, now at press.

The Table shows examples of the result (50*d*): here, for shortness—(x, y) denotes $\frac{1}{3}(x^3 - y^3) \div (x - y)$.

η	0	3	6	9	12	15
(x, y)	(2, 1)	(11, 8)	(38, 35)	(83, 80)	(146, 143)	(227, 224)
$L:M$	1:1;	7:13;	31:43;	73:7.13;	7.19:157;	211:241;
$(x_1, y_1) \cdot (x_2, y_2)$	(1,2)(2,1)	(4,1)(5,2)	(7,4)(8,5)	(10,7)(11,8)	(13,10)(14,11)	(16,13)(17,14)

The highest number of this kind completely factorisable by the larger Factor-Tables is given by $\eta = 3162$, viz.

$$\tfrac{1}{3}N_{3162^2+2} = \tfrac{1}{3}N_{3163} \cdot \tfrac{1}{3}N_{3164} = 7.19.223.337:13\,769339;$$

Special cases of much higher numbers can be formed by aid of the Congruence-Tables quoted in Art. 26.

Ex. Take* $\eta = 110^5$; $\frac{1}{3} \cdot \frac{(110^{10}+2)^3 - (110^{10}-1)^3}{(110^{10}+2) - (110^{10}-1)} = \frac{110^{15}-1}{110^5-1} \cdot \frac{110^{15}+1}{110^5+1}$

$$= 1211;541.12421.21031.150301:7.571;31.61.4021.2844761401;$$

28. *Perfect powers in $\frac{1}{3}N_{iii}$ & N_{iii}', and in their products.*

$$N_{iii}' = \frac{8^3+5^3}{8+5} = 7^2 \text{ is the } only\ square.$$

$$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} \cdot \frac{20^3 - 17^3}{20-17} = 7^3 \text{ is the } only\ cube.$$

$\frac{1}{3}N_{iii}$ & N_{iii}' have no higher powers.

But $\quad \frac{1}{3} \cdot \frac{4^3-1^3}{4-1} \cdot \frac{1}{3} \cdot \frac{20^3-17^3}{20-17} = 7^4; \quad \frac{8^3+5^3}{8+5} \cdot \frac{1}{3} \cdot \frac{20^3-17^3}{20-17} = 7^5.$

Similarly higher powers of 7 may be formed of products of $\frac{1}{3}N_{iii}$ & N_{iii}'.

Also by Result (50*b*) it is seen that

$$\tfrac{1}{3}N_{\eta^2+2} \cdot \tfrac{1}{3}N_{\eta+1} \cdot \tfrac{1}{3}N_{\eta+2} = (\tfrac{1}{3}N_{\eta^2+2})^2.$$

[Further Examples will be found among Aurifeuillians, Art. 34].

29. *Aurifeuillian, &c., forms of $\frac{1}{3}N_{iii}$.* With help of the forms (43*a*) it will be found that $\frac{1}{3}N_{iii}$ yields Aurifeuillians and Ant-Aurifeuillians of *two* kinds.

30. Case 1°. Take $Y = 3\eta^2$; then Result (43*a*) gives

$$\tfrac{1}{3}N_{iii} = \frac{Y^3 - 1^3}{Y - 1} = \frac{(3\eta^2)^3 - 1^3}{3\eta^2 - 1} = 9\eta^4 + 3\eta^2 + 1, \text{ [a } Trin\text{-}Ant\text{-}Aurifeuillian\text{]} \ldots (51),$$

$$= (3\eta^2 - 1)^2 + (3\eta)^2 = P'^2 + Q'^2 = \text{A}' \ldots\ldots\ldots\ldots\ldots\ldots (51a).$$

30*a*. Case 1°*a*. Take $X = 3\xi^2$; then Result (43*a*) gives

$$\tfrac{1}{3}N_{iii} = \frac{X^3 + 1^3}{X + 1} = \frac{(3\xi^2)^3 + 1^3}{3\xi^2 + 1} = 9\xi^4 - 3\xi^2 + 1, \text{ [a } Trin\text{-}Aurifeuillian \ldots\ldots (52),$$

$$= (3\xi^2 - 3\xi + 1)(3\xi^2 + 3\xi + 1) = L.M = \text{A} \ldots\ldots\ldots\ldots\ldots (52a).$$

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* This example is taken from the Author's MS. Factorisation-Tables of $(\eta^{15} \pm 1)$; it has 41 figures.

Now take $\xi_r = 1, 2, 3, \ldots, r$, in succession, giving

$$\xi_{r+1} = \xi_r + 1; \quad N_r = L_r . M_r, \quad N_{r+1} = L_{r+1} . M_{r+1} \ldots\ldots\ldots\ldots(53).$$

Hence $M_r = 3\xi_r^2 + 3\xi_r + 1 = 3\xi_{r+1}^2 - 3\xi_{r+1} + 1 = L_{r+1}$ *always*............(53*a*),

showing that—

The series N_r—(given by $\xi_r = 1, 2, 3, \ldots, r$)—is *in chain*, [$M_r = L_{r+1}$]...(54).

30*b*. *Factorisation of Case* 1°*a*. Table A gives the factorisation of this Case—[with $x = X + 1$, $X = 3\xi^2$]—showing ξ, x, y, z, and the Aurifeuillian Factors L, M resulting up to $\xi = 20$. It will be seen that the series of $\frac{1}{3}N_{iii}$ is *in chain* [$M_r = L_{r+1}$ throughout].

The highest number ($\frac{1}{3}N_{iii}$) factorisable by the large Factor-Tables is given by $\xi = 1825$.

$$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} . \frac{(3.1825^2+1)^3 - (3.1825^2-2)^3}{(3.1825+1) - (3.1825-2)} = \frac{(3.1825^2)^3 + 1}{3.1825^2 + 1}$$

$$= (3.1825^2 - 3.1825 + 1)(3.1825^2 + 3.1825^2 + 1)$$

$$= 1021.9781 : 7.13.61.1801;$$

The highest number certainly within the powers of the Author's Congruence Tables of $(\eta^3+1) \div (\eta+1) \equiv 0 \pmod{p}$,—[see Art. 26]—is given by $\xi = 57734$; but the labor would be considerable (from the great number of trial divisors).

Ex. Take $\xi = 2^{13}$;

$$\tfrac{1}{3}N_{iii} = \tfrac{1}{3} . \frac{24577^3 - 24574^3}{24777 - 24574} = \frac{24576^3 + 1^3}{24576 + 1}, \text{ [has 18 figures]}$$

$$= (3.2^{26} - 3.2^{13} + 1)(3.2^{26} + 3.2^{13} + 1) = 7.19.547.2767 : 31.307.21157$$

Trin-Aurifeuillian Sub-Cubans. TAB. A

$\frac{1}{3}N_{iii} = \frac{1}{3}(x^3 - y^3) \div (x - y) = (X^3 + 1^3) \div (X + 1)$

$x - y = 3, \quad X = 3\xi^2, \quad x = X + 1; \quad \frac{1}{3}N_{iii} = L.M.$

ξ	x	y	X	$L : M$
1	4,	1	3	1 : 7 ;
2	13,	10	12	7 : 19 ;
3	28,	25	27	19 : 37 ;
4	49,	46	48	37 : 61 ;
5	76,	73	75	61 : 7.13 ;
6	109,	106	108	7.13 : 127 ;
7	148,	145	147	127 : 13.13 ;
8	193,	190	192	13.13 : 7.31 ;
9	244,	241	243	7.31 : 271 ;
10	301,	298	300	271 : 331 ;
11	364,	361	363	331 : 397 ;
12	433,	430	432	397 : 7.67 ;
13	508,	505	507	7.67 : 547 ;
14	589,	586	588	547 : 631 ;
15	676,	673	675	631 : 7.103 ;
16	769,	766	768	7.103 : 19.43 ;
17	868,	866	867	19.43 : 919 ;
18	973,	970	972	919 : 13.79 ;
19	1084,	1081	1083	13.79 : 7.163 ;
20	1201,	1198	1200	7.163 : 13.97 ;

30c. *Perfect square products.* Using a subscript r to denote the value of ξ in the Aurifeuillian Series (Case 1°a), take $r = 1, 2, 3, \ldots, r$ in succession. Thus

$$\Pi\left(\frac{N_r}{3}\right) = \frac{N_1, N_2, N_3, \ldots, N_r}{3 \,.\, 3 \,.\, 3 \ldots 3} = L_1 M_1 \,.\, L_2 M_2 \,.\, L_3 M_3, \ldots, L_r M_r$$

$$= (L_1 L_2 L_3 L_4 \ldots L_r)^2 . M_r, \text{ (since the series is } \textit{in chain}) \ldots (55a),$$

$$= (L_2 L_3 L_4 \ldots L_r)^2 . M_r, \text{ (since } N_1 = 1:7, \text{ giving } L_1 = 1) \ldots (55b).$$

Now $M_r = 3\xi_r^2 + 3\xi_r + 1 = z^2$ suppose,

where $\quad (2z)^2 - 3(2\xi_r + 1)^2 = +1$ ....(56).

Comparing this with the solutions (τ, υ) of the Pellian Equation $\tau^2 - 3\upsilon^2 = +1$, gives

$$\xi_r = \tfrac{1}{2}(\upsilon - 1), \quad z = \tfrac{1}{2}\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56a).$$

Every solution (τ, υ) of the Pellian with τ *even* and υ *odd* gives a suitable value of ξ_r; $X_r = 3\xi_r^2$, $x_r = X_r + 1$, $y_r = x_r - 3$. The Table below shows the values of ξ_r, z_r, x_r, y_r arising from $\tau = \epsilon$, $\upsilon = \omega$, giving $M_r = z^2$, and $\pi(N_r) = \square$.

$\tau,\ \upsilon$	2, 1	26,15	362,209	5042,2911
$z_r,\ \xi_r$	1, 0	13,7	181,104	2521,1005
$x_r,\ y_r$	1, 2	148,145	32449,32446	$3.1005^2+1, 3.1005^2-2$

Ex. Take $r = \xi_r = 7$. The symbol (x, y) is here used to denote $\frac{1}{3}N_{\omega}$.

$$\Pi(\tfrac{1}{3}N_7) = \frac{N_1, N_2, N_3, \ldots, N_7}{3 \,.\, 3 \,.\, 3 \ldots 3} = (4, 1)(13, 10)(28, 25)(49, 46)(76, 73)(109, 106)(148, 145)$$

$$= (1.7.19.37.61.7.13.127\ 13)^2.$$

31. Case 2° (of $\frac{1}{3}N_{\omega}$). Take $X = \xi^2$, $Y = 3\eta^2$, whence

$$x = \xi^2 + 1, \quad y = 3\eta^2 - 1, \quad x - y = 3, \quad X - Y = \xi^2 - 3\eta^2 = 1 \ldots\ldots\ldots (57).$$

Formulæ 43a give

$$\tfrac{1}{3}N_{\omega} = \frac{(3\eta^2)^3 - 1^3}{3\eta^2 - 1} = \frac{(\xi^2)^3 + 1^3}{\xi^2 + 1} = \frac{(\xi^2)^3 + (3\eta^2)^3}{\xi^2 + 3\eta^2}$$

$$= 9\eta^4 + 3\eta^2 + 1 = \xi^4 - \xi^2 + 1 = \xi^4 - 3\xi^2\eta^2 + 9\eta^4 \ldots\ldots\ldots\ldots\ldots\ldots (58a),$$

$$= (3\eta^2 - 1)^2 + (3\eta)^2 = \ldots = (\xi^2 + 3\eta^2)^2 - (3\eta)^2 \ldots\ldots\ldots\ldots\ldots\ldots (58b),$$

$$= \mathsf{A}' \ldots\ldots\ldots\ldots\ldots\ldots = \mathsf{A} = LM \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58c);$$

showing that this $\frac{1}{3}N_{\omega}$ is *both an Aut-Aurifn. and an Aurifn.*

Here $\qquad L = \xi^2 - 3\xi\eta + 3\eta^2, \quad M = \xi^2 + 3\xi\eta + 3\eta^2 \ldots\ldots\ldots\ldots\ldots (59).$

Now take ξ_r, η_r successive terms of the Pellian equation $\xi^2 - 3\eta^2 = +1$, giving

$$\ldots N_r = L_r . M_r, \qquad N_{r+1} = L_{r+1} . M_{r+1} \ldots$$

Here $\xi_{r+1} = 2\xi_r + 3\eta_r$, $\eta_{r+1} = \xi_r + 2\eta_r$ ....(60),

$M_r = \xi_r^2 + 3\xi_r\eta_r + 3\eta_r^2$, $L_{r+1} = \xi_{r+1}^2 - 3\xi_{r+1}\eta_{r+1} + 3\eta_r^2$......(60*a*),

and hence, by (57 to 59) $M_r = L_{r+1}$, *always* ....(60*b*),

showing that this series of N_{iii} is *in chain*.

31*b*. *Factorisation of* $\frac{1}{3}N_{iii}$. *Case* 2°. The Table below shows the successive elements (ξ_r, η_r) of the Pellian equation $\xi^2 - 3\eta^2 = +1$, with the values of x, y, X, Y thereby given, and finally the Aurifeuillian Factors (L_r, M_r) of the successive $\frac{1}{3}N_{iii}$.

r	**0**	**1**	**2**	**3**	**4**	**5**
ξ, η	1,0	2,1	7,4	26,15	97,56	362,209
x, y	2,1	5,2	50,47	677,674	9410,9407	131045,131042
X, Y	1,0	4,3	49,48	676,675	9409,9408	131044,131043
L,M	1:1;	1:13;	13:181;	181:2521;	2521:13 37·73;	13·37·73:489061;

r	**6**	**7**	**8**
ξ, η	1351,780	5042,2911	18817,10864
x, y	1825202,1825199	$5042^2+1, 3.2911^2-1$	$18817^2+1, 3.10864^2-1$
X, Y	1825201,1825200	$5042^2, 3.2911^2$	$18817^2, 3.10864^2$
L,M	489061:6811741;	6811741:13 181;61.661;	13.181;61.661:1321442641?†

32. *Aurifeuillian, &c., forms of* N_{iii}'. With the help of the formulæ (43*b*) it will be found that N_{iii}' yields Aurifeuillians and Ant-Aurifeuillians of *one* kind.

Case 3°. Take $x = \eta^2$; then Result (43*b*) gives

$$N_{iii}' = \frac{y^3 - 3^3}{y - 3} = \frac{\eta^6 - 3^3}{\eta^2 - 3} = \eta^4 + 3\eta^2 + 9, \text{ [}a\ Trin.\ Ant\text{-}Aurifeuillian\text{]}...(61a),$$

$$= (\eta^2 - 3)^2 + (3\eta)^2 = P'^2 + Q'^2 = A' \text{ ....}(61b).$$

Case 3*a*°. Take $x = \xi^2$; then Result (43*b*) gives

$$N_{iii}' = \frac{x^3 + 3^3}{x + 3} = \frac{\xi^6 + 3^3}{\xi^2 + 3} = \xi^4 - 3\xi^2 + 9, \text{ [}a\ Trin\text{-}Aurifeuillian\text{]}.........(62a),$$

$$= (\xi^2 - 3\xi + 3)(\xi^2 + 3\xi + 3) = L.M = A \text{ ....}(62b).$$

Now form *two* series of ξ_r, increasing by 3, with two series of $N_r' = A$ corresponding,

Series 1°. $\xi_r = 1, 4, 7, 10, \ldots, r = 3\rho + 1$; $N_1', N_4', N_7', \ldots, N_r' = L_r.M_r$...(63*a*).

Series 2°. $\xi_r = 2, 5, 8, 11, \ldots, r = 3\rho + 2$; $N_2', N_5', N_8', \ldots, N_r' = L_r.M_r$...(63*b*).

Then—in each series—$N_r = L_r M_r$, $N_{r+3} = L_{r+3}.M_{r+3}$ ....(63*c*),

$M_r = \xi_r^2 + 3\xi_r + 3 = (\xi_r + 3)^2 - 3(\xi_r + 3) + 3 = L_{r+3}$, *always*......(63*d*),

showing that each series of N_{iii}' is *in chain*.

32b. *Factorisation of Case 3°,* Table B gives the factorisation of this Case—(with $x = \xi^2$)—shewing ξ, x, y and the Aurifeuillian Factors L, M of N_{iii}' up to $\xi = 25$, and also the Sub-Cuban element (line x, y) of L, M. It will be seen that both series of N_{iii}—{with ξ as in (63a, b)}—are *in chain.* ($M_r = L_{r+1}$ throughout).

The highest number N_{iii}' factorisable by the large Factor-Tables is given by $\xi = 3163$, $x = 3163^2$;

$$N_{iii}' = \frac{(3163^2)^3 + (3163^2-3)^3}{3163^2 + (3163^2-3)} = \frac{(3163^2)^3 + 3^3}{3163^2+3} = \frac{3161^3 - 1^3}{3161-1} : \frac{3164^3-1^3}{3164-1} = L.M$$

$= 7.19.223.337:2917.3433$; (14 figures).

Trin-Aurifeuillian Sub-Cubans. TAB. B

$N_{iii}' = (x^3+y^3) \div (x+y) = (x^3+3^3) \div (x+3)$

$x - y = 3$, $x = \xi^2$; $N_{iii}' = L \, . \, M$.

ξ	x, y	L : M	L	M
1	1, 2	1 : 7 ;	2, 1	4, 1
2	4, 1	1 : 13 ;	2, 1	5, 4
4	16, 13	7 : 31 :	4, 1	7, 4
5	25, 22	13 : 43 ;	5, 2	8, 5
7	49, 46	31 : 73 ;	7, 4	10, 7
8	64, 61	43 : 7.13 ;	8, 5	11, 8
10	100, 97	73 : 7.19 ;	10, 7	13, 10
11	121, 118	7.13 : 157 ;	11, 8	14, 11
13	169, 166	7.19 : 211 ;	13, 10	16, 13
14	196, 193	157 : 241 ;	14, 11	17, 14
16	256, 253	211 : 307 ;	16, 13	19, 16
17	289, 286	241 : 343 ;	17, 14	20, 17
19	361, 358	307 : 421 ;	19, 16	22, 19
20	400, 397	343 : 463 ;	20, 17	23, 20
22	484, 481	421 : 7.79 ;	22, 19	25, 22
23	529, 526	463 : 601 ;	23, 20	26, 23
25	625, 622	7.79 : 19.37 ;	25, 22	28, 25

33. *Connexion of the $\frac{1}{3}N_{iii}$, N_{iii}' series.* In the $\frac{1}{3}N_{iii}$ series take the $x_r = \xi_r$; and in the N_{iii}' series take the $x_r = \xi_r^2$, so that $N_r' = \mathbf{A}$ of Art. 19. Then, by (43a, b)

$\frac{1}{3}N_r = \xi_r^2 - 3\xi_r + 3$ = the L_r of N_r' ....(64),

$\frac{1}{3}N_{r+3} = \xi_{r+3}^2 - 3\xi_{r+3} + 3$, [here $\xi_{r+3} = \xi_r + 3$]..............(65a),

$= \xi_r^2 + 3\xi_r + 3$ = the M_r of N_r'........................(65b),

where $\frac{1}{3}N_r \cdot \frac{1}{3}N_{r+3} = L_r \cdot M_r = N_r'$, *always* ....(66).

Now arrange the ξ_r, $\frac{1}{3}N_r$, N_r', each in *two* Series as in Art. 32. Then from it is seen that *in each of the Series* 1°, 2°.

Every pair of adjacent members $(\frac{1}{3}N_r, \frac{1}{3}N_{r+3})$ of the $\frac{1}{3}N_{iii}$ series are the L, M_r of the corresponding number of the $N_r' = \mathbf{A}$ series ....(67a).

The product of every such pair $(\frac{1}{3}N_r, \frac{1}{3}N_{r+3})$ = the corresponding N_r'.(67b).

The complete N_{iii}' series is made up wholly out of the $\frac{1}{3}N_{iii}$ series, and contains the whole of the members thereof twice over ... .(67c).

34. *Perfect square products.* Result (64) shows that, taking adjacent members of either Series of $\frac{1}{3}N_{iii}$ with the corresponding N_{iii}'

$$\tfrac{1}{3}N_r \cdot \tfrac{1}{3}N_{r+3} \cdot N_r' = (\tfrac{1}{3}N_r \cdot \tfrac{1}{3}N_{r+3})^2 = N_r'^2 \quad \text{.................(68).}$$

Also, since each series of N_{iii}' is *in chain*, and since $N_1' = 1{:}7$ and $N_2' = 1{:}13$, it follows that the continued product of either series taken along with the last $M_r = \frac{1}{3}N_{r+3}$ is a perfect square.

$$(N_1', N_4', N_7', \dots, N_r') \cdot \tfrac{1}{3}N_{r+3} = \left(\frac{N_1}{3} \cdot \frac{N_4}{3} \cdot \frac{N_7}{3} \cdots \frac{N_r}{3}\right)^2, \; [r=3\rho+1] \text{..(68a)},$$

$$(N_2', N_5', N_8', \dots, N_r') \cdot \tfrac{1}{3}N_{r+3} = \left(\frac{N_2}{3} \cdot \frac{N_5}{3} \cdot \frac{N_8}{3} \cdots \frac{N_r}{3}\right)^2, \; [r=3\rho+2] \text{..(68b)}.$$

Errata in the previous Paper. Vol. xlix. 1919.

page	*Tab.*	*p.*	*Col.*	*For*	*Read*
31	C3	619	x, x	409, 201	291, 329
31	C3	877	x, x	481	491
33	C7	211	x', x	83	93

TAB. C3.

Least Roots (x, x') *of* $(x^3 \mp y^3) \div (x \mp y) \equiv 0$ $(mod\ p\ \&\ p^\kappa)$.

$[(x-y)=(x'-y')=3]$.

x	x	p	x'	x'	x	x	p	x'	x'
.	.	3	.	6	234,	256	487	212,	278
4,	6	7	1,	2	141,	361	499	82,	420
5,	11	13	4,	12	62,	464	523	183,	343
9,	13	19	5,	17	131,	413	541	154,	390
7,	27	31	16,	18	42,	508	547	123,	427
12,	28	37	7,	33	111,	463	571	244,	330
8,	38	43	21,	25	215,	365	577	65,	515
15,	49	61	22,	42	26,	578	601	75,	529
31,	39	67	23,	47	212,	398	607	26,	584
10,	66	73	27,	49	67,	549	613	198,	418
25,	57	79	10,	72	254,	368	619	140,	482
37,	63	97	11,	89	45,	589	631	132,	502
48,	58	103	38,	68	179,	467	643	112,	534
47,	65	109	29,	83	298,	366	661	230,	434
21,	109	127	60,	70	257,	419	673	95,	581
44,	98	139	13,	129	255,	439	691	71,	623
34,	120	151	55,	99	229,	483	709	28,	684
14,	146	157	39,	121	283,	447	727	119,	611
60,	106	163	14,	152	309,	427	733	191,	545
59,	134	181	37,	147	322,	420	739	518,	663
86,	110	193	62,	134	74,	680	751	219,	535
94,	108	199	80,	122	29,	731	757	84,	676
16,	198	211	45,	169	362,	410	769	314,	458
41,	185	223	106,	120	381,	409	787	353,	437
96,	136	229	56,	176	132,	682	811	393,	421
17,	227	241	48,	196	176,	650	823	301,	525
30,	244	271	87,	187	127,	705	829	378,	454
118,	162	277	74,	206	222,	634	853	193,	663
46,	240	283	135,	151	262,	600	859	79,	783
19,	291	307	54,	256	284,	596	877	31,	849
100,	216	313	19,	297	339,	547	883	131,	755
33,	301	331	96,	238	386,	524	907	248,	662
130,	210	337	50,	290	54,	868	919	159,	763
124,	228	349	20,	332	324,	616	937	32,	908
85,	285	367	118,	252	144,	826	967	541,	729
90,	286	373	109,	267	115,	879	991	342,	652
53,	329	379	156,	226	306,	694	997	85,	915
36,	364	397	105,	295					
55,	357	409	162,	259			p^κ		
22,	402	421	63,	361	20,	32	49	8,	44
200,	236	433	164,	272	20,	326	343	57,	289
173,	269	439	77,	365	24,	148	169	69,	113
135,	325	457	58,	402	70,	294	361	157,	207
23,	443	463	66,	400	441,	523	961	359,	605

TAB. C5.

Least Roots (x, x') *of* $(x^5 \mp y^5) \div (x \mp y) \equiv 0$ $(mod\ p$ & $p^\kappa)$.

$[x - y = x' - y' = 5]$.

x	x	x	x	p	x'	x'	x'	x'
2,	3,	7,	9	11	1,	4,	6,	10
10,	17,	19,	26	31	1,	4,	12,	24
1,	4,	19,	27	41	12,	22,	24,	34
7,	19,	47,	59	61	28,	31,	35,	38
20,	24,	52,	56	71	13,	16,	60,	63
34,	44,	62,	72	101	6,	52,	54,	100
16,	58,	78,	120	131	36,	51,	85,	100
64,	65,	91,	92	151	38,	63,	93,	118
28,	89,	97,	158	181	20,	38,	148,	166
25,	76,	120,	171	191	24,	31,	165,	172
9,	15,	201,	207	211	2,	3,	34,	182
70,	72,	174,	176	241	39,	74,	172,	207
61,	79,	177,	195	251	60,	97,	159,	196
16,	126,	150,	260	271	13,	99,	177,	263
24,	33,	253,	262	281	42,	76,	210,	244
6,	94,	222,	310	311	101,	134,	182,	215
78,	105,	231,	258	331	51,	53,	283,	285
67,	79,	327,	339	401	55,	186,	220,	351
161,	192,	234,	265	421	88,	203,	223,	338
80,	83,	353,	356	431	86,	198,	238,	350
54,	196,	270,	412	461	25,	93,	373,	441
89,	226,	270,	407	491	45,	53,	443,	451
91,	244,	282,	435	521	105,	129,	397,	421
115,	179,	367,	431	541	142,	243,	303,	404
113,	141,	435,	463	571	85,	186,	390,	491
58,	240,	366,	548	601	73,	129,	477,	533
89,	189,	447,	547	631	16,	174,	462,	620
45,	132,	514,	601	641	47,	304,	342,	599
204,	223,	443,	462	661	40,	242,	424,	626
70,	327,	369,	626	691	28,	192,	504,	668
109,	252,	454,	597	701	39,	98,	608,	667
78,	95,	661,	678	751	112,	246,	510,	644
219,	283,	483,	547	761	50,	347,	419,	716
65,	196,	620,	751	811	321,	327,	489,	495
35,	82,	744,	791	821	129,	147,	679,	697
170,	415,	471,	716	881	136,	226,	660,	750
291,	354,	562,	625	911	83,	228,	688,	833
197,	435,	511,	749	941	46,	121,	825,	900
263,	362,	614,	713	971	93,	461,	515,	883
273,	349,	647,	723	991	30,	197,	799,	966
				p^κ				
51,	58,	68,	75	121	34,	61,	65,	92
72,	141,	825,	894	961	210,	280,	686,	756

TAB. C7.

Least Roots (x, x') *of* $(x^7 \mp y^7) \div (x \mp y) \equiv 0$ (*mod* p & p^κ).

$[x - y = x' - y' = 7]$.

x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'
1,	6,	13,	15,	21,	23	29	10,	16,	17,	19,	20,	26
11,	12,	19,	31,	38,	39	43	8,	10,	14,	36,	40,	42
12,	21,	37,	41,	57,	66	71	17,	24,	3[illegible],	48,	54,	61
15,	24,	54,	66,	96,	105	113	25,	27,	47,	73,	93,	95
8,	14,	40,	94,	120,	126	127	29,	31,	47,	87,	103,	105
1,	6,	86,	89,	115,	118	197	38,	84,	92,	112,	120,	166
10,	33,	48,	170,	185,	208	211	18,	46,	99,	119,	172,	200
12,	111,	114,	132,	135,	234	239	19,	115,	117,	129,	131,	227
91,	121,	126,	162,	167,	197	281	61,	116,	127,	161,	172,	227
8,	60,	82,	262,	284,	336	337	38,	51,	156,	188,	293,	306
3,	4,	16,	177,	269,	370	379	28,	48,	176,	210,	338,	358
73,	108,	205,	223,	320,	355	421	84,	90,	143,	285,	338,	344
25,	26,	76,	380,	430,	431	449	24,	121,	170,	286,	335,	432
87,	98,	188,	282,	372,	383	463	21,	109,	192,	278,	361,	419
132,	157,	197,	301,	341,	366	491	53,	59,	67,	431,	439,	445
137,	142,	213,	341,	412,	417	547	61,	147,	270,	284,	407,	493
144,	245,	303,	321,	379,	480	617	10,	22,	227,	397,	602,	614
40,	102,	117,	521,	536,	598	631	29,	87,	194,	444,	551,	609
90,	203,	239,	427,	463,	576	659	15,	254,	316,	350,	412,	651
203,	212,	304,	376,	468,	477	673	120,	194,	275,	405,	486,	560
60,	266,	280,	428,	442,	648	701	153,	208,	252,	456,	500,	555
227,	284,	308,	412,	466,	523	743	12,	45,	332,	418,	705,	732
72,	229,	367,	397,	535,	692	757	42,	91,	366,	398,	673,	722
29,	130,	276,	558,	704,	805	827	225,	231,	412,	422,	663,	609
72,	272,	431,	459,	618,	818	883	100,	113,	135,	755,	777,	790
52,	120,	140,	778,	798,	866	911	160,	237,	285,	633,	681,	758
156,	386,	394,	566,	574,	804	953	214,	261,	344,	616,	699,	746
10,	452,	454,	520,	522,	964	967	76,	439,	450,	524,	535,	898
						p^κ						
71,	204,	218,	630,	644,	777	841	229,	339,	365,	483,	509,	619

TAB. C9.

Least Solutions (x, x') *of* $(x^9 \mp y^9) \div (x^3 \mp y^3) \equiv 0$ (*mod* p & p^κ).

$[x - y = x' - y' = 9]$.

x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'
2,	3,	6,	7,	12,	16	19	4,	5,	10,	11,	17,	18
16,	17,	22,	24,	29,	30	37	12,	14,	15,	31,	32,	34
12,	14,	18,	64,	68,	70	73	3,	6,	22,	31,	51,	60
29,	41,	43,	75,	77,	89	109	12,	17,	39,	79,	101,	106
41,	54,	66,	70,	82,	95	127	28,	37,	42,	94,	99,	108
29,	35,	75,	97,	137,	143	163	2,	7,	36,	84,	88,	136
14,	48,	77,	113,	142,	176	181	26,	29,	50,	140,	161,	164
14,	72,	90,	118,	136,	194	199	44,	59,	99,	109,	149,	164
20,	101,	102,	178,	179,	260	271	67,	71,	103,	177,	209,	213
90,	112,	132,	184,	204,	226	307	48,	60,	134,	182,	256,	268
39,	55,	105,	283,	333,	349	379	71,	155,	174,	214,	233,	317
55,	116,	152,	254,	290,	351	397	15,	56,	80,	326,	350,	391
26,	29,	56,	386,	413,	416	433	23,	86,	105,	337,	356,	419
60,	85,	174,	322,	411,	436	487	13,	35,	97,	399,	461,	483
118,	179,	261,	271,	353,	414	523	87,	108,	126,	406,	424,	445
38,	70,	120,	430,	480,	512	541	102,	170,	187,	363,	380,	448
119,	171,	247,	339,	415,	467	577	27,	67,	95,	491,	519,	559
84,	185,	241,	381,	437,	538	613	129,	224,	278,	344,	398,	493
37,	50,	318,	322,	590,	603	631	33,	115,	231,	409,	525,	607
147,	221,	292,	456,	527,	601	739	2,	7,	168,	363,	385,	580
294,	350,	374,	392,	416,	472	757	196,	228,	332,	434,	538,	570
64,	143,	254,	566,	677,	756	811	257,	262,	289,	531,	558,	563
13,	113,	205,	633,	725,	825	829	96,	397,	416,	422,	441,	742
336,	417,	418,	474,	475,	556	883	255,	360,	392,	560,	532,	637
86,	247,	316,	612,	681,	842	919	84,	127,	382,	546,	801,	844
184,	343,	431,	515,	603,	762	937	244,	255,	268,	678,	691,	702
42,	233,	301,	699,	767,	958	991	165,	308,	440,	560,	692,	835
						p^κ						
92,	102,	120,	250,	268,	278	361	75,	112,	138,	232,	258,	295

TAB. C11.

Least Roots (x, x') *of* $\dfrac{x^{11} \mp y^{11}}{x \mp y} \equiv 0 \pmod{p \;\&\; p^\kappa}$.

[when $x - y = 11$].

x	x	x	x	x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'	x'	x'	x'	x'
1,	4,	5,	6,	7,	10,	12,	13,	21,	22	23	2,	3,	8,	9,	14,	15,	16,	18,	19,	20
4,	7,	13,	17,	25,	31,	47,	53,	61,	65	67	3,	8,	14,	28,	30,	32,	44,	48,	50,	64
14,	22,	23,	26,	38,	62,	74,	77,	78,	86	89	20,	21,	30,	37,	47,	53,	63,	70,	79,	80
48,	85,	93,	95,	96,	114,	115,	117,	125,	162	199	32,	68,	74,	97,	101,	109,	113,	136,	142,	178
33,	61,	85,	86,	155,	187,	256,	257,	281,	309	331	41,	53,	84,	116,	139,	203,	226,	258,	289,	301
5,	6,	31,	36,	146,	155,	209,	218,	328,	333	353	32,	47,	66,	158,	164,	200,	206,	298,	317,	332
92,	104,	161,	197,	200,	208,	211,	247,	304,	316	397	24,	112,	116,	122,	147,	261,	286,	292,	296,	384
5,	6,	34,	36,	134,	182,	248,	296,	394,	396	419	66,	77,	100,	137,	181,	249,	293,	330,	353,	364
73,	111,	119,	133,	197,	277,	341,	355,	363,	401	463	13,	25,	155,	174,	199,	275,	300,	319,	449,	461
24,	41,	109,	259,	267,	361,	368,	429,	587,	604	617	135,	222,	251,	254,	270,	358,	374,	377,	406,	493
24,	95,	168,	256,	317,	355,	416,	504,	577,	648	661	25,	30,	76,	293,	325,	347,	379,	596,	644,	647
153,	224,	228,	235,	318,	376,	459,	466,	470,	541	683	22,	81,	99,	242,	282,	412,	452,	598,	613,	672
114,	129,	132,	195,	214,	524,	543,	606,	609,	624	727	44,	227,	314,	326,	356,	382,	412,	424,	511,	694
21,	129,	354,	357,	415,	455,	513,	516,	741,	849	859	44,	146,	208,	317,	385,	485,	533,	662,	724,	826
191,	219,	227,	265,	412,	480,	627,	665,	673,	701	881	37,	274,	294,	311,	333,	559,	581,	598,	618,	855
63,	90,	160,	240,	458,	500,	718,	798,	868,	895	917	230,	275,	282,	331,	426,	532,	627,	676,	683,	728
20,	190,	216,	253,	398,	604,	749,	786,	812,	982	991	242,	366,	380,	422,	486,	516,	580,	622,	633,	760
										p^κ										
33,	97,	127,	234,	266,	274,	306,	413,	443,	507	529	31,	89,	179,	199,	216,	324,	341,	361,	451,	509

TAB. C15.

Least Roots (x, x') *of* $(x^{15} \mp y^{15})(x \mp y) \div (x^5 \mp y^5)(x^3 \mp y^3) \equiv 0$ (*mod* p & p^2).

$[x - y = 15]$.

x	x	x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'	x'	x'
2,	6,	9,	13,	19,	21,	25,	27	31	1,	7,	8,	14,	17,	22,	24,	29
3,	7,	8,	12,	16,	33,	43,	60	61	17,	20,	26,	34,	42,	50,	56,	59
16,	20,	30,	71,	95,	136,	146,	150	151	3,	5,	10,	12,	23,	50,	116,	143
22,	42,	64,	76,	120,	132,	154,	174	181	7,	8,	17,	91,	93,	103,	105,	179
51,	52,	57,	59,	167,	169,	174,	175	211	47,	67,	68,	87,	139,	158,	159,	179
6,	9,	27,	89,	124,	132,	167,	229	241	45,	66,	97,	127,	129,	159,	190,	211
32,	66,	79,	82,	204,	207,	220,	254	271	61,	71,	128,	137,	149,	158,	215,	225
5,	10,	16,	20,	54,	292,	326,	330	331	3,	12,	30,	49,	92,	254,	297,	316
31,	35,	47,	111,	325,	389,	401,	405	421	82,	102,	190,	198,	238,	246,	334,	354
70,	84,	128,	169,	387,	428,	472,	486	541	44,	210,	223,	231,	325,	333,	346,	512
24,	41,	218,	277,	309,	368,	545,	562	571	56,	74,	127,	151,	435,	459,	512,	530
60,	206,	216,	228,	388,	400,	410,	556	601	111,	204,	250,	264,	352,	366,	412,	505
88,	150,	226,	245,	401,	420,	496,	558	631	65,	207,	212,	306,	340,	434,	439,	581
145,	200,	239,	267,	409,	437,	476,	531	661	23,	52,	68,	334,	342,	608,	624,	653
83,	129,	157,	350,	356,	549,	577,	623	691	135,	199,	306,	324,	382,	400,	507,	571
150,	158,	214,	300,	466,	552,	608,	616	751	166,	181,	195,	380,	386,	571,	585,	600
1,	14,	[illegible],	192,	327,	499,	634,	807	811	186,	205,	350,	365,	461,	476,	621,	640
86,	233,	283,	411,	595,	723,	773,	920	991	49,	126,	155,	469,	537,	851,	880,	957
106,	112,	316,	397,	579,	660,	864,	870	p^2 961	53,	91,	100,	107,	869,	876,	885,	923

Factorisation of

Tab. F3a.

$\frac{1}{3}N = \frac{1}{3}(x^3 - y^3) \div (x - y)$.

$[x - y = 3]$.

x, y	$\frac{1}{3}N$	A, B
1, −2	1;	1, 0
2, −1	1;	1, 0
4, 1	1:7;	2, 1
5, 2	13;	1, 2
7, 4	31;	2, 3
8, 5	43;	4, 3
10, 7	73;	5, 4
11, 8	7:13;	4, 5
13, 10	7:19;	5, 6
14, 11	157;	7, 6
16, 13	211;	8, 7
17, 14	241;	7, 8
19, 16	307;	8, 9
20, 17	343;	10, 9
22, 19	421;	11, 10
23, 20	463;	10, 11
25, 22	7.79;	11, 12
26, 23	601;	13, 12
28, 25	19:37;	14, 13
29, 26	757;	13, 14
31, 28	13.67;	14, 15
32, 29	49.19;	16, 15
34, 31	7.151;	17, 16
35, 32	1123;	16, 17
37, 34	13.97;	17, 18
38, 35	31:43;	19, 18
40, 37	1483;	20, 19
41, 38	7.223;	19, 20
43, 40	1723;	20, 21
44, 41	13.139;	22, 21
46, 43	7.283;	23, 22
47, 44	19.109;	22, 23
49, 46	37:61;	23, 24
50, 47	13.181;	25, 24
52, 49	2551;	26, 25
53, 50	7.379;	25, 26
55, 52	7.409;	26, 27
56, 53	2971;	28, 27
58, 55	31.103;	29, 28
59, 56	3307;	28, 29
61, 58	3541;	29, 30
62, 59	7.523;	31, 30
64, 61	3907;	32, 31
65, 62	37.109;	31, 32
67, 64	7.613;	32, 33
68, 65	4423;	34, 33
70, 67	13.19.19;	35, 34
71, 68	4831;	34, 35
73, 70	5113;	35, 36
74, 71	7.751;	37, 36

Tab. F3b.

$N' = (x'^3 + y'^3) \div (x' + y')$.

$[x - y = 3]$.

x, y	N'	A, B
1, −2	7;	$\bar{2}$, 1
2, −1	1:7;	$\bar{2}$, 1
4, 1	13;	1, $\bar{2}$
5, 2	19;	4, 1
7, 4	37;	5, 2
8, 5	49;	1, 4
10, 7	79;	2, 5
11, 8	97;	7, 4
13, 10	139;	8, 5
14, 11	163;	4, 7
16, 13	7:31;	5, 8
17, 14	13.19;	10, 7
19, 16	313;	11, 8
20, 17	349;	7, 10
22, 19	7.61;	8, 11
23, 20	7.67;	13, 10
25, 22	13:43;	14, 11
26, 23	607;	10, 13
28, 25	709;	11, 14
29, 26	7.109;	16, 13
31, 28	877;	17, 14
32, 29	937;	13, 16
34, 31	1063;	14, 17
35, 32	1129;	19, 16
37, 34	7.181;	20, 17
38, 35	13.103;	16, 19
40, 37	1489;	17, 20
41, 38	1567;	22, 19
43, 40	7.13.19;	23, 20
44, 41	49.37;	19, 22
46, 43	1987;	20, 23
47, 44	31.67;	25, 22
49, 46	31:73;	26, 23
50, 47	7.337;	22, 25
52, 49	2557;	23, 26
53, 50	2659;	28, 25
55, 52	19.151;	29, 26
56, 53	13.229;	25, 28
58, 55	7.457;	26, 29
59, 56	3313;	31, 28
61, 58	3547;	32, 29
62, 59	19.193;	28, 31
64, 61	7.13.43;	29, 32
65, 62	7.577;	34, 31
67, 64	4297;	35, 32
68, 65	43.103;	31, 34
70, 67	37.127;	32, 35
71, 68	7.691;	37, 34
73, 70	5119;	38, 35
74, 71	19.277;	34, 37

TAB. F5*a*.

$\frac{1}{5} N = \frac{1}{5} (x^5 - y^5) \div (x - y), \quad [x - y = 5].$

x	y	N	x	y	N	x	y	N
1,	$\bar{4}$	41;	63,	58	13,443,191?	126,	121	271.859121;
2,	$\bar{3}$	11;	64,	59	11.151.8641;	127,	122	41.71.82601;
3,	$\bar{2}$	11;	66,	61	16,309,451?	128,	123	11.22,569,751?
4,	$\bar{1}$	41;	67,	62	401.43261;	129,	124	61.4201181;
6,	1	311;	68,	63	121.41.61.61;	131,	126	
7,	2	11.61;	69,	64	11.1291.1381;	132,	127	641.439081;
8,	3	1301;	71,	66	22,075,871?	133,	128	
9,	4	211:11;	72,	67	31 31.101.241;	134,	129	11.31.877531;
11,	6	6131;	73,	68	11.2251411;	136,	131	
12,	7	9281;	74,	69	26,190,011?	137,	132	
13,	8	11 1231;	76,	71	191.153151;	138,	133	
14,	9	11.1741;	77,	72	30,874,661!	139,	134	11.31,581,241?
16,	11	131.271;	78,	73	131.331.751;	141,	136	11.31.31.61.571;
17,	12	31.1511;	79,	74	11 31.251.401;	142,	137	41.1361.6791;
18,	13	11.5521;	81,	76	31 1227431.	143,	138	31.12,578,201?
19,	14	31.41.61;	82,	77	821.48751;	144,	139	
21,	16	121421;	83,	78	41.431 2381;	146,	141	11.38,572,561?
22,	17	149351;	84,	79	11.1931.2081;	147,	142	131.3330121;
23,	18	181871;	86,	81	11.41.107981;	148,	143	
24,	19	11.71.281;	87,	82	51 072,431?	149,	144	
26,	21	31.10061;	88,	83	31.1726811;	151,	146	
27,	22	41.8971;	89,	84	181.491.631;	152,	147	11.45,437,551?
28,	23	181.2381;	91,	86	11.71.151 521;	153,	148	
29,	24	11.45631;	92,	87	151.425591;	154,	149	
31,	26	11.60901;	93,	88	67,182,581?	156,	151	11.50,497,651?
32,	27	768221;	94,	89	311.225721;	157,	152	11.51,826,111?
33,	28	281.3121;	96,	91	76,536,221?	158,	153	181.32,031,971?
34,	29	101.9871;	97,	92	11.181.40111;	159,	154	
36,	31	11.115771;	98,	93	83,292,971?	161,	156	11.421.136351;
37,	32	1431581;	99,	94	8683441?	162,	157	71.1361.6701;
38,	33	1021 1571;	101,	96	11 41.208991;	163,	158	11.101.592581;
39,	34	1791551;	102,	97	11.8921701;	164,	159	
41,	36	31.71471;	103,	98	31 3294881;	166,	161	71.99,169,611?
42,	37	11 41.5441;	104,	99	106,265,141?	167,	162	11.66,599,581?
43,	38	2710931;	106,	101	11.10,444,201?	168,	163	11 41.41.40591;
44,	39	101.29581;	107,	102		169,	164	61.12,604,421?
46,	41	11 327661;	108,	103	11.61.184831;	171,	166	191.41,222,381?
47,	42	11.61.5881;	109,	104	41.701.4481;	172,	167	121.31.220151;
48,	43	31.139091;	111,	106		173,	168	101.8370721;
49,	44	4702361;	112,	107	11.31.422041;	174,	169	11 31.241.10531;
51,	46	121.45971;	113,	108	11.571 23761;	176,	171	241.3761501;
52,	47	71.84991;	114,	109		177,	172	251.3695611;
53,	48	11.594151;	116,	111		178,	173	11.86,276,401?
54,	49	461.15331;	117,	112	11.15,640,231?	179,	174	121.541.14831;
56,	51	71.115891;	118,	113		181,	176	31.61.537071;
57,	52	11.31 25981;	119,	114	11.31.540691,	182,	177	
58,	53	121.131.601;	121,	116		183,	178	11.41.2354501;
59,	54	61.167711;	122,	117		184,	179	
61,	56	251.46831;	123,	118	11.71.270191;	186,	181	
62,	57	11.101.11321;	124,	119	11.41.483611;	187,	182	

TAB. F5*b*.

$$N' = (x^5 + y^5) \div (x + y), \quad [x - y = 5].$$

x	y	N'	x	y	N'	x	y	N'
1,	4	31‖11;	63,	58	31.41.71.151;	126,	121	
2,	3	211;	64,	59	14,542,001?	127,	122	11.21,929,701?
3,	2	211;	66,	61	31.532621;	128,	123	31.8034031;
4,	$\bar{1}$	31‖11;	67,	62	11.1597081;	129,	124	521.601.821;
6,	1	11.101;	68,	63	18,674,581?	131,	126	11.71.350431;
7,	2	1871;	69,	64	19,832881?	132,	127	
8,	3	3001;	71,	66	22,310,671?	133,	128	11.26,463,091?
9,	4	4621;	72,	67	11.2148491;	134,	129	71.311.13591;
11,	6	9931;	73,	68	601.41621;	136,	131	11.31.881.1061;
12,	7	11.31.41;	74,	69	31.241.3541;	137,	132	
13,	8	71.271;	76,	71	11.281.9551;	138,	133	11.30,749,681?
14,	9	25951;	77,	72	31,152,361?	139,	134	
16,	11	71.631;	78,	73	11.2986301;	141,	136	
17.	12	11.5231;	79,	74	34,614,791?	142,	137	11.541.63841;
18,	13	72931;	81,	76	11.3487151;	143,	138	
19,	14	91331;	82,	77	40,340,771?	144,	139	11.36,558,371?
21,	16	11.12611;	83,	78	11.911.4231;	146,	141	
22,	17	41.4111;	84,	79	71.627251;	147,	142	11.41.821.1181;
23,	18	11.18461;	86,	81	31.431.3671;	148,	143	31.181.8011;
24,	19	31.41.191;	87,	82	11.71.65851;	149,	144	11.41,997,251?
26,	21	11.30881;	88,	83	421.128021;	151,	146	
27,	22	39801;	89,	84	11.61.84131;	152,	147	
28,	23	11.61.691;	91,	86	61,833,851?	153,	148	11.61.101.7591;
29,	24	537241;	92,	87	121.61.8761;	154,	149	
31,	26	61.61.191;	93,	88	151.461.971;	156,	151	31.17,956,531!
32,	27	11.31.2381;	94,	89	11.31.41.5051;	157,	152	41.61.228421;
33,	28	923701;	96,	91	61.1261861;	158,	153	11.71.1641.2731
34,	29	121.41.211;	97,	92	31.251.10321;	159,	154	31.251.77291;
36,	31	131.10151;	98,	93	11.701.10861;	161,	156	
37,	32	11.135571;	99,	94	61.271.5281;	162,	157	
38,	33	61.151.181;	101,	96	311.304631;	163,	158	
39,	34	11.241.701;	102,	97	98,633,911?	164,	159	11.61,992,191?
41,	36	2289901;	103,	98	11.9331501;	166,	161	11.181.359761;
42,	37	281.9011;	104,	99	41.2604401;	167,	162	31.131.180731;
43,	38	11.31.8191;	106,	101	41.2815171;	168,	163	
44,	39	3073981;	107,	102	101.1187471;	169,	164	11.70,023,271?
46,	41	941.3931;	108,	103	124.578,301?	171,	166	11.73,444,961?
47,	42	641.6311;	109,	104	11.11,759,611?	172,	167	191.241.17971;
48,	43	11.401411;	111,	106	11.12,665,621?	173,	168	
49,	44	4810661;	112,	107	751.192431;	174,	169	631.1373,881;
51,	46	131.131.331;	113,	108		176,	171	41.22,147,601?
52,	47	101.60961;	114,	109	11.14,121,641?	177,	172	11.271.311681;
53,	48	41.331.491;	116,	111	11.41.369751;	178,	173	
54,	49	11.101.6481;	117,	112	31.5570911;	179,	174	31.31,368,061?
56,	51	11.761051;	118,	113	151.1184081;	181,	176	
57,	52	9008221;	119,	114		182,	177	121.131.211.311;
58,	53	1471.6581;	121,	116	941.210481;	183,	178	
59,	54	11.944561;	122,	117	11.18,619,841?	184,	179	
61,	56	121.98561;	123,	118		186,	181	121.41.401.571;
62,	57	12,754,831?	124,	119		187,	182	31.37,447,301?

TAB. F7.

Factorisation of $\frac{1}{7}N = \frac{1}{7}(x^7 - y^7) \div (x - y)$, $N' = (x^7 + y^7) \div (x + y)$,

$[x - y = 7]$.

x, y	$\frac{1}{7}N$	N'
1, −6	29.197;	55987;
2, −5	1597;	25999;
3, −4	379;	14197;
4, −3	379;	14197;
5, −2	1597;	25999;
6, −1	29.197;	55987;
8, 1	127;337;	43.5419;
9, 2	97609;	434827;
10, 3	211.967;	29 43.617;
11, 4	43.9241;	1300237;
12, 5	43.71 239;	743,2843;
13, 6	29.43961;	1499 2213;
15, 8	29.113.1051;	659.11411;
16, 9	14197‖379;	29 376853;
17, 10	8170177;	29 71.7561;
18, 11	12,096,589?	211.103237;
19, 12	43.407233;	29.239 4327;
20, 13	24,841,867?	29.1403081;
22, 15	47,418,337?	617.116747;
23, 16	29.12,207,171?	94,186,177?
24, 17	113.754223;	71.449.3823;
25, 18	449 249593;	113 1382123;
26, 19	449.324437;	29.1163 5881;
27, 20	187.354,147?	113.2210573;
29, 22	827.364127;	127,631.4831;
30, 23	29.129,944,93?	71.1583.4243;
31, 24	43 10,880,941?	127.1373.3347;
32, 25	4271.135617;	309,881,649 ‡
33, 26	211.3345259;	858,478,867?
34, 27	858,428,509?	1032,519,727?
35, 29	1,247,230,363?	43.34,208,959?
37, 30	71.21,000,827?	1,343.311,599?
38, 31	43.41,244,883?	197.337.30983;
39, 32	43 48,823,573?	29.83,336,933?
40, 33	127.631.30871;	43.65,772,043?
41, 34	71.49,882,703?	3,297,034.987?
43, 36	3,948,049,039?	4.432,093.357?
44, 37	29.157,878,9 1?	5,113,699,837?
45, 38	5,290,895.317?	29.743.272917;
46, 39	6,093,605,113?	29 211.1101773;
47, 40	6,995.573.887?	113.127.536999;
48, 41	211.37,945,069?	29.71.379 11257;
50, 43	29.358,502,033?	11,323,318,399?
51, 44	11,798,629,963?	43.337.883807;
52, 45	29.911 505513;	14,450,939,749?
53, 46	15,079,458,829?	43.491.770519;
54, 47	43.113.3495731;	71.257,362,757?
55, 48	43.443,933,029?	29.706,228,181?
57, 50	71.	
58, 51		
59, 52	29.	491.64,525,847?
60, 53	337.701.140351;	

Tab. F9.

Factorisation of $\frac{1}{3}N=\frac{1}{3}(x^9-y^9)\div(x^3-y^3)$, $N'=(x^9+y^9)\div(x^3+y^3)$.
$[x-y=9]$.

x, y	$\frac{1}{3}N$	N'
1, −8	87211;	262657;
2, −7	19.2017;	163.739;
4, −5	3907;	19.1459;
5, −4	3907;	19.1459;
7, −2	19.2017;	163.739;
8, −1	87211;	262657;
10, 1	333667;	19.52579;
11, 2	594091;	19.92683;
13, 4	829.1999;	487.9631;
14, 5	73.181 199;	37.194653.
16, 7	19 37 8677;	15489037;
17, 8	37.242479;	19.109.10567;
19, 10	18,301,609?	41186881?
20, 11	271 93997;	55123561?
22, 13	19.37.67141;	73.1298809;
23, 14		19 433.14851;
25, 16	19.1459.3907;	19691784I? ‡
26, 17	19.433.16903;	181.1362997;
28, 19		127.2979271;
29, 20	37.109.163.433;	19 181.134839;
31, 22	19 23,124,529?	37.37.73.6841;
32, 23		37.22,245.661?
34, 25	801,023,347?	37.31,751,893?
35, 26	19.163.312211;	487.2861623;
37, 28	631.2197333;	19.127.802189;
38, 29	541.3046231;	
40, 31	19.120,879,433?	
41, 32	19.109.127.10243;	
43, 34	109 100,835,877?	19.249,538,123?
44, 35	19.223,628,833?	199.27,348,319?
46, 37		
47, 38		
49, 40		19 37.14,804,767?
50, 41	19.631.806023;	181.64,972,261?
52, 43		37.403,044,661?
53, 44	37.379,302,823?	
55, 46	379.397.118189;	19.1,103,192,659?
56, 47	433 46,076,347?	19.397.3100519;
58, 49		
59, 50	19 37.118,745,347?	199.161,473,159?
61, 52	37.929,786,743?	19.2,072,402,083?
62, 53		19.2,288,581,003?
64, 55	19.73.811.41491;	
65, 56		
67, 58	37.1,686,572,479?	19.271.577.23509;
68, 59	73.	19.37.
70, 61	73.127.541.16417;	
71, 62		37.271.379.26209;
73, 64	19.	37.
74, 65		19.

TAB. F11.

Factorisation of $\frac{1}{11}N = \frac{1}{11}(x^{11} - y^{11}) \div (x - y)$,

and $N' = (x^{11} + y^{11}) \div (x + y)$.

$[x - y = 11]$.

[All factors $\not>$ 1000 cast out].

x, y	$\frac{1}{11}N$		N'	
1, −10	23.4093.8779;		21649.513239;	
2, −9	259,347,617	‡	23.194,913,401?	‡
3, −8	5171.13729;		23.67.1114829;	
4, −7	23.67.10627;		657,710.813?	
5, −6	23.353.419;		313,968,931?	
6, −5	23.353.419;		313,968,931?	
7, −4	23.67.10627;		657,710,813?	‡
8, −3	5171.13729;		23.67.1114829;	
9, −2	259,347,617	‡	23.194,913,401?	‡
10, −1	23.4093.8779;		21649 513239;	
12, 1	23 266,981,089;			
13, 2	23.67.2311.4159;		463.258,050.453?	
14, 3	89.376,039,093?		23.67.154.581,263?	
15, 4			23.19,793,501,267?	
16, 5			23.36,422,846,527?	
17, 6	67.4,227,400,211?			
18, 7			23.111,774,586,793?	
19, 8			23.187,598.790.391?	
20, 9	991.1,707,671,681?		23.89.3450 489,043?	
21, 10	23.859.146,481,193?		89.89.1426,903,171?	
23, 12	89.88,408,092,631?			
24, 13	23.617.661.1339097;		397.103.715,249,749?	
25, 14	67.293,590,258,523?		23.463.661.8699659;	
26, 15	89.340,021,796,929?		23.3901,372,473,857?	
27, 16	23.1991,185,188,829?			
28, 17	23 2967,746,162,833?		67.2762,116,538,567?	
29, 18	23.4360,848,492,859?			
30, 19	23.6323.467,388,107?		67.89.661.92,324.717?	
31, 20	67.353.8807,020,861?		23.23.949,378,290,389?	
32, 21			67.199.353.145,838,837?	
34, 23	419 1365,629.181,643?			
35, 24	23.34,145,697,531,397?			
36, 25	23.353 419‖313,968,931?	‡		
37, 26			23.89.881.1598,279.629?	
38, 27	89.21,637,384,126,381?		23.163,298,401,866,293?	
39, 28			23.211,401,263,328,971?	
40, 29				
41, 30	617.7134,151,274,713?		23.347,852,866,941,977?	
42, 31				
43, 32			23.559,007,067.539.543?	

TAB. F15.

Factorisation of N & $N' = (x^{15} \mp y^{15})(x \mp y) \div (x^5 \mp y^5)(x^3 \mp y^3)$.

[when $x - y = 15$].

x y	N	N'
1, −14	811.1948981;	31.2851.15511;
2, −13	31.39,249,421?	
4, −11	‡	‡
7, −8	61.231661;	31.181.2011;
8, −7	61.231661;	31.181.2011;
11, −4	‡	‡
13, −2	31.39,249,421?	
14, −1	811.1948981;	31.2851.15511;
16, 1	151|331||61.1321;	4562284561;L
17, 2	6,165,257,911?	31.61.181.22741;
19, 4	31.811.538561;	
22, 7	181.214,294,051?	31.2,253,263,101?
23, 8		151.661.1009651;
26, 11		61.4,468,240,531?
28, 13		
29, 14		31.20,958,289,441?
31, 16	421.1,179,672,061?	
32, 17	271.2,335,947,151?	31.45,315,372,591?
34, 19		61.36,852,425,371?
37, 22	31.62,804,861,461?	
38, 23		31.170,526,098,221?
41, 26	571.7,643,952,541?	
43, 28	61.104,405,389,521?	
44, 29	31.246,736,303,771?	541.29,845,102,561?
46, 31		
47, 32	421.30,798,416,461?	211.126,170,760,571?
49, 34		331.991.111,274,591?

ELECTROMAGNETIC POTENTIALS AND RADIATION.

By *R. Hargreaves.*

§ 1. THE vector and scalar potentials of a moving charge are derivable from its coordinates $(x'y'z't')$ by use of the harmonic operator, viz.

$$(F, G, H, \psi) = -\tfrac{1}{2}e\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right\}(x', y', z', Vt') \quad \text{......(1).}$$

The equation $\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} + \dfrac{\partial H}{\partial z} + \dfrac{1}{V}\dfrac{\partial \psi}{\partial t} = 0$ is then the correlative of

$$\frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} + \frac{\partial t'}{\partial t} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The dependence of t' and therefore of $(x'y'z')$ on $(xyzt)$ is defined by $t' = t - r'/V$, where

$$r'^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \ldots\ldots\ldots\ldots (3).$$

It is not proposed to give in detail a proof of (1), but it may be well to state that it turns on two properties of t',

$$\left.\begin{aligned} &\left(\frac{\partial t'}{\partial x}\right)^2 + \left(\frac{\partial t'}{\partial y}\right)^2 + \left(\frac{\partial t'}{\partial z}\right)^2 - \frac{1}{V^2}\left(\frac{\partial t'}{\partial t}\right)^2 = 0 \\ \text{and} \quad &\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right\} t' = -\frac{2}{Vs} \end{aligned}\right\} \ldots\ldots\ldots (4),$$

where $Vs = Vr' - \Sigma(x-x)\dot{x}'$, $\dot{x}'$ being $\dfrac{dx'}{dt'}$. Also the corresponding properties of s are

$$\left.\begin{aligned} &\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2 - \frac{1}{V^2}\left(\frac{\partial s}{\partial t}\right)^2 = \frac{V^2 - \Sigma\dot{x}'^2 + 2\Sigma(x-x')\ddot{x}'}{V^2} \\ &\text{and} \\ &\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right\} s = \frac{2\{V^2 - \Sigma\dot{x}'^2 + 2\Sigma(x-x')\ddot{x}'\}}{V^2 s} \end{aligned}\right\} \ldots\ldots (5),$$

while

$$\frac{\partial t'}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial t'}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial t'}{\partial z}\frac{\partial s}{\partial z} - \frac{1}{V^2}\frac{\partial t'}{\partial t}\frac{\partial s}{\partial t} = -\frac{1}{V} \ldots\ldots (6)$$

is a connecting link.

In virtue of these equations the harmonic operator twice applied to any function of t' yields a zero result, except at the source.

§ 2. In dealing with radiation at the surface of a very large sphere the harmonic operator may be simplified. For if (xyz) are replaced by polar coordinates $(r\theta\phi)$ the differential coefficients involving θ and ϕ contain r^{-2}, $\dfrac{\partial^2}{\partial r^2} - \dfrac{1}{V^2}\dfrac{\partial^2}{\partial t^2}$ yields no term; thus only the section $\dfrac{2}{r}\dfrac{\partial}{\partial r}$ remains and is equivalent to $-\dfrac{2}{Vr}\dfrac{\partial}{\partial t}$. Hence at a sufficiently great distance we may write

$$(F, G, H, \psi) = \frac{e}{Vr}\frac{\partial}{\partial t}(x', y', z', Vt') \ldots\ldots\ldots (7).$$

The position is now that if expressions for $(x'y'z't')$ in terms of $(xyzt)$ are obtained, the potentials are got by differentiation with regard to t only. The same applies to radiation at the surface of the sphere, for when r is great the energy-content E is given by

$$\frac{1}{V^2}\left\{\left(\frac{\partial F}{\partial t}\right)^2+\left(\frac{\partial G}{\partial t}\right)^2+\left(\frac{\partial H}{\partial t}\right)^2-\left(\frac{\partial \psi}{\partial t}\right)^2\right\} \text{......(8)},$$

an expression essentially positive in virtue of

$$\frac{\partial \psi}{\partial t}=\frac{x}{r}\frac{\partial F}{\partial t}+\frac{y}{r}\frac{\partial G}{\partial t}+\frac{z}{r}\frac{\partial H}{\partial t},$$

the form assumed by $\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}+\frac{1}{V}\frac{\partial \psi}{\partial t}=0$ when r is great. At the same time (3) is replaced by

$$t'=t-r/V+(xx'+yy'+zz')/Vr \text{...........(9)}.$$

Another form for energy-content at a great distance is

$$\frac{e^2}{V^4r^2}\left\{\left(\frac{\partial^2 x'}{\partial t^2}\right)^2+\left(\frac{\partial^2 y'}{\partial t^2}\right)^2+\left(\frac{\partial^2 z'}{\partial t^2}\right)^2-V^2\left(\frac{\partial^2 t'}{\partial t^2}\right)^2\right\} \text{...(10)}.$$

In dealing with several moving charges e (x', y', z', t') must be replaced in (7) or (10) by the sums of the several contributions, all expressed in terms of $(xyzt)$

§ 3. In illustration of the method we may deal with linear oscillatory motion, and with the circular motion of one or more charges. For the first case $z'=a\cos\omega t'$, and then (9) becomes $t'=t-rV^{-1}+aV^{-1}\cos\theta\cos\omega t'$, or, multiplying by ω,

$$\tau'=\tau+\rho\cos\tau', \text{ if } \tau'=\omega t', \ \tau=\omega(t-rV^{-1}), \ \rho=a\omega V^{-1}\cos\theta \text{......(11)}.$$

The problem of expressing τ' and $\cos\tau'$ in terms of τ and ρ may be solved by Lagrange's expansion, and the result is

$$\left.\begin{aligned}
\tau'&=\tau+2\{J_1(\rho)\cos\tau-\tfrac{1}{3}J_3(3\rho)\cos 3\tau+\ldots\}\\
&\qquad -2\{\tfrac{1}{2}J_2(2\rho)\sin 2\tau-\tfrac{1}{4}J_4(4\rho)\sin 4\tau+\ldots\}\\
\cos\tau'&=2/\rho\,\{J_1(\rho)\cos\tau-\tfrac{1}{3}J_3(3\rho)\cos 3\tau+\ldots\}\\
&\qquad -2/\rho\,\{\tfrac{1}{2}J_2(2\rho)\sin 2\tau-\tfrac{1}{4}J_4(4\rho)\sin 4\tau+\ldots\}\\
\sin\tau'&=\tfrac{1}{2}\rho+2\{J_1'(\rho)\sin\tau-\tfrac{1}{3}J_3'(3\rho)\sin 3\tau-\ldots\}\\
&\qquad +2\{\tfrac{1}{2}J_2'(2\rho)\cos 2\tau-\tfrac{1}{4}J_4'(4\rho)\cos 4\tau+\ldots\}
\end{aligned}\right\} \text{......(12)},$$

the last required in the next example. Thus

$$\left.\begin{aligned} H &= -\frac{2e}{r\cos\theta}\left[\{J_1(\rho)\sin\tau - J_3(3\rho)\sin 3\tau + \ldots\}\right. \\ &\qquad\qquad \left. + \{J_2(2\rho)\cos 2\tau - J_4(4\rho)\cos 4\tau \ldots\}\right] \\ \psi &= \frac{e}{r} - \frac{2e}{r}\left[\{J_1(\rho)\sin\tau - J_3(3\rho)\sin 3\tau + \ldots\}\right. \\ &\qquad\qquad \left. + \{J_2(2\rho)\cos 2\tau - J_4(4\rho)\cos 4\tau \ldots\}\right] \end{aligned}\right\} \quad \ldots\ldots(13),$$

expressions which show a complete series of harmonics* with coefficients small when $\omega a/V$ is small. For the fundamental period the mean value of E is $2e^2\omega^2 V^{-2}r^{-2}\tan^2\theta J_1^{\,2}(\rho)$, or when ρ is small $\frac{1}{2}e^2\omega^4 a^2 V^{-4}r^{-2}\sin^2\theta$, giving a radiation $\frac{e^2a^2\omega^4}{3V^3}$, viz. $\frac{1}{4\pi}\iint VEr^2\sin\theta\, d\theta\, d\phi$. The harmonic $\sin 2\tau$ gives $\frac{e^2a^4\omega^6}{15V^5}$ when $\omega a/V$ is small.

§ 4. For the motion of a charge e in a circle we have

$$x' = a\cos(\omega t' + \phi'), \quad y' = a\sin(\omega t' + \phi'),$$

while

$$t' = t - r/V + (xx' + yy')/Vr = t - r/V + aV^{-1}\sin\theta\cos(\omega t' + \phi' - \phi),$$

where $(r\theta\phi)$ are polar coordinates of (xyz). If then we write

$$\tau' = \omega t' + \phi' - \phi,\ \tau = \omega(t - r/V) + \phi' - \phi,\ \rho = \omega a V^{-1}\sin\theta \ldots(14),$$

the relation between $\tau'\tau\rho$ is that of (11).

It is proposed to deal with n equally spaced charges in a circle, so that ϕ' has values $\phi_0, \phi_0 + 2\pi/n, \ldots\ \phi_0 + 2\pi(n-1)/n$; and we may use $\tau_0 = \omega(t - r/V) + \phi_0 - \phi$ so that τ has values $\tau_0, \tau_0 + 2\pi/n, \ldots$ for the several charges.

It is convenient to calculate $R = F\cos\phi + G\sin\phi$ and $S = G\cos\phi - F\sin\phi$ instead of F, G; that is

$$R = \frac{e}{Vr}\frac{\partial}{\partial t}(x'\cos\phi + y'\sin\phi)$$

$$= \frac{ea}{Vr}\frac{\partial}{\partial t}\cos(\omega t' + \phi' - \phi) \text{ or } \frac{ea}{Vr}\frac{\partial}{\partial t}\cos\tau'.$$

We are therefore concerned with

$$R = \frac{ea}{Vr}\frac{\partial}{\partial t}\Sigma\cos\tau', \quad S = \frac{ea}{Vr}\frac{\partial}{\partial t}\Sigma\sin\tau', \quad \psi = \frac{e}{\omega r}\frac{\partial}{\partial t}\Sigma\tau' \ldots(15),$$

* In Hertz's solution the harmonics do not appear because the variable state at one fixed point is treated as source.

the summation referring to the separate charges. In $\Sigma \cos \tau'$ the sums containg multiples of τ other than n, $2n$, ... vanish, while the sum containing n as multiple will have all its terms equal, viz. $\cos n\tau_0$, and the sum is $n \cos n\tau_0$. If we ignore multiples $2n$, ... it is sufficient to note that in (12)

for n even $\Sigma \cos \tau'$ contains $\frac{2}{\rho}(-1)^{\frac{1}{2}n} J_n(n\rho) \sin n\tau_0$,

for n odd $\frac{2}{\rho}(-1)^{\frac{1}{2}(n-1)} J_n(n\rho) \cos n\tau_0$;

for n even $\Sigma \sin \tau'$ contains $-2(-1)^{\frac{1}{2}n} J_n'(n\rho) \cos n\tau_0$,

for n odd $2(-1)^{\frac{1}{2}(n-1)} J_n'(n\rho) \sin n\tau_0$.

The corresponding terms in the potentials are then

$$\left.\begin{aligned}
R &= \frac{2en\omega a}{Vr}(-1)^{\frac{1}{2}n}\frac{J_n(n\rho)}{\rho}\cos n\tau_0 \text{ for } n \text{ even},\\
&\quad -\frac{2en\omega a}{Vr}(-1)^{\frac{1}{2}(n-1)}\frac{J_n(n\rho)}{\rho}\sin n\tau_0 \text{ for } n \text{ odd}\\
S &= \frac{2en\omega a}{Vr}(-1)^{\frac{1}{2}n}J'(n\rho)\sin n\tau_0 \text{ for } n \text{ even},\\
&\quad +\frac{2en\omega a}{Vr}(-1)^{\frac{1}{2}(n-1)}J_n'(n\rho)\cos n\tau_0 \text{ for } n \text{ odd}\\
\psi - \frac{ne}{r} &= \frac{2ne}{r}(-1)^{\frac{1}{2}n}J_n(n\rho)\cos n\tau_0 \text{ for } n \text{ even},\\
&\quad -\frac{2en}{r}(-1)^{\frac{1}{2}(n-1)}J_n(n\rho)\sin n\tau_0 \text{ for } n \text{ odd}
\end{aligned}\right\} \ldots (16).$$

Thus the fundamental term has the period $2\pi/n\omega$ in which the electromagnetic conditions are manifestly reproduced. Since $\left(\frac{\partial F}{\partial t}\right)^2 + \left(\frac{\partial G}{\partial t}\right)^2 = \left(\frac{\partial R}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial t}\right)^2$, an expression for the mean value of the radiation for this period can be given. It is $\frac{e^2 n^4 \omega^4 a^2}{V^3}\int_0^\pi \left[\frac{\cos^2\theta J_n^2(n\rho)}{\rho^2} + \{J_n'(n\rho)\}^2\right]\sin\theta\, d\theta$.

If we were not concerned with the potentials the formula for radiation could be derived directly from (10) and (12).

PARTIAL FRACTIONS ASSOCIATED WITH QUADRATIC FACTORS.

By *Prof. E. H. Neville.*

THE calculation of the fractions associated with a quadratic factor X in a rational function $\phi(x)/X^n\psi(x)$ is usually thought to be prohibitively laborious in numerical examples. The object of this note is to describe a straightforward process that has been applied successfully to examples with n as large as 3 and with $\psi(x)$ a quartic.

Let us take X to be px^2+qx+r, and let us suppose the degrees of ϕ, ψ to be l, m. The various coefficients are assumed to be integers, and the work is arranged to prevent the entrance of numerical fractions at any stage. Some of the operations, though of course valid in any case, have no other point, and we describe the work throughout on the most unfavourable hypotheses. In particular, we suppose l and m to be not less than $2n-1$, and q to be prime to $2p$. Trivial steps are given in detail, in order that an exact idea of the whole amount of labour involved may be conveyed. Greek capital letters denote always polynomials whose actual coefficients there is no need to calculate.

Multiplication of ϕ, ψ by p^{l-1}, p^{m-1} is sufficient, though not always necessary, to provide polynomials that we can divide by X, to the point of finding remainders linear in x, without introducing numerical fractions. That is, we can determine rapidly two polynomials ϕ_1, ψ_1, and two remainders a_0x+b_0, c_0x+d_0, such that

$$p^{l-1}\phi = a_0x+b_0+X\phi_1, \quad p^{m-1}\psi = c_0x+d_0+X\psi_1.$$

Applying the same process to ϕ_1, ψ_1, which are of degrees $l-2$, $m-2$, we have

$$p^{l-3}\phi_1 = a_1x+b_1+X\phi_2, \quad p^{m-3}\psi_1 = c_1x+d_1+X\psi_2,$$

and n operations of this kind, which can be arranged in the manner familiar in the use of Horner's method, give

$$p^\lambda\phi = (a_0x+b_0)\,p^{\lambda-l+1}+(a_1x+b_1)\,p^{\lambda-2l+4}X+(a_2x+b_2)\,p^{\lambda-3l+9}X^2+\ldots$$
$$+(a_{n-1}x+b_{n-1})\,X^{n-1}+X^n\Gamma',$$

$$p^\mu\psi = (c_0x+d_0)\,p^{\mu-m+1}+(c_1x+d_1)\,p^{\mu-2m+4}X+(c_2x+d_2)\,p^{\mu-3m+9}X^2+\ldots$$
$$+(c_{n-1}x+d_{n-1})\,X^{n-1}+X^n\Delta',$$

where $$\lambda = n(l-n), \quad \mu = n(m-n),$$

and Γ', Δ' are polynomials.

We now express X, or a numerical multiple of X, in the form y^2-k, where k may of course be negative. If q is prime to $2p$, we must take

$$y=2px+q,\quad k=q^2-4pr,\quad y^2=k+4pX.$$

We write then

$$2p^{\lambda+1}\phi=(a_0p^{\lambda-l+1}+a_1p^{\lambda-2l+4}X+\ldots+a_{n-1}X^{n-1})(y-q)$$
$$+2p(b_0p^{\lambda-l+1}+b_1p^{\lambda-2l+4}X+\ldots+b_{n-1}X^{n-1})+X^n\Gamma,$$
$$2p^{\mu+1}\psi=(c_0p^{\mu-m+1}+c_1p^{\mu-2m+4}X+\ldots+c_{n-1}X^{n-1})(y-q)$$
$$+2p(d_0p^{\mu-m+1}+d_1p^{\mu-2m+4}X+\ldots+d_{n-1}X^{n-1})+X^n\Delta,$$

and we have

$$2p^{\lambda+1}\phi=Ay+B+X^n\Gamma,\quad 2p^{\mu+1}\psi=Cy+D+X^n\Delta,$$

where

$$A=a_0p^{\lambda-l+1}+a_1p^{\lambda-2l+4}X+\ldots+a_{n-1}X^{n-1},$$
$$B=(2pb_0-qa_0)p^{\lambda-l+1}+(2pb_1-qa_1)p^{\lambda-2l+4}X+\ldots$$
$$+(2pb_{n-1}-qa_{n-1})X^{n-1},$$
$$C=c_0p^{\mu-m+1}+c_1p^{\mu-2m+4}X+\ldots+c_{n-1}X^{n-1},$$
$$D=(2pd_0-qc_0)p^{\mu-m+1}+(2pd_1-qc_1)p^{\mu-2m+4}X+\ldots$$
$$+(2pd_{n-1}-qc_{n-1})X^{n-1}.$$

Thus, identically,

$$\frac{p^\lambda\phi}{p^\mu\psi}=\frac{Ay+B+X^n\Gamma}{Cy+D+X^n\Delta}=\frac{(Ay+B)(Cy-D)+X^n\Theta'}{C^2y^2-D^2+X^n\Phi'}$$
$$=\frac{(BC-AD)y+\{AC(k+4pX)-BD\}+X^n\Theta'}{C^2(k+4pX)-D^2+X^n\Phi'},$$

where Θ', Φ' denote $(Cy-D)\Gamma$, $(Cy-D)\Delta$, and are themselves ultimately polynomials in x. The various sums and products must be evaluated as far as terms in X^{n-1}, and if we write

$$BC-AD=u_0+u_1X+\ldots+u_{n-1}X^{n-1}+X^n\Lambda,$$
$$AC(k+4pX)-BD=v_0+v_1X+\ldots+v_{n-1}X^{n-1}+X^n\Upsilon,$$
$$C^2(k+4pX)-D^2=w_0+w_1X+\ldots+w_{n-1}X^{n-1}+X^n\Omega,$$

we have
$$\frac{p^\lambda\phi}{p^\mu\psi}=\frac{Lx+M+X^n\Theta}{N+X^n\Phi},$$

where

$$L = 2pu_0 + 2pu_1 X + \ldots + 2pu_{n-1} X^{n-1},$$

$$M = (qu_0 + v_0) + (qu_1 + v_1) X + \ldots + (qu_{n-1} + v_{n-1}) X^{n-1},$$

$$N' = w_0 + w_1 X + \ldots + w_{n-1} X^{n-1},$$

and Θ, Φ are polynomials into which Θ', Φ', Λ, Υ, Ω are all absorbed.

The final step is effectively a division of L and M by N', and to avoid fractions we may have to make the substitution $X = w_0 Y$ in these three polynomials. We have then

$$L = 2pu_0 + 2pu_1 w_0 Y + \ldots + 2pu_{n-1} w_0^{n-1} Y^{n-1},$$

$$M = (qu_0 + v_0) + (qu_1 + v_1) w_0 Y + \ldots + (qu_{n-1} + v_{n-1}) w_0^{n-1} Y^{n-1},$$

$$N' = w_0 N,$$

where $\qquad N = 1 + w_1 Y + w_2 w_0 Y^2 + \ldots + w_{n-1} w_0^{n-2} Y^{n-1}.$

The effect of the division of L and M by N as far as the term in Y^{n-1} is literally to determine polynomials F, G of degree $n-1$ in Y, such that

$$L = FN + Y^n \Pi, \quad M = GN + Y^n \Sigma,$$

where Π, Σ are polynomials in Y, and therefore also in x. But from these relations we have

$$w_0 (Lx + M + X^n \Theta) = (Fx + G) N' + w_0 Y^n (\Pi x + \Sigma + w_0^n \Theta)$$

$$= (Fx + G)(N' + X^n \Phi) + X^n \Xi,$$

where $\qquad \Xi = w_0^{-n+1} (\Pi x + \Sigma) + w_0 \Theta - (Fx + G) \Phi.$

Thus, finally,

$$\frac{w_0 (Lx + M + X^n \Theta)}{X^n (N' + X^n \Phi)} = \frac{Fx + G}{X^n} + \frac{\Xi}{N' + X^n \Phi};$$

that is, if explicitly

$$F = f_0 + f_1 Y + f_2 Y^2 + \ldots + f_{n-1} Y^{n-1},$$

$$G = g_0 + g_1 Y + g_2 Y^2 + \ldots + g_{n-1} Y^{n-1},$$

then identically

$$\frac{p^\lambda \phi(x)}{p^\mu X^n \psi(x)} = \frac{f_0 x + g_0}{w_0 X^n} + \frac{f_1 x + g_1}{w_0^2 X^{n-1}} + \ldots + \frac{f_{n-1} x + g_{n-1}}{w_0^n X} + \frac{\Xi (Cy - D)}{2 w_0 p^{\mu+1} \psi(x)}.$$

The form of the result shews that when $Cy - D$ and Ξ are both expressed in terms of x the one is a factor of the other, but although the steps of the algebra would lead by direct operations to Ξ, it is primarily for the calculation of the fractions involving X that the process is designed; for this purpose the only intermediate functions that have to be found explicitly are those denoted by A, B, C, D, L, M, N, all of which are polynomials in X of degree not higher than $n-1$, however high the degrees of $\phi(x)$ and $\psi(x)$.

The explanation has been arranged to establish incidentally the existence of the fractions. In practice of course the polynomials whose coefficients are not necessary are simply ignored, equivalence for the purpose in view being substituted for absolute identity.

The details of the arithmetic in an actual example will be found in an early number of the *Mathematical Gazette.*

POLYGONS INSCRIBED IN ONE CIRCLE AND CIRCUMSCRIBED TO ANOTHER.

By *E. C. Titchmarsh*, Balliol College, Oxford.

THE relation between two circles, that an infinity of polygons of a given number of sides can be inscribed in one and circumscribed to the other, can be expressed as a relation between the radii of the two circles and the distance between their centres. Some quite simple cases of such relations appear to have been overlooked.

Poncelet's general theorem for two conics can be expressed as follows. Let the conics be

$$(S) \qquad Ax^2 + By^2 - Cz^2 = 0,$$

$$(S') \qquad A'x^2 + B'y^2 - C'z^2 = 0.$$

The condition that the tangents to S at the points

$$\left\{\sqrt{\left(\frac{C}{A}\right)}\cos\alpha : \sqrt{\left(\frac{C}{B}\right)}\sin\alpha : 1\right\},$$

$$\left\{\sqrt{\left(\frac{C}{A}\right)}\cos\beta : \sqrt{\left(\frac{C}{B}\right)}\sin\beta : 1\right\}$$

should meet on S' is

$$a\cos\alpha\cos\beta + b\sin\alpha\sin\beta - c = 0,$$

where $$a = \frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C}, \quad b = -\frac{A'}{A} + \frac{B'}{B} - \frac{C'}{C},$$

$$c = -\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C}.$$

Hence, if a polygon of n sides can be circumscribed to S and inscribed in S', the n equations

$$a\cos\alpha_i\cos\alpha_{i+1} + b\sin\alpha_i\sin\alpha_{i+1} - c = 0 \quad (i = 1, 2, \ldots, n),$$

where $\alpha_{n+1} = \alpha_1$, must be simultaneously satisfied by values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ which do not differ by even multiples of π. This is only true if a certain relation exists between a, b, and c.

For $n = 6$ the relation is

$$\left(-\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = 0,$$

and for $n = 8$ it is

$$\left(-\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\left(\frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2}\right)\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) = 0,$$

If we take the limiting points of two circles to be $(\pm f, 0)$, their equations can be written

$$(S) \quad (f+g)(x+f)^2 + (f-g)(x-f)^2 + 2fy^2 = 0,$$

$$(S') \quad (f+g')(x+f)^2 + (f-g')(x-f)^2 + 2fy^2 = 0,$$

so that here

$$a = \frac{f+g'}{f+g} - \frac{f-g'}{f-g} - 1, \quad b = -\frac{f+g'}{f+g} + \frac{f-g'}{f-g} - 1,$$

$$c = -\frac{f+g'}{f+g} - \frac{f-g'}{f-g} + 1.$$

If r is the radius of S, R that of S', and d the distance between their centres, then

$$r^2 = g^2 - f^2, \quad R^2 = g'^2 - f^2, \quad d^2 = (g - g')^2.$$

Eliminating f, g, g', we have

$$a + b = -2, \quad ab = \frac{1}{r^4}\{2r^2(R^2 + d^2) - (R^2 - d^2)^2\}, \quad c = \frac{d^2 - R^2}{r^2}.$$

Using these expressions, the hexagon condition

$$\left(-\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) = 0$$

becomes

$$3(R^2 - d^2)^4 - 4r^2(R^2 + d^2)(R^2 - d^2)^2 - 16r^4R^2d^2 = 0 \ldots \text{(i)};$$

and

$$\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$$

becomes $$(R^2 - d^2)^2 - 4r^2d^2 = 0,$$

i.e. $$R^2 - d^2 = \quad 2rd \quad \text{(ii)}$$

or $$R^2 - d^2 = -2rd \quad \text{(iii)}.$$

The octagon condition

$$\left(-\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\left(\frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2}\right) = 0$$

becomes

$$\{(R^2 - d^2)^2 - 2r^2(R^2 + d^2)\}^4$$
$$= 16(R^2 - d^2)^4 r^4 \{r^4 - 2r^2(R^2 + d^2) + (R^2 - d^2)^2\} \quad \text{(iv)};$$

and $$\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} = 0$$

becomes $$(R^2 - d^2)^4 = 16R^2d^2r^4,$$

i.e. $$(R^2 - d^2)^2 = \quad 4Rdr^2 \quad \text{(v)}$$

or $$(R^2 - d^2)^2 = -4Rdr^2 \quad \text{(vi)}.$$

The formulæ (i) and (iv) are well known, but the remainder do not seem to have been stated before, at any rate explicitly.

The various cases in which real polygons exist may be represented by graphs. We need only consider positive values of R, r, and d. Take r to be unity; then the case (R, r, d) can be represented by the point $x = d$, $y = R$. In the part A of the first quadrant, bounded by the axes and $x + y = 1$, S' lies entirely inside S, which never gives a real polygon. In the part B between $x - y = 1$ and Ox the circles are separate; in the part C between $x - y = -1$ and Oy, S lies inside S'; in the remaining region D the circles intersect. Each relation between R, r, and d gives a curve; each branch of such a curve lying in one of the regions B, C, D represents a set of cases in which real polygons exist. For both hexagon and octagon, each such branch has as asymptote one or other of the lines $x - y = \pm 1$. For the hexagon, (i) gives a branch in B and one in C, (ii) gives one in D, and (iii) one in B. For the octagon, (iv) gives two branches in B and two in C, and (v) gives two in D.

Cayley* observes that for a polygon of $2m$ sides the complete relation breaks up into two factors—one corresponding to the case in which a degenerate polygon passes through a circular point at infinity and an actual intersection, and the other to that in which it passes through two circular points or two

* *Phil. Trans. Roy. Soc.*, 7 March, 1861.

actual intersections. This latter factor always breaks up again into two. If the circles are

$$(S) \qquad x^2 + y^2 = r^2,$$
$$(S') \qquad (x-d)^2 + y^2 = R^2,$$

the abscissae of their intersections, and of all the other vertices of a polygon which starts from an intersection, are rational functions of R, r, and d. If m is even, the $\frac{1}{2}m^{\text{th}}$ vertex must lie on S' on the line of centres; that is, if X is its abscissa, $(X-d)^2 = R^2$. If m is odd, the $\frac{1}{2}(m+1)^{\text{th}}$ side must be perpendicular to the line of centres and touch S; that is, if X is the abscissa of the $\frac{1}{2}(n+1)^{\text{th}}$ vertex, $X^2 = r^2$. In each case there are two factors.

Some of the above relations can also be easily obtained from a consideration of the figure when the polygon degenerates into a double line, starting from one intersection of the circles and ending at another. In the case of the hexagon let P, Q be these two intersections, A the pole of PQ with respect to S, and let AP, AQ meet S' again in M, N. Then MN must must touch S. It P, Q are the circular points at infinity, then A is the origin; AP, AQ are $x \pm iy - 0$; MN is

$$-2dx + d^2 = R^2,$$

and this touches S if $\dfrac{d^2 - R^2}{2d} = \pm r$.

ON CURVATURE, TORTUOSITY, AND HIGHER FLEXURES OF A CURVE IN FLAT SPACE OF n DIMENSIONS.

By *R. F. Muirhead.*

For simplicity I take a curve as the limit of an equilateral skew figure $ABCDE$... when the common length of the sides AB, BC, CD, etc , tends to zero.

The curvature at A is $\lim\limits_{AB\to 0} \dfrac{\sin A\hat{B}C}{AB}$. The tortuosity at A is $\lim\limits_{AB\to 0} \dfrac{\sin(A\overline{BC}D)}{AB}$, where $(A\overline{BC}D)$ denotes the dihedral angle between the planes ABC and BCD.

When the curve lies in space of higher than three dimensions it has a flexure of another kind, which is measured by $\lim\limits_{AB} \dfrac{\sin(A\overline{BCD}E)}{AB}$, where $(A\overline{BCD}E)$ denotes the angle between the three dimensional flat spaces containing $ABCD$ and $BCDE$ respectively. This may be called its flexure of the third order, or, more briefly, its third flexure, curvature and tortuosity being the flexures of the first and of the second order respectively. We may denote its measure by ϕ_3.

This nomenclature may be extended indefinitely. Thus the flexure of the n^{th} order of the curve at the point A_0 is given by

$$\phi_n \equiv \lim_{A_0A_1 \to 0} \frac{\sin(\overline{A_0A_1A_2\ldots A_nA_{n+1}})}{A_0A_1},$$

where $(\overline{A_0A_1A_2\ldots A_nA_{n+1}})$ denotes the angle between the flat n-dimensional spaces containing the figures $A_0A_1A_2\ldots A_n$ and $A_1A_2A_3\ldots A_nA_{n+1}$ respectively.

Now we have $\sin A\hat{B}C = \dfrac{2 \times \text{Area } ABC}{AB.BC}$, and if DM be the perpendicular let fall from D on the plane ABC, and DN the perpendicular let fall on the line BC, we have

$$\sin(A\overline{BC}D) = \frac{DM}{DN} = \frac{3 \times \text{Vol. } ABCD}{\text{Area } ABC} \div \frac{2 \times \text{Area } BCD}{BC}$$

$$= \frac{3! \times \text{Vol. } ABCD}{2! \times \text{Area } ABC} \div \frac{2! \times \text{Area } BCD}{BC},$$

and if EM be the perpendicular let fall from E on the space of $ABCD$, and if EN be the perpendicular let fall from E on the plane of BCD,

$$\sin(ABCDE) = \frac{EM}{EN}$$

$$= \frac{4! \times \text{Content } (ABCDE)}{3! \times \text{Vol. } (ABCD)} \div \frac{3! \times \text{Vol. } (BCDE)}{2! \times (\text{Area } BCD)}.$$

And generally

$$\sin(\overline{A_0A_1\ldots A_nA_{n+1}})$$

$$= \frac{(n+1)! \times \text{Co. } (A_0A_1\ldots A_{n+1})}{n! \times \text{Co. } (A_0A_1\ldots A_n)} \div \frac{n! \times \text{Co. } (A_1\ldots A_{n+1})}{(n-1)! \times \text{Co. } (A_1\ldots A_n)}$$

$$= \frac{(n+1)!\,(n-1)! \times \text{Co. } (A_0A_1\ldots A_{n+1}) \times \text{Co. } (A_1A_2\ldots A_n)}{(n!)^2 \times \text{Co. } (A_0A_1\ldots A_n) \times \text{Co. } (A_1A_2\ldots A_{n+1})}.$$

Thus

$$\phi_n = \lim_{A_0A_1 \to 0} \left\{ \frac{(n+1)!\,(n-1)! \times \text{Co } (A_0\ldots A_{n+1}) \times \text{Co.}(A_1\ldots A_n)}{(n!)^2\,\text{Co. } (A_0A_1\ldots A_n) \times \text{Co.}(A_1\ldots A_{n+1})} \times \frac{1}{A_0A_1} \right\}$$

$$= \lim_{A_0A_1 \to 0} \left[\frac{(n+1)!\,(n-1)! \times \text{Co.}(A_0\ldots A_{n+1}) \times \text{Co.}(A_0\ldots A_{n-1})}{\{n! \times \text{Co. } (A_0\ldots A_n)\}^3 . A_0A_1} \right],$$

since $\lim\limits_{A_0A_1\to 0} \dfrac{\text{Co.}(A_1A_2\ldots A_{p+1})}{\text{Co.}(A_0A_1\ldots A_p)} = 1$ for all values of p.

Now it is easy to show that if $x_1, x_2, \ldots, x_{n+1}$ are the co-ordinates of a point on the curve, referred to orthogonal axes in flat $(n+1)$-dimensional space (in S_{n+1} let us say for brevity), and $\dot{x}, \ddot{x}, \dddot{x} \ldots x^{(p)}$ the 1st, 2nd, 3rd...pth derivatives of x with regard to the length of the arc from a fixed point on the curve to the point in question, then

$$\lim_{A_0A_1\to 0}\left\{\frac{(n+1)!\,\text{Co.}(A_0A_1\ldots A_{n+1})}{(A_0A_1)^{(n+1)(n-2)/2}}\right\} = \begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \ldots & \dot{x}_{n+1} \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 & \ldots & \ddot{x}_{n+1} \\ \dddot{x}_1 & \dddot{x}_2 & \dddot{x}_3 & \ldots & \dddot{x}_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(n+1)} & x_2^{(n+1)} & \ldots & & x_{n+1}^{(n+1)} \end{vmatrix} \ldots(1),$$

and we can prove that

$$\lim_{A_0A_1\to 0}\left\{\frac{n!\,\text{Co.}(A_0A_1\ldots A_n)}{(A_0A_1)^{n(n-3)/2}}\right\}$$

$$= \sqrt{\left\{\begin{vmatrix} \dot{x}_2 & \dot{x}_3 & \ldots & \dot{x}_{n+1} \\ \ddot{x}_2 & \ddot{x}_3 & \ldots & \ddot{x}_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ x_2^{(n)} & x_3^{(n)} & \ldots & x_{n+1}^{(n)} \end{vmatrix}^2 + \begin{vmatrix} \dot{x}_1 & \dot{x}_3 & \dot{x}_4 & \ldots & \dot{x}_{n+1} \\ \ddot{x}_1 & \ddot{x}_3 & \ddot{x}_4 & \ldots & \ddot{x}_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(n)} & x_3^{(n)} & x_4^{(n)} & \ldots & x_{n+1}^{(n)} \end{vmatrix}^2\right.}$$

+ similarly formed squared determinants, leaving out successively the suffixes 3, 4, ..., $(n+1)\}$...(2).

In fact the content of a simplex of n dimensions (which has $n+1$ vertices) is the square root of the sum of the squares of its projections on the $n+1$ mutually orthogonal co-ordinate S_n's in S_{n+1}.

[Also the content of a simplex of p dimensions (with $p+1$ vertices) is the square root of the sum of the squares of the contents of its projections on the $(n+1)!/(n+1-p)!\,p!$ S_p's which contain respectively all possible p-combinations out of $n+1$ mutually orthogonal axes in S_{n+1}. I do not know whether these theorems are known, but I may mention that I have found a good straightforward proof of them based on the generalized theorem of Pythagoras.]

It follows that

$$\phi_n = \frac{D_{n+1}\sqrt{(\Sigma D^2_{n-1})}}{\Sigma D_n^2} \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where D_{n+1} denotes the determinant forming the right-hand member of (1); and D_n the similarly formed determinant of the n^{th} order and the Σ prefixed to it denotes summation over all determinants of the kind.

D_{n-1} denotes a similarly formed determinant of the $(n-1)^{th}$ order with any two of the $n+1$ suffixes omitted, and the Σ prefixed to it extends over all determinants of this kind.

We can combine the preceding results in various ways. Thus if $r! \times \text{Co.}\,(A_1 A_2 \ldots A_r)$ be denoted by C_r, we have

$$\phi_1 = \text{Lim}\, \frac{C_2}{C_1^{\,2} C_1},$$

$$\phi_1\phi_2 = \text{Lim}\, \frac{C_3}{C_1^{\,3} C_2},$$

$$\phi_1\phi_2\phi_3 = \text{Lim}\, \frac{C_4}{C_1^{\,4} C_3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\phi_1\phi_2\ldots\phi_r = \text{Lim}\, \frac{C_{r+1}}{C_1^{\,r+1} C_r},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\phi_1\phi_2\ldots\phi_n = \text{Lim}\, \frac{C_{n+1}}{C_1^{\,n+1} C_n} \cdots\cdots\cdots\cdots (4).$$

Again $$\phi_1^{\,n}\phi_2^{\,n-1}\ldots\phi_{n-1}^{\,2}\phi_n = \text{Lim}\, \frac{C_{n+1}}{C_1^{\,\frac{1}{2}\{(n+1)(n+2)\}}} \cdots\cdots\cdots (5).$$

When $n=2$ this becomes $\phi_1^{\,2}\phi_2 = \text{Lim}\, \dfrac{C_3}{C_1^{\,6}}$, which is well known in the form

$$\frac{1}{\rho^2\sigma} = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}.$$

$\dfrac{1}{\rho}$ being the curvature and $\dfrac{1}{\sigma}$ the tortuosity.

March, 1917.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*.

LVI.

On Fourier's series and Fourier's integral.

1. There are three familiar representations of an arbitrary function associated with Fourier's name, viz.

$$(1)\qquad f(x)=\frac{1}{\pi}\sum_{n=0}^{\infty}\epsilon_n\int_0^{2\pi}f(t)\cos n(t-x)\,dt$$

(where ϵ_n is $\frac{1}{2}$ if $n=0$ and 1 otherwise),

$$(2)\qquad f(x)=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(t-x)}{t-x}\,dt,$$

$$(3)\qquad f(x)=\frac{1}{\pi}\int_0^{\infty}dy\int_{-\infty}^{\infty}f(t)\cos y(t-x)\,dt.$$

These are (1) Fourier's *series*, (2) Fourier's *single integral*,* and (3) Fourier's *double integral*. It is with the first two only that I am concerned in this note.

The conditions under which (1) and (2) are valid are, so far as the behaviour of $f(t)$ in the neighbourhood of $t=x$ is concerned, identical. This is of course well known; but there is a simple formal relation between the two formulæ which I have not seen established generally.

2. Suppose first that $f(x)$ is a trigonometrical polynomial

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\nu}(a_n\cos nx+b_n\sin nx)=\tfrac{1}{2}A_0+\sum_{n=1}^{\nu}A_n,$$

and that λ is positive. A simple calculation shows that

$$(4)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(t-x)}{t-x}\,dt=\sum_{n\leq\lambda}\eta_n A_n,$$

where $\eta_n=\frac{1}{2}$ if $n=0$ or $n=\lambda$ and $\eta_n=1$ otherwise. The integral is defined in the ordinary manner when λ is non-

* I follow Prof. Hobson's nomenclature: see his *Theory of functions of a eal variable* (first ed.), p. 760.

integral; but when λ is an integer it must be defined as a principal value

$$(5) \qquad \lim_{T\to\infty} \int_{-T}^{T},$$

as it would otherwise be divergent.

Suppose that λ is non-integral. Then *the value of Fourier's integral is the sum of the terms of Fourier's series whose rank is less than* λ. It is this result which I wish to generalise.

3. THEOREM 1. *The formula* (4) *is true for every periodic and integrable* function* $f(x)$, *provided that the integral is interpreted as a principal value when* λ *is an integer.*

We may suppose that $0 \leqq x \leqq 2\pi$, since each side of (4) is periodic in x.

Suppose first that $N < \lambda < N+1$, where N is an integer. Then, by the ordinary formulæ for Fourier coefficients,

$$(6) \qquad \sum_{n<\lambda} \eta_n A_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin (N+\frac{1}{2})(t-x)}{\sin \frac{1}{2}(t-x)} f(t)\, dt.$$

On the other hand

$$(7) \quad \int_{-\infty}^{\infty} f(t) \frac{\sin\lambda(t-x)}{t-x}\, dt = \sum_{-\infty}^{\infty} \int_{2k\pi}^{2(k+1)\pi} f(t) \frac{\sin\lambda(t-x)}{t-x}\, dt$$

$$= \sum_{-\infty}^{\infty} \int_0^{2\pi} f(t) \frac{\sin\lambda(t-x+2k\pi)}{t-x+2k\pi}\, dt,$$

if this series is convergent. Let us assume for a moment that the order of summation and integration may be reversed. We have†

$$(8) \quad \psi(\lambda) = \sum_{-\infty}^{\infty} \frac{\sin\lambda(t-x+2k\pi)}{t-x+2k\pi} = \frac{\sin (N+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}.$$

There are certain terms in the series, namely those for which k is -1, 0, and 1, whose definition fails for particular values of t and x. It is to be understood that such a term is then to be replaced by its limiting value λ.

The formula (4) follows at once from (6), (7), and (8), and it remains only to justify the inversion.

It is sufficient for this purpose to show that the series (8) is boundedly convergent, that is to say that

$$(9) \qquad \left| \sum_{-K'}^{K} \frac{\sin\lambda(t-x+2k\pi)}{t-x+2k\pi} \right| < A,$$

* In the sense of Lebesgue.

† See, for example, Bromwich, *Infinite Series*, p. 257 (ex. 19).

where A is a constant, for $0 \leqq t \leqq 2\pi$, $0 \leqq x \leqq 2\pi$, and all positive integral values of K and K'.

As each term of the series is individually bounded, we may ignore the terms for which k is -1, 0, or 1. We have then

$$\left| \frac{1}{t-x+2k\pi} - \frac{1}{2k\pi} \right| = \left| \frac{t-x}{(t-x+2k\pi)\,2k\pi} \right| < \frac{A}{k^2},$$

$$\left| \sum_2^K \frac{\sin\lambda(t-x+2k\pi)}{t-x+2k\pi} \right| < A + \left| \sum_2^K \frac{\sin\lambda(t-x+2k\pi)}{2k\pi} \right|$$

$$= A + \frac{1}{2\pi}\left| \cos\lambda(t-x) \sum_2^K \frac{\sin 2\lambda k\pi}{k} \right| + \frac{1}{2\pi}\left| \sin\lambda(t-x) \sum_2^K \frac{\cos 2\lambda k\pi}{k} \right|$$

$$< A;$$

and the same argument may be applied to the sum from $-K'$ to -2. We thus deduce (9), which completes the proof of the theorem when λ is non-integral.

If λ is an integer N, we have

$$\sum_{n \leqq \lambda} \eta_n A_n = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\sin(N+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)} - \cos N(t-x) \right\} f(t)\,dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin N(t-x) \cot\tfrac{1}{2}(t-x) f(t)\,dt,$$

and

$$\lim_{K\to\infty} \sum_{-K}^K \frac{\sin N(t-x+2k\pi)}{t-x+2k\pi} = \sin N(t-x) \lim \sum_{-K}^K \frac{1}{t-x+2k\pi}$$

$$= \tfrac{1}{2} \sin N(t-x) \cot\tfrac{1}{2}(t-x);$$

so that our formal analysis still holds, the integral being interpreted as a principal value, and the series in the special manner indicated above. Also, if the terms for which $k=-1$, 0, 1 are omitted from the summations as before, we have

$$\left| \sum_{-K}^K \frac{\sin N(t-x+2k\pi)}{t-x+2k\pi} \right| \leqq \left| \sum_{-K}^K \frac{1}{t-x+2k\pi} \right|$$

$$= \left| 2(t-x) \sum_2^K \frac{1}{(t-x)^2 + 4k^2\pi^2} \right| < A,$$

so that the inversion is still legitimate. It is essential here that our special definitions of the series and integral should be adhered to, neither the proof nor the result being valid without them.

4. A similar argument establishes

THEOREM 2. *If*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a periodic and integrable function $f(x)$, *and* λ *is positive and non-integral, then*

$$(10)\quad \int_0^\infty f(x)\frac{\sin\lambda x}{x}dx = \tfrac{1}{2}\pi\left(\tfrac{1}{2}a_0 + \sum_{1\leq n\leq\lambda} a_n\right) + \tfrac{1}{2}\sum_1^\infty b_n \log\left|\frac{n-\lambda}{n+\lambda}\right|.$$

5. It appears then that, if $f(x)$ is any function integrable over $(0, 2\pi)$, and $F(x)$ is the function defined over $(-\infty, \infty)$ by periodic continuation, the problem of expressing $f(x)$ by a Fourier's series is identical with that of expressing $F(x)$ by a Fourier's single integral. We cannot extend this equivalence to Fourier's double integral (3), without some generalisation of the definition of an infinite integral, since

$$\int_{-\infty}^{\infty} F(t)\cos y\,(t-x)\,dt$$

is not convergent.

There are similar expressions for the Rieszian means of the Fourier's series of $f(x)$. Thus

$$(11)\quad \frac{2}{\pi}\int_{-\infty}^{\infty} f(t)\left\{\frac{\sin\frac{1}{2}\lambda(x-t)}{x-t}\right\}^2 dt = \tfrac{1}{2}A_0\lambda + \sum_{1\leq n\leq\lambda}(\lambda-n)A_n$$

is a formula for the Rieszian (or Cesàro) mean of order 1. This formula has however been established already by Young, who has given similar formulæ for the Rieszian means of any positive order.*

It should be noted that there is a serious difference between (4) and (11), or any of the formulæ given by Young. These latter formulæ are direct deductions from the general theorems which I considered in Note 55, since, for example, the function

$$g(x) = \left(\frac{\sin\frac{1}{2}\lambda x}{x}\right)^2$$

has bounded variation, and a convergent integral, over the whole interval $(-\infty, \infty)$. Neither of these conditions is satisfied when

$$g(x) = \frac{\sin\lambda x}{x},$$

so that Theorem 1 is not deducible from the theorems of Note 55.

* See his papers 'Über eine Summationsmethode für die Fouriersche Reihe' (*Leipziger Berichte*, 43 (1911), pp. 369–387) and 'On infinite integrals involving a generalisation of the sine and cosine functions' (*Quarterly Journal*, 43 (1912), pp. 161–177). Young restricts λ to be an integer.

6. There is a theorem concerning the allied series

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n$$

corresponding to Theorem 1.

THEOREM 3. *If $f(x)$ is any periodic and integrable function, and ΣB_n is the series allied to the Fourier series of $f(x)$, then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos\lambda(t-x)}{t-x}\,dt = \sum_{1\leq n\leq\lambda} \eta_n B_n,$$

where η_n is $\frac{1}{2}$ if $n=\lambda$ and 1 otherwise. The integral is an ordinary integral if $a_0 = 0$ and λ is not an integer; otherwise it is a principal value.

It is unnecessary to give the details of the proof, which will present no difficulty to anyone who has followed the proof of Theorem 1.

THE DISSECTION OF RECTILINEAL FIGURES.
(continued)

By *W. H. Macaulay, M.A.*

MAJOR MACMAHON'S study of repeating patterns has suggested a simple proof of the rule which I have given* for drawing the "broken" lines of what I called a hexagon dissection of a pair of rectilineal figures of equal area.

A repeating pattern is a figure such that any number of figures identical with it can be fitted together to form a tessellation, without any gaps. A triangle is a repeating pattern which requires reversal to form a tessellation, half the figures having one orientation, and the other half having the orientation obtained from this by turning them through two right angles. A parallelogram is one which does not require reversal, all the figures having the same orientation. A hexagon with two opposite sides equal and parallel is one which in general requires reversal; but if all pairs of opposite sides are equal and parallel there is no reversal.

Take two hexagons of equal area, each with a pair of opposite sides equal and parallel, and such that they have a common core; the core being the parallelogram formed by joining the middle points of the inclined sides. With each hexagon form a tessellation, and superpose these tessellations

* *Messenger of Mathematics*, vol. xlviii., p. 164.

in any way which secures that the angular points of the cores of one tessellation coincide with those of the other. There are four distinct ways in which this can be done; and in each of these superpositions, the lines of either figure dissect each hexagon of the other figure into parts which can be rearranged to form the other hexagon. What I have called a hexagon dissection is thus obtained. In general some of the dividing lines of each hexagon are broken, so as to be made up of several parallel portions, and the rule for drawing such lines, which I have ennuciated, is easily seen to be correct. Every dissection which is given by the hexagon rule could also be obtained by contriving a suitable pair of tessellations.

By the same procedure we obtain dissections of an endless variety of repeating patterns, rectilinear or curvilinear, which are derived from the hexagons, or other fundamental rectilineal figures, as bases, in the way which is indicated by Major MacMahon in his recent book, *New Mathematical Pastimes.**

It seems to be likely that all the dissections of pairs of independent rectilineal figures, in which the dividing lines of each figure, whether broken or unbroken, are equal and parallel to half-sides of the other, can be obtained by a single superposition of tessellations. Such dissections form a distinguishable class. Two or more successive operations of forming tessellations, and superposing them, give certain other dissections without this characteristic. By independent figures I mean figures such that the dimensions of one have no relation to those of the other except the equality of area.

If each of two hexagons, of equal area, and with a common parallelogram as core, has all pairs of opposite sides equal and parallel, so that their tessellations are formed without reversal, every superposition in which the cores are parallel to one another gives a dissection, and not only those superpositions in which the angular points of cores coincide. Thus we get a continuous series of dissections, doubly infinite as there are two degrees of freedom of adjustment of superposition. Only certain members of this series are what I have called hexagon dissections. For this pair of hexagons, triangles formed by joining alternate angular points of each hexagon can be drawn so as to be the same for both of them, and these have the same orientation with regard to the parallelogram core. Thus a dissection, which in simple cases has only three parts, obtained from consideration of this triangle,† though not

* See also *Proc. of Roy. Soc.*, Series A, vol. ci., p. 80.

† *Messenger of Mathematics*, vol. xlviii., p. 163.

itself a hexagon dissection, is a member of the continuous series, which can accordingly be established by either method.

Superposition of tessellations (assuming that this gives all cases) solves the question of the classification of the numerous dissections of a pair of independent parallelograms of equal area, for which each figure is divided by lines equal and parallel to half-sides of the other. This has not been dealt with in previous papers, the methods employed being insufficient for the purpose. Let a, b be the lengths of the sides of one parallelogram, and c, d those of the other, and Δ their common area. The whole set of lines representing two superposed tessellations of the parallelograms must include two sets of parallel lines, each equally spaced, between which the parallelograms are packed. Let sides a and c of the parallelograms be those which are in contact with these lines. Let p, q be the lengths of the sides of the least parallelogram formed by the two intersecting sets of parallel lines, and A its area. In order to secure repetition of the dissection of the parallelogram ab, p must be chosen so that $2a = np$ where n is some whole number. Now $A = p\dfrac{\Delta}{a} = q\dfrac{\Delta}{c}$, therefore $2c = nq$, and $nA = 2\Delta$. Also $pq \geq A$, so the range of values which n can have is given by $ac \geq \frac{1}{2}n\Delta$; and to obtain any dissections a and c must be chosen so that ac is not less than $\frac{1}{2}\Delta$. The choice of a value for n settles the angle between the two sets of parallel lines, for it gives p, q and A; that is to say it gives two alternative angles supplementary to one another. These yield different dissections, so each must be taken into account. The packing of successive rows of parallelograms must be that which is appropriate to this angle, so as to give repetition of intersections of lines. Each odd value of n gives, for each of the corresponding angles, a single dissection; for in this case the dissections are reversed in alternate figures, so that freedom of adjustment of superposition is lost, and all the sides b and d are bisected by other lines of the figure. Each even value of n gives, for each of the corresponding angles, a continuous series of dissections; for no reversal is necessary, and the superposition can be adjusted continuously in any way not involving rotation. Thus all the dissections are examples of one or other of the two rather distinct types which are given by odd values of n and by even values of n. Having thus enumerated the dissections arising from the choice of a and c as the sides in contact with the two sets of parallel lines, we can proceed to get the corresponding results arising from the three other possible choices, namely b and c, a and d, and b and d.

It should be noticed that the well known three-part dissection of a pair of parallelograms occurs, in this classification, as a particular member of a continuous series of dissections for which $n=2$. This is clearly its proper place.

The consideration of superposition of tessellations gives a simple explanation of the fact that, in hexagon dissections, the actual dividing lines of one figure are equal and parallel to the actual half-sides of the other, when coincident and cancelled lines are left out of account.

There is a slight mistake in my paper (*Messenger of Mathematics*, vol. xlix., p. 113). In certain cases the arrangement (ii) gives a core of which a half-side of the triangle is a diagonal. Accordingly the new dissection, p. 114, can be obtained from certain arangements (ii), (ii) and (ii), (iii), as well as from (iii), (iii). This does not in any way affect the final result.

RELATIONS BETWEEN THE NUMBERS OF BERNOULLI, EULER, GENOCCHI, AND LUCAS.

By *E. T. Bell*, University of Washington.

1. *Introduction.* The entire theory of the relations between the numbers B, E, G, R of Bernoulli, Euler, Genocchi and Lucas, can be uniformly developed by means of Blissard's umbral notation,* through trigonometric identities. In this, for $n \geq 0$, a_n is written a^n, the exponent being purely symbolic, and a letter a, b, ..., c, B, E, G, R, ϕ, ψ, χ, without a suffix is the representative (in Blissard's terminology), or *umbra*, of the whole class of like letters with zero or positive integral suffixes. Thus b is the umbra of the class b_0, b_1, ..., b_n, ...; the umbra of the Bernoulli numbers B_0, B_1, B_2, ..., B_n, ... is B, and so on. The letters h, x, y, ..., z denote ordinary algebraic quantities, or *ordinaries*. Throughout the paper $f(x)$ is the expression

$$f(x) \equiv k_0 + k_1 x + \ldots + k_n x^n + \ldots,$$

* J. Blissard, "Theory of Generic Equations", *Quarterly Journal of Pure and Applied Mathematics*, vol. iv (1861), p. 279; vol. v., pp. 58, 185. The main results of these papers so far as they relate to the Bernoulli and allied numbers, also the general method of Blissard, are reproduced in Lucas, *Theorie des Nombres* (1891), chaps. xiii., xiv.

which is subject only to the restriction that if it consists of an infinite number of terms the series on the right is absolutely convergent for some $|x| > 0$. The successive x-derivatives of $f(x)$ are $f'(x), f''(x), \ldots, f^{(r)}(x), \ldots$. The umbral multinomial theorem for $n \geq 0$ an integer, and $x, y, \ldots, z$ all different from zero, is

$$(xa + yb + \ldots + zc)^n = \Sigma \frac{n!}{\alpha!\beta!\ldots\gamma!} x^\alpha y^\beta \ldots z^\gamma a_\alpha b_\beta \ldots c_\gamma,$$

the summation extending to all integers $\alpha, \beta, \ldots, \gamma \geq 0$ such that $\alpha + \beta + \ldots + \gamma = n$. The umbral sine and cosine of ax are defined by the series, assumed absolutely convergent for some $|x| > 0$,

$$\sin ax = \sum_{n=0}^{\infty} (-1)^n a^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \equiv \sum_{n=0}^{\infty} (-1)^n a_{2n+1} \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos ax = \sum_{n=0}^{\infty} (-1)^n a^{2n} \frac{x^{2n}}{(2n)!} \equiv \sum_{n=0}^{\infty} (-1)^n a_{2n} \frac{x^{2n}}{(2n)!},$$

and we have

$$d/dx \sin ax = a \cos ax, \quad d/dx \cos ax = -a \sin ax,$$

in which, as in all umbral formulas, the indicated multiplications by a on the right are to be carried out on the *umbral* forms of $\cos ax$, $\sin ax$ respectively *before* exponents are degraded to suffixes.

The numbers B_n, E_n, G_n, R_n are defined as the respective coefficients of $x^n/n!$ in

$$\frac{x}{e^x - 1}, \quad \frac{2}{e^x + e^{-x}}, \quad \frac{2x}{e^x + 1}, \quad \frac{x}{e^x - e^{-x}},$$

and hence

$$B_1 = -\tfrac{1}{2}, \quad G_1 = 1, \quad B_{2n+1} = 0 \ (n > 0),$$

$$E_{2n+1} = 0 \ (n \geq 0), \quad G_{2n+1} = 0 \ (n > 0), \quad R_{2n+1} = 0 \ (n \geq 0),$$

and we have the umbral generators

(1) $(x/2) \cot (x/2) = \cos Bx, \qquad x \cot x = \cos 2Bx;$

(2) $\sec x = \cos Ex;$

(3) $x \tan (x/2) = \cos Gx, \quad 2x \tan x = \cos 2Gx;$

(4) $(x/2) \csc x = \cos Rx, \qquad x \csc x = 2 \cos Rx.$

These are the definitions of Blissard or Lucas, and give one of the more convenient notations for the numbers. An alternative set, preferable in some respects, uses the hyperbolic instead of the circular functions.

2. *Types of General Relations.* Between entire functions whose arguments are linear functions of h, x, B, E, G, R, and for which no umbra appears more than once in any argument, there are possible four types of fundamental relations according as 1, 2, 3, or 4 of the symbols B, E, G, R are present. The less important cases in which an umbra occurs more than once in an argument are noted in § 9. We shall derive a set of 21 fundamental relations containing at least one of each type. Six of the first type are included for completeness, although four were given by Blissard and Lucas. In his treatment Lucas follows the historical order, using alternative definitions of B, E, G, R as coefficients in certain sums of like powers of natural numbers. But if the relations alone be the chief object they can be more systematically derived as an application of Blissard's calculus to the rudiments of trigonometry. Moreover, this method generalizes (§ 9); the other does not.

3. *Addition Theorems for the Umbral Sine and Cosine.* These obviously are of the same form as those for the ordinary sine and cosine, and we have either from them or by inspection,

(5) $2\cos ax \cos bx = \cos(a+b)x + \cos(a-b)x = \cos\phi(a, b)x,$

(6) $2\cos ax \sin bx = \sin(a+b)x - \sin(a-b)x = \sin\psi(a, b)x,$

(7) $2\sin ax \cos bx = \sin(a+b)x + \sin(a-b)x = \sin\phi(a, b)x,$

(8) $2\sin ax \sin bx = \cos(a-b)x - \cos(a+b)x = -\cos\psi(a, b)x,$

where the ϕ, ψ functions are defined for the integer $n \geq 0$ by

$$\phi_n(a, b) = (a+b)^n + (a-b)^n, \quad \psi_n(a, b) = (a+b)^n - (a-b)^n.$$

The separate cases, n even, n odd, being constantly required we write them out for reference:

$$\phi_{2n}(a, b) = 2\sum_{r=0}^{n}\binom{2n}{2r} a_{2n-2r}b_{2r},$$

$$\phi_{2n+1}(a, b) = 2\sum_{r=0}^{n}\binom{2n+1}{2r} a_{2n-2r+1}b_{2r},$$

$$\psi_{2n}(a, b) = 2\sum_{r=0}^{n-1}\binom{2n}{2r+1} a_{2n-2r-1}b_{2r+1},$$

$$\psi_{2n+1}(a, b) = 2\sum_{r=0}^{n}\binom{2n+1}{2r+1} a_{2n-2r}b_{2r+1}.$$

From the last and the zero values of B, E, G, R, we see at once the following identities, useful later,

$$\phi_{2n+1}(R,\ 1)=\phi_{2n+1}(E,\ 1)=\psi_{2n}(E,\ 1)=\phi_{2n+1}(xG,\ 1)$$
$$=\psi_{2n}(R,\ 1)=\phi_{2n+1}(R,\ 2B)=0,\ (n\geqq 0),$$

and many more of the same sort, also the following,

$$\psi_{2n}(2B,\ 1)=-4n,\quad \phi_{2n+1}(2G,\ 1)=4(2n+1),$$
$$\phi_{2n+1}(B,\ R)=-(2n+1)R_{2n},\quad \text{for } n\geqq 0,$$

and others of a like kind. Note that these are mere identities which do not enable us to calculate the successive B, E, G, R, by recurrence. From the definitions we have the important identities

(9) $$[x+h(a+b)]^n+[x+h(a-b)]^n$$
$$=[x+h\phi(a,\ b)]^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}h^r\phi_r(a,\ b),$$

(10) $$[x+h(a+b)]^n-[x+h(a-b)]^n$$
$$=[x+h\psi(a,\ b)]^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}h^r\psi_r(a,\ b).$$

In these put $n=0,\ 1,\ 2,\ \ldots$, multiply the results throughout by k_0, k_1, k_2, ... respectively, and add:

(11) $$f(x+ha+hb)+f(x+ha-hb)$$
$$=f\{x+h\phi(a,\ b)\}=\sum_{r=0}\frac{h^r}{r!}\phi_r(a,\ b)f^{(r)}(x),$$

(12) $$f(x+ha+hb)-f(x+ha-hb)$$
$$=f\{x+h\psi(a,\ b)\}=\sum_{r=0}\frac{h^r}{r!}\psi_r(a,\ b)f^{(r)}(x).$$

The process by which these are obtained can often be applied more expeditiously than the formulas themselves. Except for one application, noted § 8, the above are sufficiently general with $h=1$.

A relation involving precisely r umbrae is called r-fold. By iteration of (5)–(8) the r-fold cases of all results in this section can be easily written out, or they may be derived independently from the exponential forms of the umbral sine and cosine.

4. *Derived Identities.* Take the x-derivatives of (1)–(4) and reduce the results by (1)–(4):

(13) $x^2 \csc^2 (x/2) = \quad 4 \cos Bx + 4Bx \sin Bx,$

(14) $\sec x \tan x \quad = - E \sin Ex,$

(15) $x \sec^2 (x/2) \ = - 2 \cos Gx - 2 Gx \sin Gx,$

(16) $x^2 \csc x \cot x = \quad 2 \cos Rx + 2 Rx \sin Rx.$

Multiply together, member by member, each pair of (1)–(4), and use (13)–(16) when necessary to reduce the results:

(17) $2 \cos Rx \cos Bx = \quad \cos Bx + Bx \sin Bx,$

(18) $2 \cos Bx \cos Gx = \quad x^2.$

(19) $2 \cos Rx \cos Gx = - \cos Gx - Gx \sin Gx,$

(20) $2 \cos Rx \cos Ex = \quad 2 \cos 2Rx,$

(21) $2 \cos Ex \cos Bx = \quad 2 \cos 2Rx + 2 \cos Rx,$

(22) $2 \cos Ex \cos Gx = \quad 4 \cos 2Rx - 4 \cos Rx.$

Similarly, taking products three at a time, we find

(23) $4 \cos Rx \cos Bx \cos Gx = 2x^2 \cos Rx,$

(24) $4 \cos Ex \cos Bx \cos Gx = 2x^2 \cos Ex,$

and from (3), (15) and (1), (13) respectively,

(25) $4 \cos Ex \cos Rx \cos 2\ Gx = - 2 \cos 2 Gx - 4 Gx \sin 2 Gx,$

(26) $4 \cos Ex \cos Rx \cos 2\ Bx = \quad 2 \cos 2Bx + 4Bx \sin 2Bx.$

Finally from (1)–(4),

(27) $8 \cos Rx \cos Bx \cos Gx \cos Ex = 4x^2 \cos 2Rx.$

Each right-hand member has been expressed as a linear function of umbral sines and cosines in order to give the essentially simplest relations of the several types. We proceed to find these from (1)–(4) and (13)–(27). The process in all cases is the same: we equate coefficients of like powers of x and apply (5)–(12), noting by inspection as in § 3 the values of the ϕ, ψ functions of odd index. Slightly more general results in all of the following can be obtained by replacing x by hx in each of the derivations; but as already mentioned these are of interest chiefly in one connection. They can be written down from the forms given.

5. *One-fold Relations.* Multiplying the second form of (1) throughout by $2\sin x$ we get

$$\sin \psi\,(2B,\ 1)\,x = 2x\cos x,$$

and therefore equating coefficients of like powers of x,

$$\psi_{2n+1}(2B,\ 1) = 2\,(2n+1) \quad (n \geqq 0).$$

We have also identically

$$\psi_0(2B,\ 1) = 0, \quad \psi_{2n}(2B,\ 1) = -4n \quad (n > 0),$$

and hence, combining results and using (10),

$$(x+2B+1)^n - (x+2B-1)^n = 2n\,(x-1)^{n-1} = 2d/dx\,(x-1)^n.$$

Change x into $2x+1$:

$$(x+B+1)^n - (x+B)^n = nx^{n-1}.$$

From the first of these, as in deriving (12), we get

$$\text{(I)} \qquad f(x+2B+1) - f(x+2B-1) = 2f'(x-1),$$

and from the second,

$$\text{(II)} \qquad f(x+B+1) - f(x+B) = f'(x),$$

the last of which is Blissard's and Lucas' form. In the same way from (2)–(4) on multiplying by $\cos x$, $\sin x$, respectively:

$$\text{(III)} \qquad f(x+E+1) + f(x+E-1) = 2f(x),$$

$$\text{(IV)} \qquad f(x+2G+1) + f(x+2G-1) = 4f'(x-1),$$

$$\text{(V)} \qquad f(x+G+1) + f(x+G) = 2f'(x),$$

$$\text{(VI)} \qquad f(x+R+1) - f(x+R-1) = f'(x),$$

the forms (III), (V), (VI) being those of Blissard and Lucas. As *always* the accent denotes a derivative with respect to x.

6. *Twofold Relations.* Each of (1), (3) gives a twofold relation. Write (3) in the form

$$\cos 2Gx = 2x\sin x \sec x \equiv 2x\cos Ex \sin x = 2\sin\psi\,(E,\ 1)\,x,$$

and equate coefficients,

$$\psi^{2n+1}(E,\ 1) = -\frac{(2G)^{2n+2}}{2n+2} \quad (n \geqq 0);$$

also, identically, $\psi_{2n}(E,\ 1) = 0 \ (n \geqq 0)$. Hence from (12),

$$(x+E+1)^n - (x+E-1)^n = -\tfrac{1}{2}\int\left[(x+2G)^n + (x-2G)^n\right]dx,$$

which generalizes as before into

$$\text{(VII)}\quad 2f'(x+E-1)-2f'(x+E+1)$$
$$=f(x+2G)+f(x-2G).$$

Similarly (1) gives first

$$\phi_{2n}(R,\ 1)=2^{2n}B_{2n},\quad \phi_{2n+1}(R,\ 1)=0\quad (n\geq 0),$$
$$(x+R+1)^n+(x+R-1)^n=\tfrac{1}{2}\left[(x+2B)^n+(x-2B)^n\right].$$

and hence

$$\text{(VIII)}\quad 2f(x+R+1)+2f(x+R-1)$$
$$=f(x+2B)+f(x-2B).$$

The remaining twofold relations are found in the same way from those of the formulas (13)-(22) that contain only two umbrae. As some of the derivations involve a new detail we give the first in full. Evidently from (1), (4) an equivalent of (16) is

$$\cos\phi\,(R,\ 2B)\,x\equiv 2\cos Rx\cos 2Bx=2\cos Rx+2Rx\sin Rx,$$

whence $\phi_{2n}(R,\ 2B)=2(1-2n)R_{2n}$ $(n\geq 0)$, and identically $\phi_{2n+1}(R,\ 2B)=0$ $(n\geq 0)$. From these

$$(x+R+2B)^n+(x+R-2B)^n$$
$$=2\left[x^nR_0+\binom{n}{2}x^{n-2}R_2+\binom{n}{4}x^{n-4}R_4+\dots\right]$$
$$-2\left[2\binom{n}{2}x^{n-2}R_2+4\binom{n}{4}x^{n-4}R_4+6\binom{n}{6}x^{n-6}R_6+\dots\right],$$

the right member of which can be written

$$[(x+R)^n+(x-R)]^n-Rd/dx\left[(x+R)^n-(x-R)^n\right].$$

To avoid an umbral derivative in the final result we note that

$$d/dR\left[(x+R)^n+(x-R)^n\right]=d/dR\left[(x+R)^n-(x-R)^n\right],$$

and hence generalizing as usual find

$$\text{(IX)}\quad f(x+R+2B)+f(x+R-2B)=f(x+R)$$
$$+f(x-R)-R\left[f'(x+R)-f'(x-R)\right].$$

A similar treatment of (14) gives

$$\phi_{2n}(E,\ 2G)=8nE_{2n},\quad \phi_{2n+1}(E,\ 2G)\equiv 0\quad (n\geq 0);$$

whence

$$(x+E+2G)^n+(x+E-2G)^n=2Ed/dx\,[(x+E)^n-(x-E)^n],$$

$$\text{(X)}\quad f(x+E+2G)+f(x+E-2G)$$
$$=2E[f'(x+E)-f'(x-E)].$$

In the same way from (17),

$$\phi_{2n}(R,\ B)=(1-2n)\,B_{2n},\quad \phi_{2n+1}(R,\ B)\equiv 0\quad (n\geqq 0),$$

$$(x+R+B)^n+(x+R-B)^n=\tfrac{1}{2}(1-Bd/dx)[(x+B)^n-(x-B)^n],$$

$$\text{(XI)}\qquad 2f(x+R+B)+2f(x+R-B)$$
$$=f(x+B)+f(x-B)-B[f'(x+B)-f'(x-B)].$$

The derivation of the like relation from (18) is rather more complicated. First

$$\phi_2(B,\ G)=1,\quad \phi_{2n}(B,\ G)=0\quad (n\neq 1),$$
$$\phi_{2n+1}(B,\ G)\equiv -2nG_{2n}\quad (n\geqq 0),$$

which are found as in the preceding cases, and hence after some simple reduction

$$2\,(x+B+G)^n+2\,(x+B-G)^n=\frac{d^2x^n}{dx^4}$$
$$-G\frac{d^2}{dx\,dG}[1/G\,.\int\{(x+G)^n+(x-G)^n\}\,dG],$$
$$=\frac{d^2x^n}{dx^2}+1/G\,.\,[(x+G)^n-(x-G)^n]$$
$$-d/dx\,[(x+G)^n+(x-G)^n];$$

whence the relation

$$\text{(XII)}\qquad 2G[f(x+B+G)+f(x+B-G)]$$
$$=G[f''(x)-f'(x+G)-f'(x-G)]+f(x+G)-f(x-G).$$

From (19),

$$\phi_{2n}(R,\ G)=(2n-1)\,G_{2n},\quad \phi_{2n+1}(R,\ G)\equiv 0\quad (n>0),$$

$$2\,(x+R+G)^n+2\,(x+R-G)^n$$
$$=(Gd/dG-1)[(x+G)^n+(x-G)^n];$$

$$\text{(XIII)}\qquad 2f(x+R+G)+2f(x+R-G)$$
$$=G[f'(x+G)-f'(x-G)]-f(x+G)-f(x-G).$$

From (20),

$$\phi_{2n}(R,\ E) = 2^{2n+1}R^{2n},\quad \phi_{2n+1}(R,\ E) \equiv 0 \quad (n \geq 0),$$

$$(x+R+E)^n + (x+R-E)^n = (x+2R)^n + (x-2R)^n;$$

$$\text{(XIV)}\quad f(x+R+E) + f(x+R-E) = f(x+2R) + f(x-2R).$$

7. *Threefold Relations.* From (21), as in the preceding section, we find

$$\phi_{2n}(E,\ B) = 2\,(2^{2n}+1)\,R^{2n},\quad \phi_{2n+1}(E,\ B) \equiv 0 \quad (n \geq 0),$$

$$(x+E+B)^n + (x+E-B)^n = (x+2R)^n + (x-2R)^n + (x+R)^n + (x-R)^n;$$

$$\text{(XV)}\quad f(x+E+B) + f(x+E-B) = f(x+2R) + f(x-2R) + f(x+R) + f(x-R);$$

and from (22) compared with (21) we write down

$$\text{(XVI)}\quad f(x+E+G) + f(x+E-G) = 2f(x+2R) + 2f(x-2R) - 2f(x+R) - 2f(x-R).$$

The remaining threefold relations are found from (23)–(26) by means of the extension of (5), (9), (11) to the case of three umbrae:

$$4\cos ax\cos bx\cos cx = \cos\phi\,(a,\ b,\ c)\,x,$$

where $\phi_n(a,\ b,\ c)$ is defined by

$$\phi_n(a,\ b,\ c) = \Sigma\,(a \pm b \pm c)^n,$$

the summation extending to the four possible combinations of plus and minus signs. From (23) we get thus

$$\phi_{2n}(R,\ B,\ G) = -4n\,(2n-1)\,R^{2n-2},$$

$$\phi_{2n+1}(R,\ B,\ G) \equiv 0 \quad (n \geq 0),$$

$$\Sigma\,(x+R \pm G)^n = -d^2/dx^2\left[(x+R)^n + (x-R)^n\right];$$

$$\text{(XVII)}\quad \Sigma f(x+R \pm B \pm G) = -f''(x+R) - f''(x-R).$$

From this and (24) we write down by symmetry,

$$\text{(XVIII)}\quad \Sigma f(x+E \pm B \pm G) = -f''(x+E) - f''(x-E).$$

From (25) we first find

$$\phi_{2n}(E,\ R,\ 2G) = 2\,(4n-1)\,2^{2n}G^{2n},$$

$$\phi_{2n+1}(E,\ R,\ 2G) \equiv 0 \quad (n \geq 0),$$

$$\Sigma(x+E\pm R\pm 2G)^n=-[(x+2G)^n$$
$$+(x-2G)^n]+4G d/dx[(x+2G)^n-(x-2G)^n],$$

whence as usual,

$$\text{(XIX)}\quad \Sigma f(x+E\pm R\pm 2G)=-f(x+2G)$$
$$-f(x-2G)+4G[f'(x+2G)-f'(x-2G)].$$

By this and (26) we write down from symmetry,

$$\text{(XX)}\quad \Sigma f(x+E\pm R\pm 2B)=f(x+2B)$$
$$-f(x-2B)-4B[f'(x+2B)-f'(x-2B)].$$

Before applying symmetry in such work we must always see by actual inspection whether the ϕ-functions of odd index in the two cases have corresponding values.

8. *Fourfold Relation; Taylor's Theorem Forms.* Write

$$\phi_n(a,b,c,d)=\Sigma(a\pm b\pm c\pm d)^n,$$

the summation extending to the eight combinations of the signs. Then from (27),

$$\phi_{2n}(R,B,G,E)=-8n(2n-1)(2R)^{2n-2},$$
$$\phi_{2n+1}(R,B,G,E)\equiv 0\quad(n\geqq 0);$$
$$\Sigma(x+R\pm B\pm G\pm E)^n=-8d^2/dx^2[(x+2R)^n+(x-2R)^n];$$
$$\text{(XXI)}\quad \Sigma f(x+R\pm B\pm G\pm E)=-8f''(x+2R)-8f''(x-2R).$$

Each of the relations (I)–(XXI) is equivalent to a Taylor expansion obtained at once from the slightly more general form involving h in place of 1 in the arguments of f. The latter are derived in precisely the same way as the formulas given with the exception that x is everywhere replaced in the proofs by hx. Thus in place of (II), (III), (V), (VI) we get

$$\text{(II')}\quad f(x+Bh+h)-f(x+Bh)=hf'(x),$$
$$\text{(III')}\quad f(x+Eh+h)+f(x+Eh-h)=2f(x),$$
$$\text{(V')}\quad f(x+Gh+h)+f(x+Gh)=2hf'(x),$$
$$\text{(VI')}\quad f(x+Rh+h)-f(x+Rh-h)=hf'(x).$$

The umbral form of Taylor's theorem being $f(x+h)=e^{hf(x)}$, in which $f^n(x)$ in the development of the right by the exponential theorem is $\frac{d^n}{dx^n}f(x)$, we see that (II′)–(VI′) are equivalent to the following, given by Lucas, *loc. cit.*, p. 263:

$$e^{Bhf(x+h)} - e^{Bhf(x)} = hf'(x),$$

$$e^{Ehf(x+h)} + e^{Ehf(x-h)} = 2f'(x),$$

$$e^{Ghf(x+h)} + e^{Ghf(x)} = 2hf'(x),$$

$$e^{Rhf(x+h)} - e^{Rf(x-h)} = hf'(x).$$

The first, according to Lucas, is the symbolic form of an expansion first given by Euler, the third of another due to Stirling and Boole. There is no difficulty in writing out the remaining 18 corresponding to the other relations, but to save space we omit them.

9. *Equivalents of Trigonometric Identities, Extensions.* If in the multinomial theorem of § 1 precisely s umbrae are each equal to the umbra a, they are replaced by s distinct umbrae until after the degradation of exponents, when each of the s distinct letters is replaced by a. For example $(a-a)^2$ is obtained from $(a-b)^2$, that is from $a^2b^0 - 2a^1b^1 + a^0b^2$ or $a_2b_0 - 2a_1b_1 + a_0b_2$, and hence its value is $2(a_2a_0 - a_1^2)$. This remark gives us the correct interpretation of (5)–(12) when $b = a$, and of the generalizations of these results to any number of umbrae not necessarily all distinct. For uniformity with (1)–(4) we introduce two further systems of numbers, S, T, through the definitions

$$S_{2n} = (-1)^{n-1}2n, \quad T_{2n} = 1, \quad S_{2n+1} = T_{2n+1} = 0 \quad (n \geq 0).$$

and we have

$$x \sin x = \cos Sx, \quad \cos x = \cos Tx.$$

Consider an ordinary trigonometric identity

$$(28) \quad F(\sin x, \cos x, \tan x, \cot x, \sec x, \csc x) \equiv 0,$$

in which F is a rational integral function of all its arguments, and of such a sort that on multiplying this throughout by the appropriate power of x it is reduced to the form

$$F_1(x \sin x, \cos x, x \tan x, x \cot x, \sec x, x \csc x) \equiv 0.$$

We then have the umbral identity

$$(29)\quad F_1(\cos Sx,\ \cos Tx,\ \tfrac{1}{2}\cos 2Gx,\ \cos 2Bx,\ \cos Ex,\ 2\cos Rx) \equiv 0.$$

Each umbral term in the last is of the form

$$\cos^\alpha ax\cos^\beta bx\ldots\cos^\gamma cx,$$

where α, β, ..., γ are integers > 0, and a, b, ..., c umbrae as usual. Since by the remark at the beginning of this section the case where at least one of α, β, ..., γ exceeds 1 is reducible to the case in which none exceeds 1, we need consider only $\alpha=\beta=\ldots=\gamma=1$. Defining for $n \geq 0$ the ϕ_n-function of r umbrae a, b, ..., c by

$$(30)\qquad \phi_n(a, b, \ldots, c) = \Sigma(a \pm b \pm \ldots \pm c)^n,$$

we see by repeated application of (5) that

$$2^{r-1}\cos ax\cos bx\ldots\cos cx = \cos\phi\,(a, b, \ldots, c)\,x.$$

Any trigonometric identity being reducible to the form (28), it follows from (29), (30) that such an identity is equivalent to a rational integral relation between ϕ_n-functions whose arguments are chosen from among S, T, $2G$, $2B$, E, R, some of which may be repeated. From the mode of derivation it is clear that this relation between S, T, ..., R implies (28) and is implied by (28), that is the two are formally equivalent. As in the preceding sections the ϕ_n-relation can be replaced by an f-relation, and in the argument of each f there may be one ordinary, x, or two ordinaries, x, h.

Although we have not had occasion to use it in this paper we may add for completeness that all the umbral functions necessary for a full discussion are special cases of

$$\chi_n(a, b, \ldots, c, p, q, \ldots, r) \equiv \Sigma \pm (a \pm b \pm \ldots \pm c \pm p \pm q \pm \ldots \pm r),$$

in which the Σ extends to all possible combinations of signs within the parentheses, and the outer sign is in each case the product of the signs of p, q, ..., r. When p, q, ..., r are absent, χ_n reduces to ϕ_n; when a, b, ..., c are absent, χ_n becomes ψ_n, the generalization of the ψ-function in § 3. It is interesting to note that if (and only if) we take the *special cases* of these ϕ_n, ψ_n, in which *all of the letters a, b, ..., c, p, q, ..., r are interpreted as ordinaries*, we have precisely the symmetric functions which Kronecker took as his point of departure for obtaining properties of the Bernoulli numbers.*

We need not trace the connection farther here, as it has been fully discussed in the extensions next indicated.

The Bernoulli and Euler numbers can be generalized in several ways. Blissard, in the papers cited, gave an arithmetical generalization. The two following are wholly distinct from this. First, we may apply the processes of § 3 as extended above to the polynomials in the modulus k^2 that appear as coefficients in the power series developments of $x\,\mathrm{sn}(x, k)$, $\mathrm{cn}(x, k)$, $\mathrm{dn}(x, k)$ and in the developments of the reciprocals and quotients of these functions. For $k=\pm 1$ the polynomials degenerate to B, E, G, R, or to numbers simply dependent on these. The relations between the polynomials are of remarkable simplicity; for unit values of k they degenerate to the relations of the present paper.

In the foregoing generalization polynomials in *one* variable replace B, E, G, R. But in devising a self-consistent trigonometry of *umbral* tangents, cotangents, secants, and cosecants (including incidentally sines and cosines), we are led naturally to a generalization in which B_n, E_n, G_n, R_n are replaced by polynomials in n' independent variables, where n' depends simply upon n. For unit values $(+1)$ of all the variables these polynomials degenerate to B_n, E_n, G_n, R_n, and the relations between them become those of this paper. This generalization is equivalent to the algebra of formal operations upon infinite series, and is best considered from the standpoint of the umbral hyperbolic functions. Both it and the preceding have been fully discussed in papers which I hope to publish; they are mentioned here because it is interesting that the circular, hyperbolic and elliptic functions, which in a sense are the elementary functions of analysis and which are so closely interwoven in many respects, should again be interconnected through the Bernoulli and allied numbers. When we pass beyond functions of one variable or of two periods we apparently leave these numbers and must invent others.

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* *Journal des Mathématiques* (2), vol. i. (1856), p. 385; *Journal für die r.u.a Mathematik*, vol. xciv. (1882), p. 268.

NOTE ON THE EXISTENCE OF ABEL'S LIMIT.

By *B. M. Wilson.*

§ 1. If $\Sigma a_n x^n$ is a power-series whose radius of convergence is unity, the most that can be deduced concerning the order of the coefficients a_n is that

$$(1) \qquad a_n = O(e^{\epsilon n}),$$

for every positive ϵ. In this note it is shown that no better asymptotic equation can be inferred even though it be known that, in addition, Abel's limit exists for the series as x approaches (either along the radius, or, more generally, along any "Stolz-path")* a point on the circle of convergence, or, in fact, that the limit exists for approach to every point of the circle with but one exception. In the latter case we shall suppose the exceptional point to be -1. We show therefore that, if $\phi(n)$ is any function of n which increases steadily to infinity with n, a series $\Sigma a_n x^n$ exists whose radius of convergence is unity and for which Abel's limit exists (in any of the wider or narrower senses mentioned above), and also

$$(2) \qquad |a_n| > e^{n/\phi(n)}$$

for all sufficiently large values of n. A power-series for which Abel's limit exists and

$$(3) \qquad |a_n| > \frac{n^k}{(k!)^2},$$

for all values of k, has been constructed by Bohr, and was used by him to show that the summability (C, k) of Σa_n,† known to be a sufficient condition for the existence of Abel's limit as x approaches 1 along any Stolz-path, is not also a necessary one. It is easy to modify Bohr's example so as to obtain the stronger inequality (2): for this we need to obtain a lower limit for the order, as z tends to infinity along the positive real axis, of an integral function of z of regular growth (*croissance*).

* i.e. along any path which lies entirely between two chords of the unit-circle.

† Bohr's example has been reproduced by Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (Berlin, 1916), pp. 38–39.

F 2

§ 2. *Let $\phi(n)$ be a real, positive, monotone increasing function of the real, positive variable n such that*

$$\phi(n) > 1, \quad \phi(n) \to \infty \quad as \quad n \to \infty,$$

$$0 < n\frac{\phi'(n)}{\phi(n)} < c;$$

and denote by $F(z)$ the integral function

$$F(z) = \Sigma \frac{z^n}{\{n\phi(n)\}^n}.$$

Then, as z increases by real positive values

$$F(z) > e^{kz/\phi(z)},$$

where k is a positive constant.

For if n is regarded as a continuous variable, and z as a constant, the function $\left\{\frac{z}{n\phi(n)}\right\}^n$ has its maximum value when

$$\log\left\{\frac{z}{n\phi(n)}\right\} = 1 + n\frac{\phi'(n)}{\phi(n)},$$

so that for the maximum value

$$e < \frac{z}{n\phi(n)} < k_1 e.$$

Thus, if the equations

$$(4) \qquad y = k_1 e m\phi(m), \quad m = \omega(y)$$

are equivalent to one another, it is seen that the maximum of $\left\{\frac{z}{n\phi(n)}\right\}^n$ exceeds $e^{\omega(z)}$. But, from (4),

$$\omega(y) = m = \frac{y}{k_1 e\phi(m)} > \frac{y}{k_1 e\phi(y)},$$

since $m < y$. We therefore have at once the result stated in the theorem.

§ 3. Let now $\phi(n)$ be any function of n satisfying the conditions postulated above, and let k be as before. Write

$$g(z) = \Sigma\left\{\frac{k}{\phi(n)}\right\}^n z^n,$$

so that $g(z)$ is an integral function of z; write also

$$f_m(x) = (1+x)^{-m-1} = \sum_{n=0}^{\infty} (-1)^n \binom{n+m}{m} x^n, \quad (m = 0, 1, 2 \ldots),$$

$$f(x) = \sum_{m=0}^{\infty} \left\{ \frac{k}{\phi(m)} \right\}^m f_m(x).$$

It is easily shown* that $f(x)$ is analytic for $|x| < 1$, and that, as x tends along any Stolz-path to a point x_0 on the unit-circle, Abel's limit

$$(5) \quad \lim_{x \to x_0} f(x) = \frac{1}{1+x_0}\, g\left(\frac{1}{1+x_0}\right), \quad (|x_0| = 1;\ x_0 \neq -1)$$

exists. On the other hand we have, for $|x| < 1$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where
$$(-1)^n a_n = \sum_{m=0}^{\infty} \left\{ \frac{k}{\phi(m)} \right\}^m \binom{n+m}{m};$$

and therefore

$$(2\ bis) \qquad |a_n| > \sum_{m=0}^{\infty} \left\{ \frac{k}{m\phi(m)} \right\}^m n^m > e^{n\,\phi(n)},$$

in virtue of § 2.

SOME PROBLEMS IN POTENTIAL THEORY.

By *Dr. H. Bateman.*

§ 1. In a previous note† it was shown that the potential of a surface of revolution, whose meridian curve is a limaçon, can be expressed in the form

$$V = (\cosh\sigma - \cos\chi) \sum_{n=0}^{\infty} (2n+1) \frac{P_n(\cosh\sigma)}{P_n(\cosh\sigma_0)}\, Q_n(\cosh\sigma_0)\, P_n(\cos\chi).$$

* See Landau, *loc. cit.*

† *Messenger of Mathematics*, vol. li. (February, 1922), p. 151

the potential being unity over the surface $\sigma=\sigma_0$, where

$$\left(\frac{R+X}{2}\right)^{\frac{1}{2}}=\frac{a\sinh\sigma}{\cosh\sigma-\cos\chi},\quad \left(\frac{R-X}{2}\right)^{\frac{1}{2}}=\frac{a\sin\chi}{\cosh\sigma-\cos\chi}.$$

To find the capacity of the surface we must determine the form of V at infinity, *i.e.* in the neighbourhood of $\sigma=0$, $\chi=0$. Writing

$$P_n(\cosh\sigma)=1+\frac{n(n+1)}{2}(\cosh\sigma-1)+\frac{(n-1)n(n+1)(n+2)}{1^2.2^2}\left(\frac{\cosh\sigma-1}{2}\right)^2...,$$

$$P_n(\cos\chi)=1+\frac{n(n+1)}{2}(\cos\chi-1)+\frac{(n-1)n(n+1)(n+2)}{1^2.2^2}\left(\frac{\cos\chi-1}{2}\right)^2+...,$$

$$R=a^2\frac{\cosh\sigma+\cos\chi}{\cosh\sigma-\cos\chi}\sim a^2\frac{2}{\cosh\sigma-\cos\chi},$$

$$X=a^2\frac{\cosh^2\sigma+\cos^2\chi-2}{(\cosh\sigma-\cos\chi)^2}\sim 2a^2\frac{\cosh\sigma+\cos\chi-2}{(\cosh\sigma-\cos\chi)^2},$$

we find that

$$V=\frac{2a^2}{R}\sum_{n=0}^{\infty}(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)}+\frac{2a^4X}{R^3}\sum_{n=0}^{\infty}n(n+1)(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)}$$
$$+\ldots\ldots.$$

The first term gives an expression for the capacity C, viz.,

$$C=2a^2\sum_{n=0}^{\infty}(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)},$$

while the second term enables us to determine a point where the charge C should be placed in order that its potential may agree with V at infinity up to terms of the second order in $\frac{1}{R}$. This point may be called the centre of charge.

To find the polar equation of the limaçon we write

$$r=\frac{2a^2}{\cosh\sigma_0-\cos\chi_0},\quad \cos\theta=\frac{\cosh\sigma_0\cos\chi_0-1}{\cosh\sigma_0-\cos\chi_0},\quad \sin\theta=\frac{\sinh\sigma_0\sin\chi_0}{\cosh\sigma_0-\cos\chi_0},$$

then $\quad X=a^2+r\cos\theta,\quad Y=\sqrt{(R^2-X^2)}=r\sin\theta,$

and
$$r=\frac{2a^2}{\sinh^2\sigma_0}(\cosh\sigma_0+\cos\theta).$$

The area of the surface generated by the revolution of the limaçon about its axis of symmetry is $4\pi k^2$, where

$$k=2a^2\operatorname{cosech}^2\sigma_0(\cosh^2\sigma_0+\tfrac{2}{3})^{\frac{1}{2}}.$$

With the aid of tables for $Q_n(\cosh\sigma_0)$ and $P_n(\cosh\sigma_0)$ we find that

$\cosh\sigma_0$	$C/2a^2$	$k/2a^2$
2	.718695	.722009
1.2	3.25824	3.29872

In the case of a sphere ($\cosh\sigma_0=\infty$) we have, of course, $C=k$.

Of all surfaces of given area the sphere has apparently the greatest capacity. When $\cosh\sigma_0=2$ the limaçon has a point of undulation on the axis of symmetry, the points of contact of the double tangent being consecutive. The value of C in this case differs from k by about 1 part in 200. When $\cosh\sigma_0=1.2$ the double tangent touches the limaçon in two distinct real points, and the curve bends inwards near the vertex. The capacity is slightly reduced by this hollow, C differing from k by about 1 part in 80.

§ 2. Since the author does not remember having seen any tables of spheroidal harmonics, the values of P_n, Q_n and their first derivatives are given* for a few values of $\cosh\sigma$.

$s=\cosh\sigma=1\cdot1$

n	$P_n(s)$	$Q_n(s)$
0	1	1.52226 12188
1	1.1	.67448 73407
2	1.315	.35177 35028
3	1.6775	.19525 98613
4	2.24293 75	.11204 51059
5	3 09901 625	.06564 14207
6	4.38056 81875	.03900 59434
7	6.29257 53687	.02341 94953
8	9.14543 95340	.01417 25085
9	13.40879 07039	.00862 99941
10	19.79347 69907	.00528 14300
11	29.37649 19495	.00324 55538
12	43.79141 66188	.00200 13984
13	65.51892 72018	.00123 78316
14	98.33026 58463	.00076 75299
15	147.96469 99781	.00047 69708
16	223.16514 25975	.00029 69847
17	337.26232 21552	.00018 52360
18	510.59955 43788	.00011 57137
19	774.24631 91802	.00007 23842
20	1175.68877 79816	.00004 53361

* In calculating these values use has been made of the values of $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, and $\log_e 10$, given by J. C. Adams, *Proc. Roy. Soc. London*, vol. xxvii. (1878), p. 88.

$s = \cosh \sigma = 1.1$

n	$P_n'(s)$	$Q_n'(s)$
0	0	−4.76190 47619
1	1	−3.71583 40193
2	3.3	−2.73844 27398
3	7.575	−1.95696 65053
4	15.0425	−1.37162 37107
5	27.76143 75	−0.94856 05522
6	49.13167 875	−0.64956 80830
7	84.70882 39375	−0.44148 32880
8	143.52030 92812	−0.29827 56535
9	240.18129 60152	−0.20055 06435
10	398.28733 26562	−0.13430 57656
11	655.84431 28191	−0.08964 06135
12	1073.94664 74947	−0.05965 80282
13	1750.62972 82891	−0.03960 56535
14	2842.95768 19433	−0.02623 65750
15	4602.20743 78318	−0.01734 72864
16	7429.86338 12644	−0.01145 04802
17	11966.65714 35493	−0.00754 67913
18	19234.04465 66964	−0.00496 72202
19	30858.84065 55649	−0.00326 53844
20	49429.65110 47242	−0.00214 42364

Since these values were calculated with the aid of the difference relations

$$P'_{n+1} - P'_{n-1} = (2n+1)\, P_n,$$
$$Q'_{n+1} - Q'_{n-1} = (2n+1)\, Q_n,$$

the last two or three figures in the above numbers are doubtful when n is large. The difference relations

$$P'_n - sP'_{n-1} = nP_{n-1}, \quad Q'_n - s\,Q'_{n-1} = n\,Q_{n-1}$$

are, however, satisfied to 9 decimal places when $n = 20$, so the last figure may be the only one which is wrong.

$s = \cosh \sigma = 1.2$

n	$P_n(s)$	$P_n'(s)$	$Q_n(s)$	$Q_n'(s)$
0	1	0	1.19894 76364	−2.27272 72727
1	1.2	1	.43873 71637	−1.52832 50908
2	1.66	3.6	.19025 30764	−0.95651 57816
3	2.52	9.3	.08801 47104	−0.57705 97088
4	4.047	21.24	.04214 10845	−0.34041 28088
5	6.72552	45.723	.02061 29742	−0.19778 99483
6	11.423644	95.22072	.01023 09729	−0.11367 00926
7	19.6936752	194.230372	.00513 21902	−0.06478 73006
8	34.3150807	390.625848	.00259 53267	−0.03668 72396
9	60.27536052	777.5867439	.00132 07936	−0.02066 67467
10	106.5442493556	1535.85769788	.00067 56155	−0.01159 21612

$s = \cosh\sigma = 2$

0	1	0	.54930 61443	−0.33333 33333
1	2	1	.09861 22886	−0 11736 05223
2	5.5	6	.02118 37938	−0.03749 64673
3	17	28.5	.00487 11203	−0 01144 15531
4	55.375	125	.00116 10758	−0.00339 86249
5	185.75	526.875	.00028 29767	−0.00099 18706
6	634.9375	2168.25	.00007 00180	−0.00028 58810
7	2199.125	8781.0625	.00001 75157	−0.00008 16355
8	7691.1484375	35155.125	.00000 44181	−0.00002 31451
9	27100.671875	139530.5859375	·00000 11212	−0.00000 65271
10	96060.51953125	550067.890625	.00000 02843	−0.00000 18419

$s = \cosh\sigma = 3$

0	1	0	.34657 35903	−0.125
1	3	1	.03972 07708	−0.02842 64097
2	13	9	.00545 66736	−0.00583 76874
3	63	66	.00080 28543	−0.00114 30415
4	321	450	.00012 24799	−0.00021 77073
5	1683	2955	.00001 91079	−0.00004 07227
6	8989	18963	.00000 30267	−0.00000 75209
7	48639	119812	.00000 04847	−0.00000 13759
8	265729	748548	.00000 00783	−0.00000 02499
9	1462563	4637205	.00000 00127	−0.00000 00451
10	8097453	28537245	.00000 00021	−0.00000 00081

§ 3. To obtain a potential function V which satisfies the condition

$$\frac{\partial V}{\partial N} = U \frac{\partial X}{\partial N}$$

over the surface $\sigma = \sigma_0$, we assume for points outside the body

$$V = a^2 U (\cosh\sigma - \cos\chi) \sum_{m=0}^{\infty} (2m+1) A_m P_m(\cosh\sigma) P_m(\cos\chi)$$

$$= a^2 U \sum_{m=0}^{\infty} (m+1)(A_{m+1} - A_m) \times \{P_m(\cosh\sigma) P_{m+1}(\cos\chi) - P_{m+1}(\cosh\sigma) P_m(\cos\chi)\}.$$

Now

$$X = a^2 \frac{\sinh^2\sigma - \sin^2\chi}{(\cosh\sigma - \cos\chi)^2}$$

$$= a^2 (\cosh\sigma - \cos\chi) \sum_{m=0}^{\infty} (2m+1)[m(m+1)+1] Q_m(\cosh\sigma) P_m(\cos\chi)$$

$$= a^2 + 2a^2 \sum_{m=0}^{\infty} (m+1)^2 \{Q_m(\cosh\sigma) P_{m+1}(\cos\chi) - Q_{m+1}(\cosh\sigma) P_m(\cos\chi)\};$$

hence the boundary condition at $\sigma = \sigma_0$ will be satisfied for all values of χ if

$$U \sum_{m=0}^{\infty} (m+1)(A_{m+1} - A_m)$$
$$\times \{P'_m(\cosh\sigma_0) P_{m+1}(\cos\chi) - P'_{m+1}(\cosh\sigma_0) P_m(\cos\chi)\}$$
$$= 2U\Sigma (m+1)^2 \{Q'_m(\cosh\sigma_0) P_{m+1}(\cos\chi) - Q'_{m+1}(\cosh\sigma_0) P_m(\cos\chi)\}.$$

This leads to the system of equations

$$m(A_m - A_{m-1}) P'_{m-1}(\cosh\sigma_0) - (m+1)(A_{m+1} - A_m) P'_{m+1}(\cosh\sigma_0)$$
$$= 2m^2 Q'_{m-1}(\cosh\sigma_0) - 2(m+1)^2 Q'_{m+1}(\cosh\sigma_0).$$

The left-hand side of the typical equation becomes a perfect difference when multiplied by $P'_m(\cosh\sigma_0)$, while the right-hand side may be transformed with the aid of the identity

$$Q'_m(\cosh\sigma_0) P'_{m-1}(\cosh\sigma_0) - Q'_{m-1}(\cosh\sigma_0) P'_m(\cosh\sigma_0) = m \operatorname{cosech}^2\sigma_0.$$

Consequently the typical equation may be written in the form

$$m(A_m - A_{m-1}) P'_m(\cosh\sigma_0) P'_{m-1}(\cosh\sigma_0)$$
$$-(m+1)(A_{m+1} - A_m) P'_m(\cosh\sigma_0) P'_{m+1}(\cosh\sigma_0)$$
$$= 2m^2 Q'_m(\cosh\sigma_0) P'_{m-1}(\cosh\sigma_0)$$
$$-2(m+1)^2 Q'_{m+1}(\cosh\sigma_0) P'_m(\cosh\sigma_0) - 2m^3 \operatorname{cosech}^2\sigma_0.$$

Summing from $m = 1$ to $m = n$, we get

$$(n+1)(A_{n+1} - A_n) P'_n(\cosh\sigma_0) P'_{n+1}(\cosh\sigma_0)$$
$$= 2(m+1)^2 Q'_{m+1}(\cosh\sigma_0) P'_n(\cosh\sigma_0) + \frac{n^2(n+1)^2}{2\sinh^2\sigma_0},$$

therefore

$$A_{m+1} - A_m = 2(m+1)\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}$$
$$+ \frac{m^2}{2}\left[\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - \frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right]$$
$$= \tfrac{1}{2}(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - \tfrac{1}{2}m^2 \frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}.$$

Hence finally we obtain the following expression for V

$$V = \tfrac{1}{2}a^2 U \Sigma (m+1)\left[(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - m^2 \frac{Q_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right]$$
$$\times [P_m(\cosh\sigma) P_{m+1}(\cos\chi) - P_{m+1}(\cosh\sigma) P_m(\cos\chi)].$$

We may deduce from this expression the form which Φ takes at infinity by writing for small values of σ and χ the expansions for $P_m(\cosh\sigma)$ and $P_m(\cos\chi)$ used before. The coefficient of $\cosh\sigma - \cos\chi$ is then

$$a^2 U \sum_{m=0}^{\infty} (m+1)^2 \left[(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - m^2 \frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)} \right],$$

and this is zero. The most important term in the expansion is thus

$$\tfrac{1}{8} a^2 U (\cosh\sigma - \cos\chi)(\cosh\sigma + \cos\chi - 2) \sum_{m=0}^{\infty} m(m+1)^2(m+2)$$
$$\times \left[(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - m^2 \frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)} \right].$$

Now

$$\frac{X}{R^3} = \frac{1}{a^4} \frac{\cosh^2\sigma + \cos^2\chi - 2}{(\cosh\sigma - \cos\chi)^2} \cdot \frac{(\cosh\sigma + \cos\chi)^3}{(\cosh\sigma + \cos\chi)^3}$$
$$= \frac{1}{a^4} \frac{(\cosh\sigma + \cos\chi - 2)(\cosh\sigma + \cos\chi) - 2(\cosh\sigma - 1)(\cos\chi - 1)}{(\cosh\sigma + \cos\chi)^3} \times (\cosh\sigma - \cos\chi)$$
$$= \frac{1}{4a^4}(\cosh\sigma - \cos\chi)(\cosh\sigma + \cos\chi - 2)$$
$$+ \text{ terms of the 3rd and higher orders;}$$

hence the most important part of the expansion is equal to

$$\tfrac{1}{2} a^6 U \frac{X}{R^3} \sum_{m=0}^{\infty} m(m+1)^2(m+2)$$
$$\times \left[(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - m^2 \frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)} \right]$$
$$= -\tfrac{1}{2} a^6 U \frac{X}{R^3} \sum_{m=0}^{\infty} (2m+3)(m+1)^2(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}.$$

This gives the moment of the doublet whose potential is a first approximation to the value of V at infinity. The apparent mass of the fluid may be found by means of a theorem due to Munk,* and is

$$\rho B \left[\frac{2\pi a^6}{B} \sum_{m=0}^{\infty} (2m+3)(m+1)^2(m+2)^2 \frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)} - 1 \right],$$

* "Notes on aerodynamic forces, Technical Note No. 104, National Advisory Committee for Aeronautics", Washington, July, 1922.

where B is the volume of the fluid displaced by the solid, a ρ the density of the fluid. Since

$$B = \frac{4\pi}{3} \cdot 8a^6 \frac{\cosh^3\sigma_0}{\sinh^6\sigma_0} \left[1 + \frac{1}{\cosh^2\sigma_0}\right],$$

we find that the apparent mass is $k\rho B$, where $k = .5$ for the sphere. When

$\cosh\sigma_0 = 1.2$ we find $k = .5688$,

$\cosh\sigma_0 = 2$,, $k = .548$,

$\cosh\sigma_0 = 3$,, $k = .527$.

A GENERAL FORM OF THE REMAINDER IN TAYLOR'S THEOREM.

By *G. S. Mahajani*, St. John's College, Cambridge.

1. An examination of the various extant accounts of Taylor's theorem reveals that, for the most part, they obtain the particular form of the remainder with which they happen to be concerned by utilising what we may call the *simple* form of the mean value theorem, which states that if $f(x)$ is continuous in the interval (a, b), end points included, and differentiable in the same interval, end points not necessarily included, then

$$f(b) - f(a) = (b - a) f'(\xi),$$

where ξ is some number between a and b and not coinciding with either.

Now it is well known that the mean value theorem can be expressed in a form more general than the above. If $\phi(x)$ satisfies the same conditions as $f(x)$ and, in addition, is such that $\phi'(x)$ does not vanish anywhere in (a, b), then

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)},$$

where ξ, not necessarily the same as before, lies between a and b and does not coincide with either of them.

We propose to show that, by utilising this more general form of the mean value theorem, we can obtain an extremely general form of the remainder in Taylor's theorem.

2. We suppose that $f(x)$ satisfies the strict conditions of order $n+1$ at a, being such that it and its first $n+1$ derivatives exist in some neighbourhood of a; and that $\phi(x)$ satisfies

the conditions of order $p+1$ at a. Further, we suppose that $\phi^{p+1}(x)$ does not vanish.

3. Let

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^n}{n!}f^n(a)+R_n,$$

so that R_n is the usual remainder. Evidently

$$R_n=f(a+h)-f(a)-hf'(a)-\ldots-\frac{h^n}{n!}f^n(a)\ldots.(1).$$

4. Write now

$$\psi(x)=f(a+h)-f(x)-(a+h-x)f'(x)-\ldots-\frac{(a+h-x)^n}{n!}f^n(x)\quad\ldots\ldots(2),$$

$$\chi(x)=\phi(a+h)-\phi(x)-(a+h-x)\phi'(x)-\ldots-\frac{(a+h-x)^p}{p!}\phi^p(x)\quad\ldots\ldots(3).$$

Then, as is easily seen,

$$\psi'(x)=-\frac{(a+h-x)^n}{n!}f^{n+1}(x),$$

$$\chi'(x)=-\frac{(a+h-x)^p}{p!}\phi^{p+1}(x).$$

5. By the mean value theorem in its general form,

$$\frac{\psi(a+h)-\psi(a)}{\chi(a+h)-\chi(a)}=\frac{\psi'(\xi)}{\chi'(\xi)},$$

where ξ lies between a and $a+h$ and coincides with neither.

In the usual way we have

$$\xi=a+\theta h,$$

where $$0<\theta<1.$$

Further, as is easily seen,

$$\psi(a+h)=\chi(a+h)=0.$$

Thus

$$\frac{\psi(a)}{\chi(a)}=\frac{\psi'(a+\theta h)}{\chi'(a+\theta h)}=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}.$$

6. But (1) and (2) give at once $\psi(a)=R_n$. Thus

$$R_n=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}\chi(a)$$

$$=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}\left\{\phi(a+h)-\phi(a)-\ldots-\frac{h^p}{p!}\phi^p(a)\right\}.$$

This is the form of the remainder we set out to obtain.

7. The above form of the remainder contains as special cases all the hitherto recognised forms, including those of Lagrange, Cauchy and Schlömilch; and, in addition, other forms of which we give a specimen.

8. Take any integer q not less than p and put

$$\phi(x) = (x-a)^{q+1}.$$

Then
$$\phi(a) = \phi'(a) = \ldots = \phi^p(a) = 0,$$
$$\phi(a+h) = h^{q+1},$$
$$\phi^{p+1}(a+\theta h) = (q+1)q\ldots(q-p+1)(\theta h)^{q-p},$$

and the remainder takes the form

$$\text{(A)} \qquad R_n = \frac{p!}{n!}\,\frac{(1-\theta)^{n-p}}{\theta^{q-p}}\,\frac{h^{n+1}f^{n+1}(a+\theta h)}{(q+1)q\ldots(q-p+1)}.$$

9. From this form, by taking special values of p and q, we can deduce the respective forms of Lagrange, Cauchy and Schlömilch.

Put $q=p=n$, and we obtain

$$\text{(B)} \qquad R_n = \frac{h^{n+1}f^{n+1}(a+\theta h)}{(n+1)!},$$

which is *Lagrange's form*.

Put $q=p=0$, and we obtain

$$\text{(C)} \qquad R_n = \frac{(1-\theta)^n h^{n+1} f^{n+1}(a+\theta h)}{n!},$$

which is *Cauchy's form*.

Put $q=p$, and we obtain

$$\text{(D)} \qquad R_n = \frac{(1-\theta)^{n-p} h^{n+1} f^{n+1}(a+\theta h)}{(p+1)!\,n!},$$

which is *Schlömilch's form*. This last, of course, includes the first two.

10. Put $q=p-\frac{1}{2}$, and we obtain, after a little reduction

$$\text{(E)} \qquad R_n = \frac{p!}{n!}\,\frac{\theta^{\frac{1}{2}}(1-\theta)^{n-p}h^{n+1}f^{n+1}(a+\theta h)}{1.3.5\ldots(2p+1)}.2^{p+1}.$$

This is a new form.

THE TRANSFORMATION OF THE ELLIPTIC FUNCTION OF THE SEVENTH ORDER.

By *Sir G. Greenhill.*

THE Transformation of the Third and Fifth Order is completed in Jacobi's *Fundamenta nova*, and the extension of his method to the Seventh Order is sketched by Cayley,* but not carried out to a finish there or in his other memoirs.† Then there is a recent article by A. Berry ‡ on the Septimic Transformation. And the general theory of Transformation has been discussed by Prof. J. H. McDonald.§ The algebraical work is completed in the present note, and connected up with the corresponding formulas of Kiepert || and Joubert.¶

1. The object is to determine the constants α, β, γ in the relation

$$(1)\qquad \frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\alpha x+\beta x^2-\gamma x^3}{1+\alpha x+\beta x^2+\gamma x^3}\right)^2$$

or

$$y=\frac{(1+2\alpha)x+(2\beta+2\gamma+\alpha^2+2\alpha\beta)x^3+(\beta^2+2\beta\gamma+2\alpha\gamma)x^5+\gamma^2x^7}{1+(2\alpha+2\beta+\alpha^2)x^2+(2\gamma+\beta^2+2\alpha\beta+2\alpha\gamma)x^4+(2\beta\gamma+\gamma^2)x^6},$$

connecting $x=\sin\phi=\operatorname{sn}(u,\kappa)$ and $y=\sin\Phi=\operatorname{sn}\left(\frac{u}{M},\lambda\right)$, satisfying the differential relation

$$(2)\qquad \frac{Md\Phi}{\sqrt{(1-\lambda^2\sin^2\Phi)}}=\frac{d\phi}{\sqrt{(1-\kappa^2\sin^2\phi)}},$$

or

$$\frac{Mdy}{\sqrt{(1-y^2.1-\lambda^2y^2)}}=\frac{dx}{\sqrt{(1-x^2.1-\kappa^2x^2)}}=du,$$

and to express κ, λ as well as α, β, γ in terms of a parameter, θ.

Then, as Cayley shows in his *Elliptic Functions*, the conditions to be satisfied in order that x, y should change into $\frac{1}{\kappa x}$, $\frac{1}{\lambda y}$ in (1) are, with $\gamma^2=\Omega$,

$$(3)\qquad \kappa^2(\beta^2+2\beta\gamma+2\alpha\gamma)=\Omega(2\alpha+2\beta+\alpha^2),$$

$$(4)\qquad \kappa^4(2\beta+2\gamma+\alpha^2+2\alpha\beta)=\Omega(2\gamma+\beta^2+2\alpha\beta+2\alpha\gamma),$$

$$(5)\qquad \kappa^6(1+2\alpha)=\Omega(2\beta\gamma+\gamma^2),$$

* *Elliptic Functions* (1876), chap. viii, p. 192.
† *Phil. Trans* (1894–98).
‡ *Messenger of Mathematics* (April, 1921).
§ *Bulletin A.M.S.* (May, 1921), p. 366.
|| *Math. Ann.* (1885–87). ¶ *Comptes rendus* (1858).

where $\Omega^2 = \frac{\kappa^7}{\lambda}$; and writing $\kappa = u^4$, $\lambda = v^4$, we have

$$\Omega = \frac{u^{14}}{v^4}, \quad \gamma = \sqrt{\Omega} = \frac{u^7}{v}, \quad \frac{\kappa^2}{\gamma} = uv = \theta,$$

suppose.

The relations (3), (4), (5) may be rewritten

$$\text{(I)}\quad \theta = \frac{2\alpha\gamma + 2\beta\gamma + \alpha^2\gamma}{\beta^2 + 2\beta\gamma + 2\alpha\gamma}, \qquad \text{(II)}\quad \theta^2 = \frac{2\gamma + \beta^2 + 2\alpha\beta + 2\alpha\gamma}{2\beta + 2\gamma + \alpha^2 + 2\alpha\beta},$$

$$\text{(III)}\quad \theta^3 = \frac{2\beta + \gamma}{1 + 2\alpha}.$$

From Cayley's equation on his p. 193, and from (III),

$$(6)\qquad \frac{1}{1+2\alpha} = M = \frac{v(1-uv)(1-uv+u^2v^2)}{v-u^7} = \frac{1-\theta}{1+\theta}\cdot\frac{1+\theta^3}{1-\gamma},$$

$$\text{(IV)}\quad \frac{1-\theta}{1+\theta} = \frac{M(1-\gamma)}{1+\theta^3} = \frac{1-\gamma}{1+2\alpha+2\beta+\gamma}, \quad \theta = \frac{\alpha+\beta+\gamma}{\alpha+\beta+1};$$

and the elimination of θ between (I) and (II), or more simply between (I) and (IV), leads to the relation, after division by $\alpha + \beta$,

$$(7)\qquad (2\beta+\gamma)^2 - \gamma(1+2\alpha)^2 - 7\gamma(1-\gamma) = 0,$$

$$\frac{\theta^6}{\gamma} - 1 - 7(1-\gamma)M^2 = 0,$$

the equivalent of Cayley's expression on his p. 201,

$$(8)\qquad 7M^2 = -\frac{v(u-v^7)}{u(v-u^7)} = \frac{\frac{v^7}{u} - 1}{1 - \frac{u^7}{v}} = \frac{\frac{\theta^6}{\gamma} - 1}{1-\gamma},$$

with
$$M = \frac{1}{1+2\alpha}, \quad \theta^3 = \frac{2\beta+\gamma}{1+2\alpha},$$

2. I have to thank Major Chepmell for carrying out, with his invincible courage, the heavy algebraical work of determining α, β, γ from these equations in terms of θ as the parameter. Writing (IV),

$$(1)\qquad (1-\theta)(2\beta+\gamma) = (1+\theta)(1-\gamma) - (1-\theta)(1+2\alpha),$$

and substituting in (7), § 1,

$$(2)\quad (1-\theta)^2\gamma\,[7(1-\gamma) + (1+2\alpha)^2]$$
$$- [(1+\theta)(1-\gamma) - (1-\theta)(1+2\alpha)]^2 = 0,$$

in which $1-\gamma$ is a factor, fortunately, to be divided out, leaving

(3) $$\gamma = \frac{[1+\theta-(1-\theta)(1+2\alpha)]^2}{4(2-3\theta+2\theta^2)},$$

and then in (8), § 1,

(4) $$\theta^6 = \left[1 + \frac{7(1-\gamma)}{(1+2\alpha)^2}\right]\gamma,$$

in which

(5) $$1 + \frac{7(1-\gamma)}{(1+2\alpha)^2} = \frac{[7(1-\theta)+(1+\theta)(1+2\alpha)]^2}{4(2-3\theta+2\theta^2)(1+2\alpha)^2},$$

so that, multiplying and taking the square root of θ^6,

(6) $$\theta^3 = \frac{[7(1-\theta)+(1+\theta)(1+2\alpha)][1+\theta-(1-\theta)(1+2\alpha)]}{4(2-3\theta+2\theta^2)(1+2\alpha)},$$

a quadratic for $1+2\alpha$ in terms of θ, from which the factor $1-\theta^2$ can be divided out, leaving

(7) $$(1+2\alpha)^2 + 2(1-2\theta)(3-2\theta+2\theta^2)(1+2\alpha) - 7 = 0,$$

(8) $$7M,\ 1+2\alpha = \pm(1-2\theta)(3-2\theta+2\theta^2) + 2\sqrt{(1-\theta+2\theta^2 . 2-\theta+\theta^2 . 2-3\theta+2\theta^2)},$$

$$\alpha = -2+4\theta-3\theta^2+2\theta^3+\sqrt{(1-\theta+2\theta^2 . 2-\theta+\theta^2 . 2-3\theta+2\theta^2)},$$

$$1+2\beta = (1-2\theta+2\theta^2)(1+2\alpha);$$

and the rest follows, in terms of θ;

(9) $$\sqrt{\gamma} = (1-\theta+\theta^2)\sqrt{(2-3\theta+2\theta^2)} - (1-\theta)\sqrt{(1-\theta+2\theta^2 . 2-\theta+\theta^2)},$$

and, with $\kappa = \sqrt{\gamma}\sqrt{\theta}$, $\kappa\lambda = \theta^4$,

(10) $$\kappa,\ \lambda = \sqrt{\theta}\,[(1-\theta+\theta^2)\sqrt{(2-3\theta+2\theta^2)} \mp (1-\theta)\sqrt{(1-\theta+2\theta^2 . 2-\theta+\theta^2)}]$$
$$= \theta^2\,[(\phi-1)\sqrt{(2\phi-3)} \mp \sqrt{(\phi-2 . 2\phi^2-3\phi+2)}],$$

with $$\theta+\theta^{-1} = \phi,$$

(11) $$\kappa',\ \lambda' = \sqrt{(1-\theta)}\,[(1-\theta+\theta^2)\sqrt{(1-\theta+2\theta^2)} \pm \theta\sqrt{(2-\theta+\theta^2 . 2-3\theta+2\theta^2)}],$$

(12) $\kappa'\lambda' = (1-\theta)^4$, thence $\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = 1$,

Gutzlaff's modular equation. And

(13) $$\kappa\lambda',\ \kappa'\lambda = \sqrt{(\theta-\theta^2)}\,[\sqrt{(1-\theta+2\theta^2 . 2-3\theta+2\theta^2)} \mp (1-\theta+\theta^2)\sqrt{(2-\theta+\theta^2)}],$$

(14) $$2(\kappa\kappa')^{\frac{1}{3}},\ 2(\lambda\lambda')^{\frac{1}{3}} = (\theta-\theta^2)^{\frac{1}{6}}\,[\sqrt{(1-\theta+2\theta^2 . 2-3\theta+2\theta^2)} \mp (1-2\theta)\sqrt{(2-\theta+\theta^2)}].$$

3. In Schläfli's Modular Equations,* where u, v now denote $\left(\frac{4}{\kappa\kappa'}\right)^{\frac{1}{12}}$, $\left(\frac{4}{\lambda\lambda'}\right)^{\frac{1}{12}}$, so that $u^3v^3 = \frac{2}{\theta - \theta^2} = \frac{2}{a}$, if a denotes $\theta - \theta^2$,

$$(1)\quad B = u^3v^3 + \frac{8}{u^3v^3} = \frac{2}{a} + 4a,$$

$$(2)\quad A = \left(\frac{u}{v}\right)^4 + \left(\frac{v}{u}\right)^4 = B - 7 = \frac{2 - 7a + 4a^2}{a},$$

$$(3)\quad A - 2 = \left(\frac{u^2}{v^2} - \frac{v^2}{u^2}\right)^2 = \frac{(1-4a)(2-a)}{a},$$

$$A + 2 = \left(\frac{u^2}{v^2} + \frac{v^2}{u^2}\right)^2 = \frac{2 - 5a + 4a^2}{a},$$

$$(4)\quad \left(\frac{\kappa\kappa'}{\lambda\lambda'}\right)^{\frac{1}{6}}, \left(\frac{\lambda\lambda'}{\kappa\kappa'}\right)^{\frac{1}{6}} = \frac{v^2}{u^2},$$

$$\frac{u^2}{v^2} = \frac{\sqrt{(2 - 5a + 4a^2)} \mp \sqrt{(1 - 4a \,.\, 2 - a)}}{2\sqrt{a}}.$$

Then with x^{12}, $y^{12} = \kappa\kappa'$, $\lambda\lambda'$, and x^8, $y^8 = (\kappa\kappa')^{\frac{2}{3}}$, $(\lambda\lambda')^{\frac{2}{3}}$, $xy = a^{\frac{1}{3}}$,

$$(5)\quad 2(\kappa\kappa')^{\frac{1}{3}},\ 2(\lambda\lambda')^{\frac{1}{3}} = a^{\frac{1}{6}}\left[\sqrt{(2 - 5a + 4a^3)} \mp \sqrt{(1 - 4a \,.\, 2 - a)}\right],$$

as in (14), § 2,

$$(6)\quad x^8,\ y^8 = \tfrac{1}{2}a^{\frac{1}{3}}\left[2 - 7a + 4a^2 \mp \sqrt{(1 - 4a \,.\, 2 - a \,.\, 2 - 5a + 4a^2)}\right],$$

$$(7)\quad x^8 + y^8 = a^{\frac{1}{3}}(2 - 7a + 4a^2),$$

a form of the Modular Equation, equivalent on rationalisation to Joubert's form,† where his

$$U,\ V = (\kappa\kappa')^{\frac{1}{4}},\ (\lambda\lambda')^{\frac{1}{4}} = x^3,\ y^3.$$

With Kiepert's notation,‡

$$(8)\quad L(2)^{24} = -\frac{16}{\kappa'^2\kappa^4} = \frac{\xi_3}{\xi_4^{\ 4}},\quad \left[\frac{L(14)}{L(7)}\right]^{24} = -\frac{16}{\lambda^2\lambda'^2} = \frac{\xi_3^{\ 7}}{\xi_4^{\ 4}},$$

$$(9)\quad 4a = 4(\kappa\lambda\kappa'\lambda')^{\frac{1}{4}} = -\frac{\xi}{\xi_3} = \frac{7\xi + w}{2(\xi - 1)},$$

$$\frac{\xi_3}{\xi} = \frac{7\xi - w}{2(\xi - 8)},\quad \xi = 8\xi_4,\quad w^2 = 4\xi^3 + 13\xi^2 + 32\xi,$$

and so on; hence Kiepert's ξ and η parameters all in terms of a or θ. Thence the equation

$$(10)\quad 7\xi + w + 8a(\xi - 1) = 0,$$

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* Given in Weber's *Elliptic Functions*, p. 272.

† *Math. Ann.*, xxxii. (1887), p. 87. ‡ *Comptes rendus* (1858).

having a factor $\xi - 1$ when rationalised, and then

(11) $\xi^2 - 4(2 - 7a + 4a^2)\xi + 16a^2 = 0,$

(12) $\xi = 2(2 - 7a + 4a^2) - 2\sqrt{(1 - 4a.2 - a.2 - 5a + 4a^2)}$

$$= [\sqrt{(2 - 5a + 4a^2)} - \sqrt{(1 - 4a.2 - a)}]^2 = \frac{4(\kappa\kappa')^{\frac{2}{3}}}{a},$$

(13) $1 - \xi = -(1 - 4a)(3 - 2a) + 2\sqrt{(1 - 4a.2 - a.2 - 5a + 7a^2)}$

$$= (1 + 2a)\sqrt{(1 - 4a)},$$

(14) $1 - a = 1 - \theta + \theta^2 = \frac{1 + \theta^3}{1 + \theta} = \frac{1 + \alpha + \beta}{1 + 2\alpha},$

$$1 + 2\beta = (1 - 2a)(1 + 2\alpha).$$

In Klein's Modular Equation,

(15) $\tau = L(7)^4 = \eta_5 = \frac{\eta_1}{\xi_3^{\,2}}, \quad \text{and} \quad \sqrt{\eta} = \frac{1}{M} = 1 + 2\alpha,$

(16) $\tau = \left(\frac{1 + 2a}{\xi_3}\right)^2 = \frac{1}{1 - 4a}\left(\frac{1 - \xi}{\xi_3}\right)^2 = \frac{16a^2}{1 - 4a}\left(\frac{1}{\xi} - 1\right)^2$

$$= \left[\frac{\sqrt{(2 - \alpha, 2 - 5a + 4a^2)} \pm (2 + a)\sqrt{(1 - 4a^2)}}{2a}\right],$$

and then $\tau\tau' = 49$,

(17) $1728\, J = (\tau + \tau' + 13)(\tau^2 + 5\tau + 1)^3 = \frac{4(1 - a)^3}{a^4}(\tau^2 + 5\tau + 1)^3.$

4. As a numerical test take

(I) $\xi = 0, \ a = 0, \ \theta = 0$ or 1; and $\theta = 0, \ 2\kappa\kappa', \ 2\lambda\lambda' = 0,$

$$1 + 2\alpha = -7, \ \alpha = -4, \ \beta = -4, \ \gamma = 8,$$

$$y = -7x + 56x^2 - 112x^5 + 64x^7,$$

connecting $x = \sin\phi = \sin u$, and $y = -\sin 7\phi$.

Or with $a = 0, \ \theta = 1, \ \kappa'\lambda' = 0, \ \kappa = 1, \ \lambda = 1,$

$$1 + 2\alpha = 3 \pm 4 = 7 \text{ or } -1,$$

$$\alpha = \beta = +3 \text{ or } -1, \ \gamma = 1,$$

$$\frac{1 - y}{1 + y} = \left(\frac{1 - x}{1 + x}\right)^7 \text{ or } \frac{1 + x}{1 - x}, \ y = -x;$$

$$x = \operatorname{th} u, \ y = \operatorname{th} 7u, \text{ or } -\operatorname{th} u.$$

(II) $\xi = 1, \ \alpha = \frac{1}{4}, \ \theta = \frac{1}{2}, \ 2\kappa\kappa' = \frac{1}{8}, \ \frac{K'}{K} = \sqrt{7},$

$1 + 2\alpha = \sqrt{7}, \ 1 + 2\beta = \frac{1}{2}\sqrt{7}, \ 2\beta + \gamma = \frac{1}{8}\sqrt{7}, \ \sqrt{\gamma} = \frac{1}{4}(3 - \sqrt{7}).$

(III) $\xi = 8,\ a = 2,\ \theta = \frac{1}{2}(1 - i\sqrt{7}),\ 2\kappa\kappa',\ 2\lambda\lambda' = 8$ (in another region),

$1 + 2\alpha = i\sqrt{7},\ 1 + 2\beta = 3i\sqrt{7},\ \gamma = \frac{1}{2}(-5 - i\sqrt{7}) = [\frac{1}{2}(1 + i\sqrt{7})]^3$,

a Complex Multiplication.

Generally, if $\gamma = \alpha\beta$, a factor $1 + \beta x^2$ cancels, and

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left(\frac{1-\alpha x}{1+\alpha x}\right)^2,$$

the degenerate Cubic Transformation.

5. To connect up with the treatment for the Seven-section Division-Values of the Elliptic Function of the second stage,* take

$$(1)\quad c + 2 = \phi = \theta + \frac{1}{\theta},\quad 2c + 1 = 2\phi - 3 = -\frac{1}{p} = \frac{1}{1-2m},$$

$$(2)\quad \xi = \frac{(p+1)(p-1)^2}{8p^2} = \frac{(\phi-2)(\phi-1)^2}{2\phi-3} = \frac{(1-\theta.1-\theta+\theta^2)^2}{(2-3\theta+2\theta^2)\theta^2},$$

$$(3)\quad P = 4p^3 + (p-1)^2 = \frac{4(\phi-2)(2\phi^2-3\phi+2)}{(2\phi-3)^3},$$

$$w = \frac{(p-1)(p^2+p+2)\sqrt{P}}{p^3},$$

$$(4)\quad C = c(2c+1)(2c^2+5c+4) = \frac{(p+1)(4p^2-3p+1)}{4p^4} = \frac{P}{4p^4}, \ldots,$$

and so on: thence the expression in terms of θ or $a = \theta - \theta^2$, but in a different region of ξ.

In Transformation the coefficients are symmetrical functions of the Division-Values.

6. Similar Transformations can be given between the other Elliptic Functions, connecting

$$c = \text{cn}\, u \text{ and } C = \text{cn}\frac{u}{M};\quad d = \text{dn}\, u \text{ and } D = \text{dn}\frac{u}{M};$$

$$t = \text{tn}\, u \text{ and } T = \text{tn}\frac{u}{M};\quad z = \tan\tfrac{1}{2}\phi \text{ and } Z = \tan\tfrac{1}{2}\Phi;$$

$$v = e^{\phi i} \text{ and } V = e^{\Phi i};\quad z = i\frac{1-v}{1+v},\ \frac{2zi}{1-z^2} = \frac{1-v^2}{1+v^2}.$$

Thus we find

$$(1)\quad \frac{1-C}{1+C} = \frac{1-c}{1+c}\left[\frac{1+\alpha+\beta+\gamma+(\alpha+\gamma)c-(\beta+\gamma)c^2-\gamma c^3}{1+\alpha+\beta+\gamma-(\alpha+\gamma)c-(\beta+\gamma)c^2+\gamma c^3}\right]^2,$$

* *Proc. L.M.S.* (1893), p. 257; *Phil. Trans.* (1904), p. 266.

(2) $\frac{1-D}{1+D} = \frac{1-d}{1+d} \left[\frac{1+\alpha+\beta+\gamma+(1+\beta)d-(1+\alpha)d^2-d^3}{1+\alpha+\beta+\gamma-(1+\beta)d-(1+\alpha)d^2+d^3}\right]^2,$

(3) $\frac{1}{M} = 1+2\alpha,\ \alpha = \operatorname{dn}\frac{2}{7}K + \operatorname{dn}\frac{4}{7}K + \operatorname{dn}\frac{8}{7}K$ (Cayley, p. 265),

(4) $\frac{1+Ti}{1-Ti} = \left(\frac{1-ti}{1+ti}\right)\left[\frac{1+(1+\alpha)ti+(1+\beta)t^2+(1+\alpha+\beta+\gamma)t^3i}{1-(1+\alpha)ti+(1+\beta)t^2-(1+\alpha+\beta+\gamma)t^3i}\right]^2$

$= V^2 = \left(\frac{1+Zi}{1-Zi}\right)^2,$

(5) $Z = \frac{(1+2\alpha)z+(3+4\alpha+4\beta+8\gamma)z^3+(3+2\alpha+4\beta)z^5+z^7}{1+(3+2\alpha+4\beta)z^2+(3+4\alpha+4\beta+8\gamma)z^4+(1+2\alpha)z^6},$

(6) $V = v\Pi\frac{1-d_r+(1+d_r)v^2}{1+d_r+(1-d_r)v^2},\quad d_r = \operatorname{dn}\frac{2rK}{7}.$

7. In the general transformation of order $n = 2m+1$ between s and t of the Weierstrassian Elliptic Function of the First Stage,

(1) $t = M^2\frac{s^n - A_1 s^{n-1}+\ldots}{(s^m - G_1 s^{m-1}+\ldots)^2},\ \frac{Mdt}{\sqrt{T}} = \frac{ds}{\sqrt{S}} = du,$

where S and T are cubic in s and t, supposed at first irreducible.

Then $s = s(u)$ is either the Weierstrass function $\wp u$ or differs by a constant; and employing the formula

(2) $s(u+v)+s(u-v)-2s(v) = \frac{s''v}{s-sv}+\frac{s'^2v}{(s-sv)^2},$

the transformation can be written, with $v = \frac{2r\omega}{n}$,

(3) $\frac{t}{M^2} = s+A+\sum_{r=1}^{m}[s(u+v)+s(u-v)-2s(v)]$

$= s+A+\Sigma\frac{s''v}{s-sv}+\Sigma\frac{s'^2v}{(s-sv)^2},$

in a resolution of (1) into partial fractions.

The notation employed,* following Abel, with Halphen's x, y, was taken as

(4) $S = 4s(s-x)^2+[(1+y)s-xy]^2,$

(5) $s(v) = x,\ s'^2v = x^2,\ s''v = x(1+y),$

* *Proc. L.M.S.* (1893).

(6) $s(2v) = 0,\ s'^2 2v = x^2 y^2,\ s'' 2v = 2x^2 - xy(1+y),$

(7) $s(3v) = x - y,\ s'^2 3v = (x - y + y^2)^2,\ s'' 3v = x(1-3y) - y(1-4y+y^2)$;

and so on; and the relation between x and y was given by the curve of Halphen's $\gamma_n = 0$.

Thus for the Seventh Order,

(8) $\gamma_7 = 0,\ xy - x^2 - y^3 = 0,\ y = z(1-z),\ x = z(1-z)^2,$

(9) $s(v) = z(1-z)^2,\ s'^2 v = z^2(1-z)^4,\ s'' v = z(1-z)^2(1+z-z^2),$

(10) $s(2v) = 0,\ s'^2 2v = z^4(1-z)^6,\ s'' 2v = z^2(1-z)^3(1-3z+z^2),$

(11) $s(3v) = -z^2(1-z),\ s'^2 3v = z^6(1-z)^2,\ s'' 3v = z^3(1-z)(1-z-z^2).$

The transformation can then be written down and verified.

8. When a factor $s = e,\ t - f$ can be assigned of S, T, then $\sqrt{(s-e)}$, $\sqrt{(t-f)}$ is a single-valued Elliptic Function of the Second Stage, and the transformation of Weierstrass can be written

(1)
$$\sqrt{(t-f)} = M\sqrt{(s-e)}\,\frac{s^m - B_1 s^{m-2} + \dots}{s^m - G_1 s^{m-1} + \dots},$$

and the various transformations above in (1), (3), § 7 are all included in this general form, depending on the region separated by the roots of the cubic S in their sequence.

Thus in the Seventh Order, to construct a factor of S, $s + b^2$,* it is determined from

(2) $2b^3 - (1+y)b^2 + 2xb - xy = 0,$

(3) $2b^3 - (1+z-z^2)b^2 + 2bz(1-z)^2 - z^2(1-z)^3 = 0$;

and, putting $b = z(1-z)(c+1) = y(c+1)$,

(4) $(c+1)^2(2c+1)z^2 - c(c+2)(2c+1)z + c^2 = 0,$

(5)
$$z = \frac{c(c+2)(2c+1) + c\sqrt{C}}{2(c+1)^2(2c+1)},$$

$$1 - z = \frac{(c^2+2c+2)(2c+1) - c\sqrt{C}}{2(c+1)^2(2c+1)},\quad C = c(2c+1)(2c^2+5c+4),$$

(6)
$$y = z(1-z) = \frac{(\sqrt{C}+c)^2}{4(c+1)^4(2c+1)} = -\frac{p[p(p+1)-\sqrt{P}]^2}{(p-1)^4},$$

$$b = \frac{(\sqrt{C}+c)^2}{4(c+1)^3(2c+1)} = -\frac{1}{2}\frac{[p(p+1)-\sqrt{P}]^2}{(p-1)^3},$$

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* *Proc. L.M.S* (1839), p. 257.

in reducing to the expressions* with

$$2c+1=-\frac{1}{p}=\frac{1}{2m-1}, \ldots,$$

as above in § 5.

9. The Transformation of the Ninth Order may be sketched out here and left for completion as a research for the young mathematician, referring to the formulas of Kiepert.† But there is some confusion here in Kiepert's notation with the use of ξ on his p. 85. And Gierster may be consulted.‡

An extension is made of Jacobi's notation to

$$(1) \qquad \frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\alpha x+\beta x^2-\gamma x^3+\delta x^4}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4}\right)^2,$$

and in Kiepert's relations

$$(2) \qquad \frac{1}{M}=1+2\alpha=\xi_1{}^2=\frac{L(18)^4}{L(9)^2\,L(2)^4},$$

where in his notation §

$$(3) \qquad \xi_1=\frac{q^3+3q^2+0-1-\sqrt{Q}}{2q(q+1)},$$

$$Q=q^6+2q^5+5q^4+10q^3+10q^2+4q+1$$
$$=(q^3+q^2-2q-1)^2+8q^2(q+1)^3,$$

$$\tau=\tau_{18}=\frac{q^3-3q^2-6q-1+\sqrt{Q}}{2q(q+1)},$$

$$\frac{1}{\xi_3}=\tau+2=\frac{q^3+q^2-2q-1+\sqrt{Q}}{2q(q+1)}, \quad \xi_3=\frac{-q^3-q^2+2q+1+\sqrt{Q}}{4q(q+1)},$$

$$\frac{\xi_1}{\xi_2}=\frac{q^3+0-3q-1}{q(q+1)}, \ldots$$

$$(4) \qquad L(2)^{24}=-\frac{16}{\kappa^2\kappa'^2}=\frac{1-8\xi_3{}^3}{\xi_3{}^4(1+\xi_3{}^3)}=\frac{(\tau+2)^9\left[(\tau+2)^3-8\right]}{(\tau+2)^3+1},$$

$$\left[\frac{L(18)}{L(9)\,L(2)}\right]^3=\left(\frac{\kappa\kappa'}{\lambda\lambda'}\right)^{\frac{1}{4}}=\xi_1\xi_2,$$

and so on, to be developed and completed.

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* *Phil. Trans.* (1904), p. 266.
† *Math. Ann.*, vol. xxxii., p. 127; *Phil. Trans.* (1904), p. 269.
‡ *Math. Ann.*, vol. xiv., p. 541.
§ *Phil. Trans.* (1904), p. 270.

A PROOF OF BURNSIDE'S FORMULA FOR $\log\Gamma(x+1)$ AND CERTAIN ALLIED PROPERTIES OF RIEMANN'S ζ-FUNCTION.

By *J. R. Wilton.*

IN vol. xlvi.* (p. 159) Prof. Burnside has given a rapidly converging series for $\log N!$. His proof holds only if N is a positive integer, but the formula naturally holds for a much wider range of values. A simple proof may be obtained from the elementary properties of Riemann's ζ-function as follows.

We define $\zeta(x;\ s)$ when σ, the real part of s, is greater than 1, by the series

$$\zeta(x;\ s) = \sum_{n=1}^{\infty}(n+x)^{-s}.$$

Then it is known that

$$\zeta(x;\ 0) = -\tfrac{1}{2} - x,$$

$$\zeta'(x;\ 0) \equiv (d\zeta/ds)_{s=0} = \log\Gamma(x+1) - \tfrac{1}{2}\log 2\pi,$$

$$\operatorname*{Lt}_{s\to 1}\left[\zeta(x;\ s) - \frac{1}{s-1}\right] = -\frac{\Gamma'(x+1)}{\Gamma(x+1)}.$$

It follows from the definition that, when $\sigma > 1$ and $|z| < |1+x|$,

$$\zeta(x+z;\ s) = \Sigma(n+x+z)^{-s} = \Sigma(n+x)^{-s}\{(1+z)/(n+x)\}^{-s}$$

$$= \zeta(x;\ s) - sz\zeta(x;\ s+1) + \frac{s(s+1)}{2!}z^2\zeta(x;\ s+2) - \ldots (1),$$

on expanding by the binomial theorem and rearranging the resulting absolutely convergent double series. And this result, proved when $\sigma > 1$, is, by the principle of analytical continuation, valid for all values of s. In the series (1) put $z = \frac{1}{2}$, $z = -\frac{1}{2}$, and subtract

$$(x+\tfrac{1}{2})^{-s} = \zeta(x-\tfrac{1}{2};\ s) - \zeta(x+\tfrac{1}{2};\ s)$$

$$= s\zeta(x;\ s+1) + \frac{s(s+1)(s+2)}{3!}\,\frac{\zeta(x;\ s+3)}{2^2} + \ldots (2).$$

Rewrite this equation in the form

$$\frac{(x+\frac{1}{2})^{-s} - s\zeta(x;\ s+1)}{s(s+1)} = \frac{s+2}{3!}\,\frac{\zeta(x;\ s+3)}{2^2} + \ldots,$$

* "A rapidly converging series for $\log N!$", *Messenger of Mathematics*, vol. xlvi. (1917), pp. 157–159.

and then make $s \to -1$. The resulting equation is Burnside's formula,

$$\log\Gamma(x+1) = (x+\tfrac{1}{2})\log(x+\tfrac{1}{2}) - x - \tfrac{1}{2} + \tfrac{1}{2}\log 2\pi - \sum_{n=1}^{\infty} \frac{\zeta(x;\, 2n)}{2n(2n+1)\,2^{2n}} \quad \ldots\ldots(3).$$

The method of proof shows that the expansion is correct if $Rx > -\tfrac{1}{2}$; and it is, in fact, correct when $x = -\tfrac{1}{2}$, for which value it gives

$$\sum_{1}^{\infty} \zeta(2n)/2n(2n+1) = \tfrac{1}{2}\log 2\pi - \tfrac{1}{2}.$$

[The left-hand side is

$$1 - \log 2 + \Sigma[\zeta(2n) - 1]/2n(2n+1)$$
$$= 1 - 0.6931471806 + 0.1120857138$$
$$= 0.4189385332 = \tfrac{1}{2}\log 2\pi - \tfrac{1}{2}].$$

Differentiation of Burnside's formula (3) leads to

$$\psi(x+1) \equiv \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \log(x+\tfrac{1}{2}) + \sum_{n=1}^{\infty} \frac{\zeta(x;\, 2n+1)}{(2n+1)\,2^{2n}} \quad \ldots(4),$$

$$\psi'(x+1) = \frac{d^2}{dx^2}\log\Gamma(x+1) = \frac{1}{x+\tfrac{1}{2}} - \sum_{1}^{\infty} \frac{\zeta(x;\, 2n+2)}{2^{2n}}.$$

The last equation, as is immediately evident on putting $s = 1$ in (2), is merely a variant of the customary formula

$$\psi'(x+1) = \zeta(x;\, 2).$$

Differentiating equation (2) with respect to s,

$$-(x+\tfrac{1}{2})^{-s}\log(x+\tfrac{1}{2}) = s\zeta'(x;\, s+1) + \frac{s(s+1)(s+2)}{3!}\,\frac{\zeta'(x;\, s+3)}{2^2} + \ldots$$

$$+ \zeta(x;\, s+1) + \frac{s(s+1)(s+2)}{3!}\left(\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right)\frac{\zeta(x;\, s+3)}{2^2} + \ldots.$$

Making $s \to 0$, we obtain (4); $s \to -1$ gives Burnside's formula. Making $s \to -2$,

$$\zeta'(x;\, -1) = \tfrac{1}{2}(x+\tfrac{1}{2})^2\log(x+\tfrac{1}{2}) - \tfrac{1}{24}\psi(x+1) - \tfrac{1}{4}x^2 - \tfrac{1}{4}x - \tfrac{5}{48}$$
$$+ \frac{2!}{5!}\,\frac{\zeta(x;\, 3)}{2^4} + \frac{4!}{7!}\,\frac{\zeta(x;\, 5)}{2^6} + \ldots\ldots(5).$$

Making $s \to -3$,

$$\zeta'(x; -2) = \tfrac{1}{3}(x+\tfrac{1}{2})^3 \log(x+\tfrac{1}{2}) - \tfrac{1}{12}\log\Gamma(x+1)$$
$$+ \tfrac{1}{24}\log 2\pi - \tfrac{1}{9}x^3 - \tfrac{1}{6}x^2 - \tfrac{5}{24}x - \tfrac{11}{144}$$
$$- \frac{1\,!}{5\,!}\frac{\zeta(x;\,2)}{2^3} - \frac{3\,!}{7\,!}\frac{\zeta(x;\,4)}{2^5} \ldots (6),$$

and there would be no difficulty in deriving the corresponding formulæ for $\zeta(x; 1-2n)$ and $\zeta'(x; -2n)$.

From Riemann's relation,

$$\zeta'(1-s) = 2(2\pi)^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\,\zeta(s),$$

it follows that, if n is a positive integer (not zero),

$$(-)^n \zeta'(-2n) = \tfrac{1}{2}(2\pi)^{-2n}\, 2n!\, \zeta(2n+1),$$

$$(-)^n\zeta'(1-2n) = \frac{B_n}{2n}\left(\log 2\pi + \gamma - 1 - \tfrac{1}{2} - \tfrac{1}{3} - \ldots - \frac{1}{2n-1}\right) - 2\frac{2n-1\,!}{(2\pi)^{2n}}\zeta'(2n),$$

where B_n is the n^{th} Bernoulli number and γ is Euler's constant. Making use of these equations when necessary, and putting $x=0$ in (4), (3), (5) and (6),

$$\log 2 - \gamma = \tfrac{1}{3}\frac{\zeta(3)}{2^2} + \tfrac{1}{5}\frac{\zeta(5)}{2^4} + \ldots,$$

$$\log\pi - 1 = \frac{1}{2\,.\,3}\frac{\zeta(2)}{2} + \frac{1}{4\,.\,5}\frac{\zeta(4)}{2^3} + \ldots,$$

$$-\frac{\zeta'(2)}{\pi^2} = \tfrac{3}{8} - \tfrac{1}{6}\log\pi - \tfrac{1}{4}\gamma + \tfrac{1}{12}\log 2 - \frac{2\,!}{5\,!}\frac{\zeta(3)}{2^3} - \frac{4\,!}{7\,!}\frac{\zeta(5)}{2^5} - \ldots,$$

$$\frac{\zeta(3)}{\pi^2} = \tfrac{11}{36} - \tfrac{1}{6}\log\pi + \frac{1\,!}{5\,!}\frac{\zeta(2)}{2} + \frac{3\,!}{7\,!}\frac{\zeta(4)}{2^3} + \ldots .$$

A more general formula of this type,* valid when n is a positive integer, is

$$\frac{\zeta(2n+1)}{\pi^{2n}} = \frac{\zeta(2n-1)}{3\,!\,\pi^{2n-2}} - \frac{\zeta(2n-3)}{5\,!\,\pi^{2n-4}} + \ldots + (-)^n\frac{\zeta(3)}{2n-1\,!\,\pi^2} + (-)^n\frac{\log\pi}{2n+1\,!}$$
$$+ (-)^{n-1}\left\{\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{2n+1}\right)\frac{1}{2n+1\,!} + \sum_{m=1}^{\infty}\frac{2m-1\,!}{2m+2n+1\,!}\frac{\zeta(2m)}{2^{2m-1}}\right\}.$$

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* The connection of the series here obtained with those given by Ramanujan, "A series for Euler's constant, γ", *Messenger of Mathematics*, vol. xlvi., pp 73–80, especially p. 78, is, apparently, less close than might have been anticipated.

Returning now to equation (1), again put $z=\frac{1}{2}$, $z=-\frac{1}{2}$, and take the sum of the resulting series

$$\tfrac{1}{2}(x+\tfrac{1}{2})^{-s}+\zeta(x+\tfrac{1}{2};\ s)=\zeta(x;\ s)+\frac{s(s+1)}{2!}\ \frac{\zeta(x;\ s+2)}{2^{2}}+\ldots.$$

Differentiating with respect to s and then making $s \to 0$, we obtain the expansion *

$$\log\Gamma(x+1)=\log\Gamma(x+\tfrac{1}{2})+\tfrac{1}{2}\log(x+\tfrac{1}{2})-\sum_{1}^{\infty}2^{-2n}\zeta(x;\,2n)/2n.$$

Putting $x=0$, we have

$$\log\tfrac{1}{2}\pi=\Sigma\, 2^{-2n}\zeta(2n)/n,$$

and putting $x=1$, we obtain the same equation in the slightly different form

$$\log\pi-3\log 2+\log 3=\Sigma\, 2^{-2n}[\zeta(2n)-1]/n,$$

in which form it is readily verified numerically.

Again returning to equation (1), we may, if $|x|<|1+x|$, *i.e.* if $Rx>-\frac{1}{2}$, put $z=-x$, and obtain

$$\zeta(s)=\zeta(x;\ s)+sx\zeta(x;\ s)+\{s(s+1)/2!\}\,x^{2}\zeta(x;\ s+2)+\ldots(7).$$

Making $s \to 1$,

$$\psi(x+1)=-\gamma+\sum_{1}^{\infty}x^{n}\zeta(x;\ n+1)\ldots\ldots\ldots\ldots(8).$$

Differentiating (7) with respect to s, and making $s \to 0$,

$$\log\Gamma(x+1)=x\psi(x+1)-\sum_{2}^{\infty}x^{n}\zeta(x;\ n)/n\ldots.(9).$$

[Differentiation of either of these equations (8) and (9) leads to $\psi'(x+1)=\zeta(x;\ 2)$].

By putting $x=1$ in (8) and (9) the well-known results† follow

$$\sum_{2}^{\infty}[\zeta(n)-1]=1,$$

$$\sum_{2}^{\infty}[\zeta(n)-1]/n=1-\gamma.$$

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* Cf. Bromwich, *Infinite Series*, p. 477, ex. 51.

† Bromwich, *loc. cit.*, p. 479, ex. 6, and the reference there given.

A DIRECT PROOF OF THE BINOMIAL THEOREM FOR A RATIONAL INDEX.

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For many purposes of presentation it seems convenient to have a proof of the Binomial Theorem which does not rely on propositions in the theory of infinite series, but which obtains the fundamental inequality $|R_n(x)| < \epsilon$ as a theorem of algebra. The way thereto is indicated, if we write the classical exponential inequality as

$$px(1-x)^{-p-1} \gtrless (1-x)^{-p} - 1 \gtrless px,$$

for this gives the first two terms of the expansion of $(1-x)^{-p}$. We appear then to need an appropriate generalisation of the exponential inequality. I employ the expression

$$\frac{1-u^p}{p(1-u)} - n\frac{1-u^{p+1}}{(p+1)(1-u)} + \frac{n(n-1)}{2!}\cdot\frac{1-u^{p+2}}{(p+2)(1-u)} - \ldots (-)^n \frac{1-u^{p+n}}{(p+n)(1-u)},$$

where u and p are positive and p rational, and I show that, if $u > 1$, it has the sign of $(-)^n$, but if $u < 1$, it is positive and diminishes as p increases.

§ 1. I consider first the case of a negative binomial index and write

$$S_{n,p}(x) \equiv 1 + px + \frac{p(p+1)}{2!}x^2 + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!}x^n \qquad \ldots\ldots(1).$$

If $x = 1$, $S_{n,p} = \dfrac{(p+1)(p+2)\ldots(p+n)}{n!}$.

If we write $x \equiv 1-u$, the coefficient of u^r in $S_{n,p}(1-u)$ is

$$(-)^r \frac{p(p+1)\ldots(p+r-1)}{r!}\left\{1 + (p+r) + \frac{(p+r)(p+r+1)}{2!} + \ldots + \frac{(p+r)(p+r+1)\ldots(p+n-1)}{(n-r)!}\right\}$$

$$= (-)^r \frac{p(p+1)\ldots(p+r-1)}{r!}\cdot\frac{(p+r+1)(p+r+2)\ldots(p+n)}{(n-r)!}.$$

Hence

$$S_{n,p}(1-u)=\frac{p(p+1)\dots(p+n)}{n!}\sum_{r=0}^{n}(-)^n\frac{n!}{r!\,(n-r)!}\frac{u^r}{p+r}.$$

Put $u=1$, then

$$1=S_{n,p}(0)=\frac{p(p+1)\dots(p+n)}{n!}\sum_{r=0}^{n}(-)^r\frac{n!}{r!\,(n-r)!}\frac{1}{p+r}.$$

Thus

$$1-u^pS_{n,p}(1-u)=\frac{p(p+1)\dots(p+n)}{n!}\sum_{r=0}^{n}(-)^r\frac{n!}{r!\,(n-r)!}\frac{1-u^{p+r}}{p+r} \quad \dots\dots(2).$$

Write $U_p\equiv\dfrac{1-u^p}{p\,(1-u)}$, and let E be the operator which charges p into $p+1$. Then the expression on the right of (2) may be written

$$(1-u)\frac{p(p+1)\dots(p+n)}{n!}(1-E)^nU_p.$$

The expression on the left is

$$(1-x)^p\{(1-x)^{-p}-S_{n,p}(x)\}=(1-x)^pR_{n,p}(x), \text{ say.}$$

Then we may write (2) as

$$x^{-1}(1-x)^pR_{n,p}(x)=\frac{p(p+1)\dots(p+n)}{n!}(1-E)^nU_p\dots(3).$$

This holds for every rational p (not a negative integer) and for every x (other than zero). I shall however suppose, naturally, that $1-x$, *i.e.* u is positive.

§ 2. Now restrict p to be a *positive integer*. Then $(1-x)^{p-1}S_{n,p}(x)$ is the coefficient of t^n in

$$t^n(1-x+tx)^{p-1}+t^{n-1}(1-x+tx)^p+\dots+(1-x+tx)^{p+n-1}.$$

We are, of course, at liberty to add any multiple of t^{n+1}, and so we may take it as the coefficient of t^n in

$$t^{p+n-1}+t^{p+n-2}(1-x+tx)+\dots+(1-x+tx)^{p+n-1},$$

i.e. in

$$\frac{(1-x+tx)^{p+n}-t^{p+n}}{(1-x)(1-t)}.$$

Put $x=0$; then 1 is the coefficient of t^n in $\dfrac{1-t^{p+n}}{1-t}$.

Hence $1-(1-x)^pS_{n,p}(x)$ is the coefficient of t^n in

$$\frac{1-(1-x+tx)^{p+n}}{1-t};$$

or $x^{-1}(1-x)^p R_{n,p}(x)$ is the coefficient of t^n in

$$\frac{1-(1-x+tx)^{p+n}}{1-(1-x+tx)},$$

i.e. in $\quad 1+(1-x+tx)+(1-x+tx)^2+\ldots+(1-x+tx)^{p+n-1}$.

Thus $$x^{-1}(1-x)^p R_{n,p}(x) = x^n \sum_{r=0}^{p-1} \frac{(n+r)!}{n!\,r!}(1-x)^r \ldots\ldots(4).*$$

Therefore, from (3), $(1-E)^n U_p$ *has the sign of* x^n, *i.e. of* $(1-u)^n$, *when* p *is a positive integer.* In particular, if $1-u$ is positive, so is $(1-E)^n U_p$, and so also is $(1-E)^{n+1} U_p$. But

$$(1-E)^{n+1} U_p = (1-E)^n (1-E) U_p = (1-E)^n U_p - (1-E)^n U_{p+1}.$$

Hence, *if* $1-u$ *is positive,* $(1-E)^n U_p$ *increases as* p *diminishes.*

§ 3. I now extend these results to the case of any rational positive p' ($\equiv p/q$, where p, q are positive integers).

$$U_{p'} \text{ is } \frac{1-u^{p/q}}{(p/q)(1-u)} = \frac{q}{p}\,\frac{1-v^p}{1-v^q}, \text{ if } u \equiv v^q.$$

$$U_{p'+1} \text{ is similarly } \frac{q}{p+q}\,\frac{1-v^{p+q}}{1-v^q}, \text{ and so on.}$$

Hence, if E' changes p' into $p'+1$, $(1-E')^n U_{p'}$ is

$$\frac{q(1-v)}{1-v^q}(1-E^q)^n \frac{1-v^p}{p(1-v)} \equiv \frac{q(1-v)}{1-v^q}(1-E^q)^n V_p, \text{ say.}$$

It therefore has the sign of $(1-E^q)^n V_p$, and increases with it.

Now $\quad (1-E^q)^n V_p = (1-E)^n (1+E+E^2+\ldots+E^{q-1})^n V_p$.

But $(1+E+\ldots+E^{q-1})^n$ expands into a series of powers of E with positive coefficients. Hence $(1-E^q)^n V_p$ can be expressed as a series in $(1-E)^n V_{p+r}$ with positive coefficients. Since every $(1-E)^n V_{p+r}$ has the sign of $(1-v)^n$, so $(1-E^q)^n V_p$ has the sign of $(1-v)^n$, *i.e.* of $(1-u)^n$.

Moreover, if $1-u$ is positive, every $(1-E^q)^n V$ is positive, and a like argument shows that $(1-E^q)^n (1-E^r) V_p$ is positive, *i.e.* that $(1-E^q)^n V_p > (1-E^q)^n V_{p+r}$. The results of § 2 are thus extended to every positive, rational p. Hence, by

* It is perhaps interesting to state this result in the symmetrical form

$$(1-x)^p S_{q-1,p}(x) + (1-y)^q S_{p-1,q}(y) = 1,$$

where p, q are integers and $x+y=1$.

equation (3), $R_{n,p}(x)$ *has the sign of* x^{n+1}, and, if x is positive, $(1-E)^n U_p$ increases as p diminishes, *i.e.* if $p < 1$,

$$(1-E)^n U_p < (1-E)^n U_1.$$

By (3), (4) $$(1-E)^n U_1 = x^n/(n+1).$$

Hence, if $p > 1$ *and* $x > 0$,

$$R_{n,p}(x) < \frac{p(p+1)\ldots(p+n)}{(n+1)!} x^{n+1}(1-x)^{-p},$$

If x is negative, $R_{n,p}(x)$ and $R_{n+1,p}(x)$ have opposite signs and consequently $R_{n,p}(x)$ is numerically less than their difference

$$\frac{p(p+1)\ldots(p+n)}{(n+1)!}|x|^{n+1} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

Thus in either case it is enough to prove that, if $p > 1$ and $|x| < 1$,

$$\frac{p(p+1)\ldots(p+n)}{(n+1)!}|x|^{n+1} \to 0 \text{ as } n \to \infty.$$

§4. If $|x| < 1$, suppose $|x| = 1/(1+c)$, where $c > 0$. Now

$$\frac{(p+1)(p+2)\ldots(p+n)}{n!} = \left(1+\frac{p}{1}\right)\left(1+\frac{p}{2}\right)\ldots\left(1+\frac{p}{n}\right).$$

But by the Exponential Inequality

$$1+\frac{p}{r} < \left(1+\frac{1}{r}\right)^p, \text{ if } p > 1.$$

Hence $$\frac{(p+1)(p+2)\ldots(p+n)}{n!} < (1+n)^p.$$

This $$< |x|^{-n}, \text{ if } 1+n < (1+c)^{n/p}.$$

Now $$(1+c)^{n/p} = \{(1+c)^{n/2p}\}^2$$
$$> \{1+cn/2p\}^2, \text{ if } n/2p > 1,$$

i.e. $$> 1 + cn/p + \tfrac{1}{4}c^2n^2/p^2.$$

Thus $$(1+c)^{n/p} > 1+n, \text{ if } n > 4p(p-c)/c^2.$$

Hence, if n is sufficiently great to satisfy these two inequalities,

$$\frac{p(p+1)\ldots(p+n)}{(n+1)!}|x|^{n+1} < \frac{p|x|}{n+1}.$$

Thus $$R_{n,p}(x) \to 0, \text{ as } n \to \infty, \text{ if } |x| < 1,\ p > 1.$$

If p is negative or less than unity, then

$$|p(p+1)\ldots(p+n)| < (n+1)!,$$

provided only that $p+n$ is positive. In these cases also, then,

$$\frac{p(p+1)\dots(p+n)}{(n+1)!}|x|^{n+1} \to 0, \text{ as } n \to \infty,$$

with this remark we can extend our proof of the Binomial Theorem to the cases $p<1$.

For it will be found that

$$R_{n,p-1}(x) = (1-x)\,R_{n,p}(x) - \frac{p(p+1)\dots(p+n-1)}{n!}x^{n+1} \dots (6).$$

The right-hand side tends to zero as n tends to infinity, if $p>1$: hence $R_{n,p-1}(x) \to 0$, if $p-1>0$.

We have thus extended the proof to the case $p>0$.

A second application of (6) brings us to $p<-1$, and so on. Thus inductively we at length cover any negative p.

§ 5. The extreme case of $|x|=1$ is best discussed independently. When $x=1$, as we have seen,

$$S_{n,p} = \frac{(p+1)(p+2)\dots(p+n)}{n!}.$$

Suppose p negative and equal to $-q$. By the exponential inequality

$$1-\frac{q}{n} < \left(1+\frac{1}{n}\right)^{-q}.$$

Hence if s is a fixed integer greater than q,

$$|S_{n,p}| < \left|\left(1-\frac{q}{1}\right)\left(1-\frac{q}{2}\right)\dots\left(1-\frac{q}{s-1}\right)\right|$$
$$\times\left(1+\frac{1}{s}\right)^{-q}\left(1+\frac{1}{s+1}\right)^{-q}\dots\left(1+\frac{1}{n}\right)^{-q},$$

i.e.
$$< \left|\left(1-\frac{q}{1}\right)\left(1-\frac{q}{2}\right)\dots\left(1-\frac{q}{s-1}\right)\right|\left(\frac{s}{n+1}\right)^{q}.$$

Hence
$$S_{n,p} \to 0 \text{ as } n \to \infty.$$

But $(1-x)^{-p} = (1-1)^q = 0$. Hence $R_{n,p} \to 0$, when $x=1$, $p<0$.

When $x=-1$ suppose $p<1$, i.e. $p-1<0$. We have then just proved that

$$\frac{p(p+1)\dots(p+n-1)}{n!} \quad \text{or} \quad \frac{p(p+1)\dots(p+n)}{(n+1)!}$$

tends to zero as $n \to \infty$. But by (5), if $p>0$ and $x=-1$,

$$R_{n,p}(-1) < \frac{p(p+1)\dots(p+n)}{(n+1)!}, \text{ i.e. } \to 0, \text{ if } p<1.$$

Hence $R_{n,p}(-1) \to 0$, for $0<p<1$. Applying the formula (6) repeatedly it follows that $R_{n,p}(-1) \to 0$ as $n \to \infty$ for any $p<1$.

ADAPTATION OF CURVILINEAR ISOTHERMAL COORDINATES TO INTEGRATE THE EQUATIONS OF EQUILIBRIUM OF ELASTIC PLATES.

By *Prof. B. G. Galerkin*, Polytechnical Institute, Petrograd.

THIS article is dedicated to the investigation by means of isothermal coordinates of the flexure of elastic plates, especially of elliptic and semi-elliptic plates.

The elastic (middle) surface of the plate in this case can be expressed in cartesian rectangular coordinates x and y, or in curvilinear isothermal coordinates, ξ and η, connected with x and y by the equation

$$x + yi = \psi(\xi + \eta i) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

§ 1. *General equations.* The equation of equilibrium of a flat plate under pressure can be presented in curvilinear isothermal coordinates as follows

$$\frac{Eh^3}{12(1-\sigma^2)} h_1^2 \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) \left[h_1^2 \left(\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\eta^2}\right)\right] = p \ldots (2),$$

where w is the normal displacement of the middle plane, h the thickness of the plate, σ Poisson's ratio, h_1 a differential parameter, p the pressure per unit area applied to one face of the plate. Then we have

$$h_1 = \sqrt{\left\{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right\}} \ldots\ldots\ldots\ldots\ldots (3),$$

and w can be expressed in the form

$$w = f(\xi, \eta) + \phi(\xi, \eta),$$

where f is a particular solution of equation (2), and ϕ satisfies the equation

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) \left[h_1^2 \left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2}\right)\right] = 0 \ldots\ldots\ldots (4).$$

It is evident that the function ϕ satisfies the last equation in case either

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0,$$

or

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = \frac{1}{h_1^2}\psi.$$

where
$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0.$$

Regarding the deformation as infinitely small, it is not difficult to obtain the formulas for the curvature of the middle surface. The curvature in a plane, passing through the point (ξ, η) parallel to the axis z and containing the normal to $\xi =$ const. is

$$\left.\begin{aligned} \frac{1}{\rho_1} &= -\frac{1}{h_1^2}\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right] \\ &= -\frac{1}{h_1^2}\left\{h_1^4 \frac{\partial^2 w}{\partial \xi^2} + \left[\left(2\left(\frac{\partial \xi}{\partial x}\right)^2 - h_1^2\right)\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 \xi}{\partial x \partial y}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right]\frac{\partial w}{\partial \xi}\right. \\ &\quad \left. - \left[\left(2\left(\frac{\partial \xi}{\partial x}\right)^2 - h_1^2\right)\frac{\partial^2 \xi}{\partial x \partial y} - 2\frac{\partial^2 \xi}{\partial x^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right]\frac{\partial w}{\partial \eta}\right\} \end{aligned}\right\} \quad \ldots\ldots(5).$$

The curvature in a plane, passing through the point (ξ, η) parallel to the axis z and containing the normal to $\eta =$ const. is

$$\left.\begin{aligned} \frac{1}{\rho_2} &= -\frac{1}{h_1^2}\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial \xi}{\partial x}\right)^2 - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right] \\ &= -\frac{1}{h_1^2}\left\{h_1^4 \frac{\partial^2 w}{\partial \eta^2} - \left[\left(2\left(\frac{\partial \xi}{\partial x}\right)^2 - h_1^2\right)\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 \xi}{\partial x \partial y}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right]\frac{\partial w}{\partial \xi}\right. \\ &\quad \left. + \left[\left(2\left(\frac{\partial \xi}{\partial x}\right)^2 - h_1^2\right)\frac{\partial^2 \xi}{\partial x \partial y} - 2\frac{\partial^2 \xi}{\partial x^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\right]\frac{\partial w}{\partial \eta}\right\} \end{aligned}\right\} \quad \ldots\ldots(6).$$

§ 2. *Plates, bounded by two arcs of concentric circles and two radii.* Using polar isothermal coordinates we can obtain

$$x + yi = ce^{\xi + \eta i} = ce^{\xi}(\cos \eta + i \sin \eta) \ldots\ldots\ldots(7),$$

whence
$$x = ce^{\xi} \cos \eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and
$$y = ce^{\xi} \sin \eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

c being constant.

From equations (8) and (9) follows

$$\frac{y}{x} = \tan \eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

and
$$x^2 + y^2 = c^2 e^{2\xi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Supposing η to be constant, we obtain a straight line issuing from the origin of coordinates; taking ξ as constant, we obtain a circle.

From (11) can be derived

$$\xi = \tfrac{1}{2} ln \frac{x^2+y^2}{c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and the formula (3) gives

$$h_1^2 = \frac{e^{-2\xi}}{c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

From equations (5) and (6) we obtain

$$\frac{1}{\rho_1} = -\frac{e^{-2\xi}}{c^2}\left(\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial w}{\partial \xi}\right) \quad \ldots\ldots\ldots\ldots(14),$$

and

$$\frac{1}{\rho_2} = -\frac{e^{-2\xi}}{c^2}\left(\frac{\partial^2 w}{\partial \eta^2} + \frac{\partial w}{\partial \xi}\right) \quad \ldots\ldots\ldots\ldots(15).$$

The equation (2) transforms itself into

$$\frac{Eh^3}{12(1-\sigma^2)}\,\frac{e^{-2\xi}}{c^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\left[\frac{e^{2\xi}}{c^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right)\right] = p \ldots(16).$$

If we can find a function $f(\xi, \eta)$ satisfying this equation, we shall obtain for a plate, bounded by two arcs of concentric circles and two radii containing an angle 2α, an equation of the deformed middle surface, under the condition that the radial edges are supported, as follows

$$w = f(\xi, \eta) + \sum_1^\infty A_n e^{\frac{n\pi\xi}{2\alpha}} \sin\frac{n\pi(\alpha+\eta)}{2\alpha} + \sum_1^\infty B_n e^{-\frac{n\pi\xi}{2\alpha}} \sin\frac{n\pi(\alpha+\eta)}{2\alpha}$$

$$+ e^{2\xi}\left\{\sum_1^\infty C_n e^{\frac{n\pi\xi}{2\alpha}} \sin\frac{n\pi(\alpha+\eta)}{2\alpha} + \sum_1^\infty D_n e^{-\frac{n\pi\xi}{2\alpha}} \sin\frac{n\pi(\alpha+\eta)}{2\alpha}\right\} \ldots(17).$$

If $f(\xi, \eta) = 0$ and $\frac{\partial^2 f}{\partial \eta^2} = 0$, when $\eta = \pm\alpha$, the whole expression w satisfies the condition that the radial edges are supported. Having written the conditions on the circular edges of the plate, namely, those which hold when

$$\xi = ln\frac{a}{c} \text{ and } \xi = ln\frac{a_0}{c},$$

where a = external radius and a_0 = internal radius of the plate, we can determine the coefficients A_n, B_n, C_n and D_n. The results will coincide with the results, obtained by means of polar coordinates in our article, "Equilibrium of elastic plates, bounded by two arcs of concentric circles and two radii".*

* *Bulletin of Russian Academy of Sciences* (1919), No. 8–11.

and therefore the transition from the formula in isothermal polar coordinates to the same in polar coordinates will not present any difficulty.

§ 3. *Elliptic plate.* For an elliptic plate with a clamped edge the solution is known for pressure proportional to distance from any fixed plane. It is expressed in cartesian rectangular coordinates.

Here will be given the solution for the case where the edge is supported and the pressure is uniform. It can be extended to the case of pressure proportional to distance from any fixed plane.. The middle surface of the plate we shall express in elliptic coordinates. Suppose

$$x + yi = c\cosh(\xi + \eta i) = c(\cosh\xi\cos\eta + i\sinh\xi\sin\eta) \ldots (18),$$

$$x = c\cosh\xi\cos\eta \text{ and } y = c\sinh\xi\sin\eta,$$

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1 \ \ldots\ldots\ldots\ldots\ldots (19)$$

and

$$\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

When $\xi =$ constant the equation (19) gives an ellipse, when $\eta =$ constant the equation (20) gives a hyperbola.

The differential parameter is given by

$$h_1 = \frac{\sqrt{2}}{c\sqrt{(\cosh 2\xi - \cos 2\eta)}} \ \ldots\ldots\ldots\ldots\ldots (21).$$

The equation (2) transforms itself into

$$\frac{Eh^3}{3(1-\sigma^2)}\frac{1}{c^4(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)$$
$$\times\left\{\frac{1}{\cosh 2\xi - \cos 2\eta}\left(\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\eta^2}\right)\right\} = p \ldots \ldots (22).$$

From the formulas (5) and (6) we obtain

$$\frac{1}{\rho_1} = -h_1^2\left(\frac{\partial^2 w}{\partial\xi^2} - \frac{c^2h_1^2}{2}\sinh 2\xi\frac{\partial w}{\partial\xi} + \frac{c^2h_1^2}{2}\sin 2\eta\frac{\partial w}{\partial\eta}\right) \ldots\ldots (23),$$

$$\frac{1}{\rho_2} = -h_1^2\left(\frac{\partial^2 w}{\partial\eta^2} + \frac{c^2h_1^2}{2}\sinh 2\xi\frac{\partial w}{\partial\xi} - \frac{c^2h_1^2}{2}\sin 2\eta\frac{\partial w}{\partial\eta}\right) \ldots\ldots (24).$$

We take an elliptic plate with semi-axes a and b. The boundary will be expressed in elliptic coordinates by $\xi = \alpha$, where α is determined from the formula $c\cosh\alpha = a$ or $c\sinh\alpha = b$ (c the distance of a focus from the centre).

Supposing the plate bent by pressure p, uniformly distributed over one face, we can take as the equation of the deformed middle surface

$$w = C(3\cosh 2\alpha \cosh 4\alpha - 4\cosh 4\alpha \cosh 2\xi + \cosh 2\alpha \cosh 4\xi)$$
$$\times (3\cosh 2\alpha \cosh 4\alpha - 4\cosh 4\alpha \cos 2\eta + \cosh 2\alpha \cos 4\eta)$$
$$+ A_0(\cosh 2\xi - \cosh 2\alpha)(\cosh 2\alpha - \cos 2\eta)$$
$$+ A_1(\cosh 4\alpha \cosh 2\xi - \cosh 2\alpha \cosh 4\xi)(\cosh 2\alpha - \cos 2\eta)$$
$$\times (1 + 2\cosh 2\alpha \cos 2\eta)$$
$$+ \sum_2^\infty A_k \left\{ \left(\frac{\cosh 2k\alpha \cosh 2(k-1)\xi}{(2k-1)\cosh 2(k-1)\alpha} - \frac{\cosh 2k\xi}{2k-1} \right) \cos 2(k-1)\eta \right.$$
$$- \left[\left(\frac{\cosh 2(k+1)\alpha}{2k+1} - \frac{\cosh 2(k+1)\alpha}{2k-1} \right) \frac{\cosh 2k\xi}{\cosh 2k\alpha} \right.$$
$$\left. - \left(\frac{\cosh 2(k+1)\xi}{2k+1} - \frac{\cosh 2(k-1)\xi}{2k-1} \right) \right] \cos 2k\eta$$
$$\left. + \left(\frac{\cosh 2k\xi}{2k+1} - \frac{\cosh 2k\alpha \cosh 2(k+1)\xi}{(2k+1)\cosh 2(k+1)\alpha} \right) \cos 2(k+1)\eta \right\} \ldots (25).$$

If p is constant, w satisfies the equation (22) if

$$C = \frac{pc^4(1-\sigma^2)}{128Eh^3 \cosh^2 2\alpha \cosh 4\alpha}.$$

At the edge of the plate (where $\xi = \alpha$) w becomes zero.

If the plate is supported at the edge it is necessary for the normal stresses on the edge to be zero, and therefore we must have, at the elliptic boundary,

$$\frac{1}{\rho_1} + \frac{\sigma}{\rho_2} = 0.$$

Since on the elliptic boundary $\frac{\partial w}{\partial \eta} = 0$ and $\frac{\partial^2 w}{\partial \eta^2} = 0$, it follows, by using the formulas (23) and (24), that

$$\frac{\partial^2 w}{\partial \xi^2} - (1-\sigma)\frac{\partial w}{\partial \xi} \cdot \frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\eta} = 0 \ldots\ldots (26).$$

We shall take the first approximation, limiting ourselves to the coefficient A_0, and supposing $A_1 = A_2 = \ldots = 0$. The equation (26) gives, if $\xi = \alpha$,

$$4A_0 \cosh 2\alpha(\cosh 2\alpha - \cos 2\eta) + (1-\sigma)\sinh 2\alpha [16C \sinh^3 2\alpha$$
$$\times (3\cosh^2 2\alpha - \cosh 2\alpha \cos 2\eta - 2) - 2A_0 \sinh 2\alpha] = 0.$$

Making the constant member equal to zero, we obtain

$$A_0\left[2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha\right]+8C(1-\sigma)\sinh^4 2\alpha(3\cosh^2 2\alpha-2)=0,$$

whence

$$A_0=-\frac{8(1-\sigma)(3\cosh^2 2\alpha-2)\sinh^4 2\alpha}{2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha}\,C \quad \ldots\ldots(27).$$

w will be

$$w=\frac{p(1-\sigma^2)c^4}{128Eh^3\cosh^2 2\alpha\cosh 4\alpha}\Big[(3\cosh 2\alpha\cosh 4\alpha-4\cosh 4\alpha\cosh 2\xi$$
$$+\cosh 2\alpha\cosh 4\xi)(3\cosh 2\alpha\cosh 4\alpha-4\cosh 4\alpha\cos 2\eta+\cosh 2\alpha\cos 4\eta)$$
$$-\frac{8(1-\sigma)(3\cosh^2 2\alpha-2)\sinh^4 2\alpha}{2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha}(\cosh 2\xi-\cosh 2\alpha)(\cosh 2\alpha-\cos 2\eta)\Big]$$
$$\ldots\ldots(28).$$

The deflexion at the middle ($\xi=0$ and $\eta=\frac{1}{2}\pi$) is given by

$$w_0=\frac{p(1-\sigma^2)c^4}{128Eh^3\cosh^2 2\alpha\cosh 4\alpha}\Big[(3\cosh 2\alpha\cosh 4\alpha-4\cosh 4\alpha$$
$$+\cosh 2\alpha)(3\cosh 2\alpha\cosh 4\alpha+4\cosh 4\alpha+\cosh 2\alpha)$$
$$-\frac{8(1-\sigma)(3\cosh^2 2\alpha-2)\sinh^4 2\alpha}{2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha}(1-\cosh 2\alpha)(\cosh 2\alpha+1)\Big]$$
$$=\frac{p(1-\sigma^2)c^4\sinh^4 2\alpha}{32Eh^3\cosh^2 2\alpha\cosh 4\alpha}\left[9\cosh^2 2\alpha-4+\frac{2(1-\sigma)(3\cosh^2 2\alpha-2)\sinh^2 2\alpha}{2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha}\right]$$
$$=\frac{p(1-\sigma^2)c^4\sinh^4 2\alpha}{64Eh^3\cosh^2 2\alpha\cosh 4\alpha}\left[9\cosh 4\alpha+1+\frac{2(1-\sigma)(3\cosh 4\alpha-1)\sinh^2 2\alpha}{2\cosh^2 2\alpha-(1-\sigma)\sinh^2 2\alpha}\right]$$
$$\ldots\ldots(29).$$

For a circular plate ($c=0$ and $\alpha=\infty$)

$$w_0=\frac{3p(1-\sigma)(5+\sigma)b^4}{16Eh^3},$$

where b is the semi-diameter of the circle.

For a rectangular strip, supported on its long sides, we must have $c=\infty$ and $\alpha=0$. Since in this case $c\alpha=b$, where b is a half of the width,

$$w_0=\frac{5p(1-\sigma^2)b^4}{2Eh^3}.$$

At the centre of the plate the moment of normal stresses about the x axis is given by

$$M_x=-\frac{Eh^3}{12(1-\sigma^2)}\left[h_1^2\left(\frac{\partial^2 w}{\partial\xi^2}+\sigma\frac{\partial^2 w}{\partial\eta^2}\right)\right]_{\xi=0,\ \eta=\frac{1}{2}\pi}$$

$$= -\frac{Eh^3}{12(1-\sigma^2)c^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \sigma\frac{\partial^2 w}{\partial \eta^2}\right)_{\xi=0,\ \eta=\frac{1}{2}\pi}$$

$$= \frac{pc^2}{96\cosh^2 2\alpha\cosh 4\alpha}\Big\{\big[(\cosh 4\alpha - \cos 2\alpha)(3\cosh 2\alpha\cosh 4\alpha + 4\cosh 4\alpha + \cosh 2\alpha)\big]$$

$$+ \frac{2(1-\sigma)(3\cosh^2 2\alpha - 2)\sinh^4 2\alpha}{2\cosh^2 2\alpha - (1-\sigma)\sinh^2 2\alpha}(\cosh 2\alpha + 1)$$

$$+ \sigma(3\cosh 2\alpha\cosh 4\alpha - 4\cosh 4\alpha + \cosh 2\alpha)(\cosh 4\alpha + \cosh 2\alpha)$$

$$+ \frac{2(1-\sigma)(3\cosh^2 2\alpha - 2)\sinh^4 2\alpha}{2\cosh^2 2\alpha - (1-\sigma)\sinh^2 2\alpha}(\cosh 2\alpha - 1)\Big\}$$

$$= \frac{pc^2\sinh^2 2\alpha}{48\cosh^2 2\alpha\cosh 4\alpha}\Big\{(\cosh 2\alpha + 1)\Big[(2\cosh 2\alpha + 1)(3\cosh 2\alpha - 2)$$

$$+ \frac{(1-\sigma)(3\cosh^2 2\alpha - 1)\sinh^2 2\alpha}{2\cosh^2 2\alpha - (1-\sigma)\sinh^2 2\alpha}\Big]$$

$$+ \sigma(\cosh 2\alpha - 1)\Big[(2\cosh 2\alpha - 1)(3\cosh 2\alpha + 2)$$

$$+ \frac{(1-\sigma)(3\cosh^2 2\alpha - 2)\sinh^2 2\alpha}{2\cosh^2 2\alpha - (1-\sigma)\sinh^2 2\alpha}\Big]\Big\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

For a circle (when $c=0$ and $\alpha=\infty$)

$$M_x = \tfrac{1}{16}\{pa^2(3+\sigma)\}.$$

For a rectangular strip, supported along the long sides ($c=\infty$ and $\alpha=0$), $M_x = \frac{1}{2}pb^2$.

The shearing force is given by

$$V_{\xi z} = -\frac{Eh^3}{12(1-\sigma^2)}h_1\frac{\partial}{\partial \xi}\left[h_1^2\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right)\right]$$

$$= \frac{pc\sqrt{2}\sinh 2\xi\,(\cosh 2\alpha - \cos 2\eta)}{4\cosh 2\alpha\sqrt{(\cosh 2\xi - \cos 2\eta)}} \ldots\ldots(31).$$

At the centre of the plate $V_{\xi z}=0$. At the support

$$V_{\xi z} = -\tfrac{1}{4}(pc\sqrt{2}\tanh 2\alpha)\sqrt{(\cosh 2\alpha - \cos 2\eta)}.$$

For a circle the cross-shearing force at the support is $V_{rz} = -\frac{1}{2}pa$. For a strip, supported along its long sides, the shearing force at any point on the y-axis ($\eta = \frac{1}{2}\pi$), is given by

$$V_{yz} = -\frac{pc\sqrt{2}\sinh 2\xi\,(\cosh 2\alpha + 1)}{4\cosh 2\alpha\sqrt{(1+\cosh 2\xi)}} = -pc\sinh\xi = -py.$$

§ 4. *Semi-elliptic plate.* There are two cases (1) when an elliptic plate is cut along the minor axis, (2) when it is cut along the major axis. In each case suppose the plate to be supported along the diameter in question.

First case.

For the displacement w we can write the expression

$$w = C(3+4\cosh 2\xi + \cosh 4\xi)(3+4\cos 2\eta + \cos 4\eta)$$
$$+\sum_1^\infty A_k \cosh(2k-1)\xi \cos(2k-1)\eta + B_1(\cosh 3\xi \cos\eta + \cosh\xi \cos 3\eta)$$
$$+\sum_2^\infty B_k\left[-\frac{\cosh(2k-1)\xi\cos(2k-3)\eta}{k-1}\right.$$
$$+\left\{\frac{\cosh(2k+1)\xi}{k} - \frac{\cosh(2k-3)\xi}{k-1}\right\}\cos(2k-1)\eta$$
$$\left.+\frac{\cosh(2k-1)\xi}{k}\cos(2k-1)\eta\right] \quad\text{..............................(32).}$$

The expression w satisfies the equation (22) when p is constant if

$$C = \frac{pc^4(1-\sigma^2)}{128Eh^3}.$$

If $\eta = \frac{1}{2}\pi$ or $\eta = -\frac{1}{2}\pi$ the expressions w and $\frac{\partial^2 w}{\partial \eta^2}$ are equal to zero; therefore the conditions that the plate may be supported along the minor axis are satisfied.

Supposing the elliptic edge to be clamped, we must have, when $\xi = \alpha$,

$$w = 0 \text{ and } \frac{\partial w}{\partial \xi} = 0,$$

or

$$8C\cosh 4\alpha(3+4\cos 2\eta + \cos 4\eta) + \sum_1^\infty A_k \cosh(2k-1)\alpha\cos(2k-1)\eta$$
$$+B_1(\cosh 3\alpha\cos\eta + \cosh\alpha\cos 3\eta) + \sum_2^\infty B_k\left[-\frac{\cosh(2k-1)\alpha\cos(2k-3)\eta}{k-1}\right.$$
$$+\left\{\frac{\cosh(2k+1)\alpha}{k} - \frac{\cosh(2k-3)\alpha}{k-1}\right\}\cos(2k-1)\eta$$
$$\left.+\frac{\cosh(2k-1)\alpha\cos(2k+1)\eta}{k}\right] = 0 \quad\text{......................(33),}$$

and

$$32C\sinh\alpha\cosh^3\alpha(3+4\cos 2\eta + \cos 4\eta)$$
$$+\sum_1^\infty A_k(2k-1)\sinh(2k-1)\alpha\cos(2k-1)\eta$$

$$+ B_1 (3 \sinh 3\alpha \cos \eta + \sinh \alpha \cos 3\eta)$$

$$+ \sum_2^\infty B_k \left[- \frac{(2k-1)\sinh (2k-1)\alpha \cos (2k-3)\eta}{k-1} \right.$$

$$+ \left\{ \frac{(2k+1)\sinh (2k+1)\alpha}{k} - \frac{(2k-3) \sinh (2k-3)\alpha}{k-1} \right\} \cos (2k-1)\eta$$

$$\left. + \frac{(2k-1)\sinh (2k-1)\alpha}{k} \cos (2k+1)\eta \right] = 0 \quad \text{..............(34).}$$

But, when η lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, we can express $3 + 4\cos 2\eta + \cos 4\eta$ in the form

$$3+4\cos 2\eta+\cos 4\eta = \frac{768}{\pi} \sum_1^\infty \frac{(-1)^{k+1} \cos (2k-1)\eta}{(2k-5)(2k-3)(2k-1)(2k+1)(2k+3)}.$$

The equations (33) and (34) will therefore yield the following system of equations:

$$\left.\begin{array}{l}
(1) \ \dfrac{6144\, C \cosh^4\alpha}{\pi.3.1.1.3.5} + A_1 \cosh \alpha + B_1 \cosh 3\alpha - B_2 \cosh 3\alpha = 0 \\
(2) \ \dfrac{6144\, C \cosh^4\alpha}{\pi.1.1.3.5.7} + A_2 \cosh^3\alpha + B_1 \cosh \alpha \\
\qquad\qquad + B_2 (\frac{1}{2} \cosh 5\alpha - \cosh \alpha) - \frac{1}{2} B_3 \cosh 5\alpha = 0 \\
(3) \ \dfrac{6144\, C \cosh^4\alpha}{\pi.1.3.5.7.9} + A_3 \cosh 5\alpha + \frac{1}{2} B_2 \cosh 3\alpha \\
\qquad\qquad + B_3 (\frac{1}{3} \cosh 7\alpha - \frac{1}{2} \cosh 3\alpha) - \frac{1}{3} B_4 \cosh 7\alpha = 0 \\
(4) - \dfrac{6144\, C \cosh^4\alpha}{\pi.3.5.7.9.11} + A_4 \cosh 7\alpha + \frac{1}{3} B_3 \cosh 5\alpha \\
\qquad\qquad + B_4 (\frac{1}{4} \cosh 9\alpha - \frac{1}{3} \cosh 5\alpha) - \frac{1}{4} B_4 \cosh 9\alpha = 0 \\
\dotfill \\
(k) \ \dfrac{6144\, C (-1)^{k+1} \cosh^4\alpha}{\pi (2k-5)(2k-3)(2k-1)(2k+1)(2k+3)} \\
\qquad\qquad + A_k \cosh (2k-1)\alpha + \dfrac{1}{k-1} B_{k-1} \cosh (2k-3)\alpha \\
\qquad\qquad + B_k \left\{ \dfrac{1}{k} \cosh (2k+1)\alpha - \dfrac{1}{k-1} \cosh (2k-3)\alpha \right\} \\
\qquad\qquad - \dfrac{1}{k} B_{k+1} \cosh (2k+1)\alpha = 0 \\
\dotfill
\end{array}\right\} (33')$$

$$
\left.
\begin{aligned}
&(1)\ \frac{24576\,C \sinh\alpha \cosh^3\alpha}{\pi.3.1.1.3.5} \\
&\qquad + A_1 \sinh\alpha + 3B_1 \sinh 3\alpha - 3B_2 \sinh^3\alpha = 0 \\
&(2)\ \frac{24576\,C \sinh\alpha \cosh^3\alpha}{\pi.1.1.3.5.7} \\
&\qquad + 3A_2 \sinh 3\alpha + B_1 \sinh\alpha + B_2 (\tfrac{5}{2} \sinh 5\alpha - \sinh\alpha \\
&\qquad - \tfrac{5}{2} B_3 \sinh 5\alpha = 0 \\
&(3)\ \frac{24576\,C \sinh\alpha \cosh^3\alpha}{\pi.1.3.5.7\ 9} \\
&\qquad + 5A_3 \sinh 5\alpha + \tfrac{3}{2} B_2 \sinh 3\alpha \\
&\qquad + B_3 (\tfrac{7}{3} \sinh 7\alpha - \tfrac{3}{2} \sinh 3\alpha) - \tfrac{7}{3} B_4 \sinh 7\alpha = 0 \\
&(4)\ - \frac{24576\,C \sinh\alpha \cosh^3\alpha}{\pi.3.5.7.9.11} \\
&\qquad + 7A_4 \sinh 7\alpha + \tfrac{5}{3} B_3 \sinh 5\alpha \\
&\qquad + B_4 (\tfrac{9}{4} \sinh 9\alpha - \tfrac{5}{3} \sinh 5\alpha) - \tfrac{9}{4} B_5 \sinh 9\alpha = 0 \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&(k)\ \frac{24576\,C \sinh\alpha \cosh^3\alpha}{\pi (2k-5)(2k-3)(2k-1)(2k+1)(2k+3)} \\
&\qquad + (2k-1) A_k \sinh(2k-1)\alpha + \frac{2k-3}{k-1} B_{k-1} \sinh(2k-3)\alpha \\
&\qquad + B_k \left\{ \frac{2k+1}{k} \sinh(2k+1)\alpha - \frac{2k-3}{k-1} \cosh(2k-3)\alpha \right\} \\
&\qquad - \frac{2k+1}{k} B_{k+1} \sinh(2k+1)\alpha = 0 \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\right\} \quad (34')
$$

From these equations we can determine the coefficients A_k and B_k, and therefore obtain the solution for a semi-elliptic plate supported along the minor axis and clamped along the elliptic boundary.

Second case.

When the plate is cut along its major axis the expression for w can be written as follows:

$$w = C(3 - 4\cosh 2\xi + \cosh 4\xi)(3 - 4\cos 2\eta + \cos 4\eta)$$

$$+ \sum_1^\infty A_k \sinh(2k-1)\xi \sin(2k-1)\eta + B_1(\sinh 3\xi \sin\eta + \sinh\xi \sin 3\eta)$$

$$+ \sum_2^\infty B_k \left[- \frac{\sinh(2k-1)\xi \sin(2k-3)\eta}{k-1} \right.$$

$$+\left\{\frac{\sinh(2k+1)\xi}{k}-\frac{\sinh(2k-3)\xi}{k-1}\right\}\sin(2k-1)\eta$$
$$\left.+\frac{\sinh(2k-1)\xi\sin(2k+1)\eta}{k}\right] \quad \ldots\ldots\ldots\ldots(35).$$

Suppose the plate supported along the major axis. For this it is necessary that, on the part of this axis that is given by $\xi=0$, $\pi>\eta>0$, w and $\frac{\partial^2 w}{\partial\xi^2}$ should vanish, and that on the remaining parts of this axis, where $\eta=0$ or π, w and $\frac{\partial^2 w}{\partial\eta^2}$ should vanish.

The chosen expressions for w satisfy completely these conditions. Observing that we can put

$$3-4\cos 2\eta+\cos 4\eta=\frac{768}{\pi}\sum_1^\infty\frac{\sin(2k-1)\eta}{(2k-5)(2k-3)(2k-1)(2k+1)(2k+3)},$$

it will be evident that the coefficients A_k and B_k can be determined, as in the *First Case*, so that at the clamped elliptic boundary, where $\xi=\alpha$, the conditions $w=0$ and $\frac{\partial w}{\partial\xi}=0$ may be satisfied.

Petrograd, *August*, 1920.

SUR QUELQUES SERIES ET PRODUITS INFINIS.

Par *S. P. Sörensen.*

1. Désignons par p et r deux nombres positifs entiers et posons $1\leq r\leq p-1$. Posons ensuite pour abréger

$$(p,r)_n=\sum_{s=0}^{s=\infty}\frac{1}{(ps+r)^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

il suit par application des formules bien connues

$$\left.\begin{aligned}\psi(x)&=D_x\log\Gamma(x)=-C+\sum_{s=0}^{s=\infty}\left(\frac{1}{s+1}-\frac{1}{x+s}\right)\\ \psi^{(n)}(x)&=(-1)^{n+1}n!\sum_{s=0}^{s=\infty}\frac{1}{(x+s)^{n+1}}\end{aligned}\right\}\ldots\ldots(2)$$

le développement

$$\log\Gamma\left(\frac{r}{p}+x\right)=\log\Gamma\left(\frac{r}{p}\right)+\frac{x}{1}-C$$
$$+\sum_{s=0}^{s=r}\left(\frac{1}{s+1}-\frac{p}{ps+r}\right)+\sum_{n=2}^{n=\infty}(-1)^n\frac{(px)^n}{n}(p,r)_n$$
$$|x|<\frac{r}{p}\ldots\ldots(3).$$

Remplaçant en (3) x de $-x$, r de $p-r$, on trouve

$$\log\Gamma\left(\frac{p-r}{p}-x\right)=\log\Gamma\left(\frac{p-r}{p}\right)$$

$$-\frac{x}{1}-C+\sum_{s=0}^{s=\infty}\left(\frac{1}{s+1}-\frac{p}{ps+p-r}\right)+\sum_{n=2}^{n=\infty}\frac{(px)^n}{n}(p,\,p-r)_n \quad \ldots\ldots(4).$$

L'addition des équations (3) et (4) donne

$$\log\frac{\sin\pi r/p}{\sin\pi(r/p+x)}=\sum_{s=1}^{s=\infty}\frac{(px)^{2s}}{2s}P_{2s}-\sum_{s=0}^{s=\infty}\frac{(px)^{2s+1}}{2s+1}P_{2s+1}\ldots\ldots(6),$$

car on a $\qquad \Gamma(x)\,.\,\Gamma(1-x)=\dfrac{\pi}{\sin\pi x}$,

en posant ensuite

$$(p,\,r)_{2q}\;+(p,\,p-r)_{2q}\;=P_{2q},$$
$$(p,\,r)_{2q+1}-(p,\,p-r)_{2q+1}=P_{2q+1}.$$

Soit encore une fonction $f(x)$ quelconque donné par son developpement convergent

$$f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

et remplaçant x de $e^{2s\pi i/n}$, $s=0,1,2,3,\ldots,n-1$, on trouve l'équation suivante

$$\frac{1}{n}\sum_{s=0}^{n-1}f(x\,.\,e^{2s\pi i/n})=a_0+a_nx^n+a_{2n}x^{2n}+a_{3n}x^{3n}+\ldots+a_{rn}x^{rn}+\ldots \quad \ldots\ldots(6a).$$

Par application des formules (6) et (6a) on trouve aisément

$$\frac{1}{2q}\sum_{s=0}^{2q-1}\log\frac{\sin\pi r/p}{\sin\pi(r/p+x\,.\,e^{2s\pi i/2q})}=\sum_{s=1}^{s=\infty}\frac{(px)^{2sq}}{2sq}P_{2sq};\qquad |x|<\frac{r}{p} \quad \ldots\ldots(7),$$

ou ce qui est le même

$$\sum_{s=1}^{s=\infty}\frac{(px)^{2sq}}{2sq}P_{2sq}=\log\sin\frac{\pi r}{p}-\frac{1}{2q}\log\prod_{s=1}^{s=2q-1}\sin\pi\left(\frac{r}{p}+x\,.\,e^{2s\pi i/2q}\right) \quad \ldots\ldots(8).$$

En posant $\qquad P=\prod_{s=1}^{2q-1}\sin\pi\left(\dfrac{r}{p}+x\,.\,e^{2s\pi i/2q}\right)$

on trouve encore

$$P=\sin\left(\pi x+\frac{\pi r}{p}\right)\sin\left(\frac{\pi r}{p}-\pi x\right)\prod_{s=1}^{q-1}\sin\pi\left(\frac{r}{p}+x\,.\,e^{2s\pi i/2q}\right)$$
$$\times\sin\pi\left(\frac{r}{p}+x\,.\,e^{-2s\pi i/2q}\right),$$

$$P=\frac{\sin(\pi r/p+\pi x)\sin(\pi r/p-\pi x)}{2^{q-1}}$$

$$\times\prod_{s=1}^{q-1}\left\{\cosh\left(2\sin\frac{s\pi}{q}\right)-\cos\left(\frac{2\pi r}{p}+2\pi x\cos\frac{s\pi}{q}\right)\right\}.$$

L'équation (8) peut donc s'écrire

$$\sum_{s=1}^{s=\infty}\frac{(px)^{2sq}}{2sq}P_{2qs}=\log\sin\frac{\pi r}{p}-\frac{1}{2q}\log P\ldots\ldots..(9).$$

Par application de l'équation

$$\frac{1}{c}+\frac{y}{c^2}+\frac{y^2}{c^3}+\ldots=\frac{1}{c-y};\quad\left|\frac{c}{y}\right|>1,$$

on tire en posant $c=(ps+r)^{2q}$, $y=(px)^{2q}$,

$$\frac{1}{(ps+r)^{2q}}+\frac{(px)^{2q}}{(ps+r)^{4q}}+\frac{(px)^{4q}}{(ps+r)^{6q}}+\ldots=\frac{1}{(ps+r)^{2q}-(px)^{2q}}$$
$$\ldots\ldots(10),$$

et encore en supposant $c=(ps+p-r)^{2q}$, $y=(px)^{2q}$,

$$\frac{1}{(ps+p-r)^{2q}}+\frac{(px)^{2q}}{(ps+p-r)^{4q}}+\frac{(px)^{4q}}{(ps+p-r)^{6q}}+\ldots$$
$$=\frac{1}{(ps+p-r)^{2q}-(px)^{2q}}\ldots\ldots(11).$$

De (10) et (11) on tire aisément

$$P_{2q}+(px)^{2q}P_{4q}+(px)^{4q}P_{6q}+\ldots$$
$$=\sum_{s=0}^{s=\infty}\left\{\frac{1}{(ps+r)^{2q}-(px)^{2q}}+\frac{1}{(ps+p-r)^{2q}-(px)^{2q}}\right\}\ldots\ldots(12).$$

Multiplication de (12) par $p^{2q}x^{2q-1}$ et l'intégration de 0 à x donnent

$$\frac{(px)^{2q}}{2q}P_{2q}+\frac{(px)^{4q}}{4q}P_{4q}+\ldots$$
$$=-\frac{1}{2q}\log\prod_{s=0}^{s=\infty}\left\{1-\frac{x^{2q}}{(s+r/p)^{2q}}\right\}\left\{1-\frac{x^{2q}}{(s+1-r/p)^{2q}}\right\},$$

$$\sum_{s=1}^{s=\infty}\frac{(px)^{2sq}}{2sq}P_{2sq}=-\frac{1}{2q}\log\prod_{s=0}^{s=\infty}\left\{1-\frac{x^{2q}}{(s+r/p)^{2q}}\right\}\left\{1-\frac{x^{2q}}{(s-r/p)^{2q}}\right\}$$
$$+\frac{1}{2q}\log\left\{1-\left(\frac{px}{r}\right)^{2q}\right\}.$$

Puis, on trouve, par application de la formule (8)

$$\prod_{s=0}^{s=\infty}\left\{1-\frac{x^{2q}}{(s+r/p)^{2q}}\right\}\left\{1-\frac{x^{2q}}{(s-r/p)^{2q}}\right\}=\frac{1-(px/r)^{2q}}{(\sin\pi r/p)^{2q}}\cdot P;$$
$$|x|<r/p.$$

Ex. $r=1$, $p=2$, $q=1$,

$$\prod_{s=0}^{s=\infty}\left\{1-\frac{4x^2}{(2s+1)^2}\right\}=\cos\pi x.$$

2. Considérons encore la formule

$$\sum_{s=1}^{s=\infty}(-1)^{s+1}\frac{(px)^{2sq}}{2sq}P_{2sq}=\sum_{s=1}^{s=\infty}\frac{(px)^{2sq}}{2sq}P_{2sq}-2\sum_{s=1}^{s=\infty}\frac{(px)^{4sq}}{4sq}P_{4sq}\ldots(13).$$

La valeur de $\sum\limits_{s=1}^{s=\infty}\frac{(px)^{4sq}}{4sq}P_{4sq}$ peut s'écrire aisément, quand on dans (9) remplace q par $2q$. On trouve ainsi

$$\sum_{s=1}^{s=\infty}\frac{(px)^{4qs}}{4sq}P_{4qs}=\log\sin\frac{\pi r}{p}-\frac{1}{4q}\cdot P_1.$$

La valeur de P_1 se trouve, quand on dans l'expression de P remplace q par $2q$. On trouve

$$P_1=\frac{\sin(\pi r/p+\pi x)\cdot\sin(\pi r/p-\pi x)}{2^{2q-1}}$$
$$\times\prod_{s=1}^{2q-1}\left\{\cosh\left(2\sin\frac{s\pi}{2q}\right)-\cos\left(\frac{2\pi r}{p}+2\pi x\cos\frac{s\pi}{2q}\right)\right\}.$$

La formule (13) donne

$$\sum_{s=1}^{s=\infty}(-1)^{s+1}\frac{(px)^{2sq}}{2sq}P_{2sq}=\frac{1}{2q}\log\frac{P_1}{P_1\left(\sin\frac{\pi r}{p}\right)^{2q}}\ldots\ldots(14),$$

et on a ensuite

$$\frac{1}{c}-\frac{y}{c^2}+\frac{y^2}{c^3}-\frac{y^3}{c^4}+\ldots=\frac{1}{c+y},\quad\left|\frac{c}{y}\right|<1.$$

Posant ici $c=(ps+r)^{2q}$, $y=(px)^{2q}$ on trouve par le même procedé, que nous avons indiqué précédemment la formule

$$\prod_{s=0}^{s=\infty}\left\{1+\frac{x^{2q}}{(s+r/p)^{2q}}\right\}\left\{1+\frac{x^{2q}}{(s-r/p)^{2q}}\right\}$$
$$=\frac{P_1\{1+(px/r)^{2q}\}}{P(\sin\pi r/p)^{2q}},\quad|x|<r/p.$$

De ces formules générales on peut déduire des resultats spéciales, mais je me réserve de revenir à cette question par une autre occasion.

THE CONNEXION BETWEEN THE SUM OF THE SQUARES OF THE DIVISORS AND THE NUMBER OF THE PARTITIONS OF A GIVEN NUMBER.

By *Major P. A. MacMahon.*

IN various papers* I have considered the partitions of a number as determined by a succession of integers in *descending order of magnitude.*

I have, from this point of view, dealt with an array of numbers ordered in such wise that a descending order of magnitude is in evidence in each column and in each row of the array and have defined such an array as a two-dimensional partition.

The enumerating generating function I found to be

$$F(q) = \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3 \ldots (1-q^s)^s \ldots},$$

when the partitions are unrestricted both in regard to number and magnitude.

Thus the 13 partitions of 4 are

4	31	3	22	2	211	21	2	1111	111	11	11	1
		1		2		1	1		1	11	1	1
							1				1	1
												1

and the expansion of the above fraction gives a term $13q^4$.

We find that

$$q\frac{d}{dq}\log F(q) = \frac{q}{1-q} + \frac{2^2q^2}{1-q^2} + \frac{3^2q^3}{1-q^3} + \ldots = \Sigma\sigma_2(n)\,q^n,$$

where $\sigma_2(n)$ denotes the sum of the squares of the divisors of n.

Writing

$$F(q) = 1 + B(1)\,q + B(2)\,q^2 + \ldots + B(n)\,q^n + \ldots,$$

and operating with $q\dfrac{d}{dq}\log$, we find by comparison

$$nB(n) = \sigma_2(n) + B(1)\,\sigma_2(n-1) + B(2)\,\sigma_2(n-2) + \ldots$$
$$+ B(s)\,\sigma_2(n-s) + \ldots.$$

* *Combinatory Analysis* (Camb. Univ. Press, 1915–16).

The denominator of $F(q)$ may be written

$$\begin{aligned}(1-q)(1-q^2)(1-q^3)(1-q^4)&\ldots\\ (1-q^2)(1-q^3)(1-q^4)&\ldots\\ (1-q^3)(1-q^4)&\ldots\\ (1-q^4)&\ldots\\ &\ldots\end{aligned}$$

and I have shown (*loc. cit.*) that if s rows only are retained the function enumerates the partitions when the array is restricted to have at most s rows. Thus, when s is 1, we have the case of ordinary linear partitions for which $\sigma_1(n)$, the sum of the divisors of n, is in evidence, and we have seen above that, when s is ∞, $\sigma_2(n)$ presents itself. It is interesting to enquire concerning the intermediate stages when s has some value between unity and infinity. For s rows,

$$F_s(q)=\frac{1}{(1-q)(1-q^2)^2(1-q^3)^3\ldots(1-q^s)^s\{(1-q^{s+1})(1-q^{s+2})\ldots\}^s}.$$

and
$$q\frac{d}{dq}\log F_s(q)$$

$$=\frac{q}{1-q}+\frac{2^2q^2}{1-q^2}+\frac{3^2q^3}{1-q^3}+\ldots$$
$$+\frac{s^2q^s}{1-q^s}+\frac{s(s+1)q^{s+1}}{1-q^{s+1}}+\frac{s(s+2)}{1-q^{s+2}}+\ldots \textit{ ad inf.}$$

As regards the coefficient herein of q^n, if d be a divisor $\leq s$, let it be squared, but if it be $>s$, let it be multiplied by s. The coefficient of q^n is then

$$\sum_{\ngtr s} d^2 + s\sum_{>s} d\,;$$

and we may write this

$$\sigma_{2\,\ngtr s}(n)+s\,\sigma_{1\,>s}(n).$$

Hence
$$q\frac{d}{dq}\log F_s(q)=\sum_1^\infty n\,\{\sigma_{2\,\ngtr s}(n)+s\,\sigma_{1\,>s}(n)\}\,q^n\,;$$

and it will be noted that the arithmetical function becomes $\sigma_1(n)$, $\sigma_2(n)$ for $s=1$ and ∞ respectively.

Writing

$$F_s(q)=1+B_s(1)\,q+B_s(2)\,q^2+\ldots+B_s(n)\,q^n+\ldots,$$

we find, as before,

$$nB_s(n) = \{\underset{\leqq s}{\sigma_2}(n) + s\underset{>s}{\sigma_1}(n)\} + B_s(1)\{\underset{\leqq s}{\sigma_2}(n-1) + s\underset{>s}{\sigma_1}(n-1)\}$$
$$+ B_s(2)\{\underset{\leqq s}{\sigma_2}(n-2) + s\underset{>s}{\sigma_1}(n-2)\} + \dots .$$

The value of $\underset{\leqq s}{\sigma_2}(n) + s\underset{>s}{\sigma_1}(n)$ is given in the annexed Table so far as $s = 10$, $n = 10$, and in the second Table the value of $B_s(n)$.

I.

$s . n =$ 1	2	3	4	5	6	7	8	9	10	...
1	1	3	4	7	6	12	8	15	13	18	...
2		5	7	13	11	23	15	29	25	35	...
3			10	17	16	32	22	41	37	50	...
4				21	21	38	29	53	46	65	...
5					26	44	36	61	55	80	...
6						53	43	69	64	90	...
7							50	77	73	100	...
8								85	82	110	...
9									91	120	...
10										130	...
											...

II.

$n =$	1	2	3	4	5	6	7	8	9	10	...
1	1	2	3	5	7	11	15	22	30	42	...
2		3	5	10	16	29	45	75	115	181	...
3			6	12	21	40	67	117	193	319	...
4				13	23	45	78	141	239	409	...
5					24	47	83	152	263	457	...
6						48	85	157	274	481	...
7							86	159	279	492	...
8								160	281	497	...
9									282	499	...
10										500	...
											...

In Table I. the values of $\underset{\leqq s}{\sigma_2}(n) + s\underset{>s}{\sigma_1}(n)$, for a fixed value of s and successive values of n, are obtained by reading down the slanting side of the Table as far as the s^{th} row and then proceeding along that row to the right.

Similarly in Table II. the values of $B_s(n)$, for a fixed value of s and successive values of n, are obtained by reading down the slanting side as far as the s^{th} row and then proceeding along that row.

It may be added that if

$$f(q) = \{(1-q)(1-q^2)(1-q^3)\ldots\}^{-1},$$

and, after Euler, if ϕ_m denote the number of primitive m^{th} roots of unity,

$$F(q) = \prod_{m_1}^{\infty} \{f(q^m)\}^{\phi_m},$$

this establishes a connection with Elliptic Functions.

ELECTROMAGNETISM AND DYNAMICS.

By *Dr. H. Bateman.*

DIFFERENT pictures of physical phenomena may be obtained by adopting different conventions with regard to the types of discontinuity that are to be regarded as admissible in the mathematical specification of physical quantities. A picture of considerable interest is based on the idea that, when all types of energy and momentum are taken into consideration, these physical quantities are distributed throughout space in such a manner that we can speak of densities of energy and momentum that are continuous functions of the rectangular coordinates (x, y, z), used to specify the position of a point, and of the time t. A different picture is obtained if the densities of energy and momentum are allowed to change suddenly in value as the point (x, y, z) crosses the boundary of an electron or some other entity such as a hypothetical light quantum of limited size.

It is doubtful whether the first picture is adequate for a complete description of all the physical phenomena with which man is acquainted, but in any case it is well worth while to give it a fair trial. The type of analysis associated with this picture of phenomena will be called *continuous analysis*. The main principles of the analysis are already familiar, as they play an important part in Maxwell's electromagnetic theory, the theory of electrons and the theory of relativity in both the restricted and general forms.*

* In all these theories there seems, however, to be some type of discontinuity at the boundary of a particle of matter when the density of electricity is not zero at the boundary.

We are interested here in the application of continuous analysis to the case in which the changes of the energy and momentum associated with a region of space are produced entirely by the fluxes of energy and momentum across the boundary of the region. In this case there are no body forces. This means that if we write the equations governing the changes of energy and momentum in the usual form

$$\left.\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial Y_x}{\partial y} + \frac{\partial Z_x}{\partial z} - \frac{\partial G_x}{\partial t} &= F_x \\
\frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Z_y}{\partial z} - \frac{\partial G_y}{\partial t} &= F_y \\
\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} - \frac{\partial G_z}{\partial t} &= F_z \\
\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z} + \frac{\partial W}{\partial t} &= -F_t
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned}
Y_z &= Z_y, \quad Z_x = X_z, \quad X_y = Y_x \\
S_x &= c^2 G_x, \quad S_y = c^2 G_y, \quad S_z = c^2 G_z
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2),$$

where W is the density of energy, (S_x, S_y, S_z) the components of a vector S specifying the flow of energy, (G_x, G_y, G_z) the components of a vector G representing the density of linear momentum, $X_x, Y_x, Z_x, X_y, Y_y, Z_y, X_z, Y_z, Z_z$ are the components of stress or minus the components of three vectors p_x, p_y, p_z representing the fluxes of the three components of linear momentum, then the body force F with components (F_x, F_y, F_z), and the quantity F_t, which specifies the rate at which this force does work, are supposed to vanish on account of the field equations and the dynamical equations of motion of the electric charges producing the field or fields.

In such a case we can say also that in virtue of (1) and (2)

$$\left.\begin{aligned}
\frac{\partial}{\partial x}(yX_z - zX_y) + \frac{\partial}{\partial y}(yY_z - zY_y) + \frac{\partial}{\partial z}(yZ_z - zZ_y) - \frac{\partial}{\partial t}(yG_z - zG_y) &= 0 \\
\frac{\partial}{\partial x}(xS_x + c^2tX_x) + \frac{\partial}{\partial y}(xS_y + c^2tY_x) + \frac{\partial}{\partial z}(xS_z + c^2tZ_x) + \frac{\partial}{\partial t}(xW - c^2tG_y) &= 0
\end{aligned}\right\} \quad \ldots\ldots(3).$$

The first equation and the two similar equations may be interpreted to mean that any change in the angular momentum associated with a region is produced entirely by a flux of angular momentum across the boundary of the region.* The

* For further details see M. Abraham, *Theorie der Elektrizität*, Bd 2.

quantity $yG_z - zG_y$ may indeed be regarded as the density of the angular momentum about the axis of x and $yX_z - zY_y$, $yY_z - zY_z$, $yZ_z - zZ_y$ as the components of a vector q_x which specifies the flux of the angular momentum about the axis of x.

The second equation gives a generalisation of the theorem that the centre of mass of a complete mechanical system moves along a straight line * provided we regard mass m and total energy ϵ as being connected by the relation

$$\epsilon = mc^2 \text{..........................} (4),$$

where c is the velocity of light (which is assumed here to be constant). A set of 16 quantities X_x, X_y, X_z, Y_x, Y_y, Y_z, Z_x, Z_y, Z_z, G_x, G_y, G_z, S_x, S_y, S_z, W satisfying equations (2) will be regarded as the components of *a symmetrical tensor* T. The familiar tensor T_e of electromagnetic theory gives $F = 0$ and $F_t = 0$ in regions not occupied by electricity, but when there is a continuous volume distribution of electric charges and convection currents it is difficult to satisfy these equations and get motions that are of physical interest.† The situation is not improved much by adding a mass tensor T_m which is different from zero only within the charged particles, for the equations of motion obtained by equating the total body force to zero and integrating over the electron do not satisfy the requirements of quantum theory.

An attempt has been made recently to generalise electromagnetic theory by supposing that an electric charge produces both an electromagnetic field and a scalar field, the latter being specified by a retarded potential ψ which is an invariant under the transformations of the theory of relativity. A new tensor T_s, depending on ψ, was added to the usual electromagnetic tensor T_e, and it was found that non-radiating electronic orbits were possible,‡ the flow of electromagnetic energy to infinity being balanced by a flow to infinity of negative energy of a new type, provided that the flow is calculated for an interval of time between two instants at which the velocity of the moving electron is a maximum or minimum, and that the electron is treated as a point charge

* A. Einstein, *Ann. d. Phys.* (4), Bd. 20 (1906), p. 627; G. Herglotz, *Ann. d. Phys.* (4), Bd. 36 (1911), p. 493; E. Bessel Hagen, *Math. Ann.*, Bd. 84 (1921), p. 258.

† The matter has been discussed by Levi Civita and his co-workers. For references see *Messenger of Mathematics*, vol. xlvi (1917), p. 140.

‡ *Physical Review*, vol. xx (1922), p. 243. A long calculation has also shown that there is a continual radiation of negative energy to infinity when two negative electric poles with the same charge move uniformly in a circle at opposite ends of a diameter, the controlling field being electrostatic. It is thought that this type of steady motion may be impossible because there is this continual radiation of negative energy.

moving under the influence of an electrostatic field which is not altered by the field of the electron. The introduction of negative energy is a defect in the analysis, but the same defect appears when the idea of a continuous distribution of stress is applied to gravitation, as Maxwell noticed long ago. The tensor $T_e + T_s$ appeared also to give satisfactory results when applied to the interior of a charged particle with a spherical boundary, but a subsequent investigation revealed a difficulty with regard to the boundary conditions at the surface of the particle, and it is now believed that other tensors must be added.

The object of this note is to point out that many important requirements may be satisfied by using a tensor T made up of four parts:

$$T = T_e + T_s + T_c + T_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The first part T_e is the usual tensor connected with the electromagnetic field (E, H). Its components are of types

$$\left.\begin{aligned} W &= \tfrac{1}{2}(E^2 + H^2), & S_x &= c\,(E_y H_z - E_z H_y) \\ X_x &= E_x^2 + H_x^2 - W, & X_y &= E_x E_y + H_x H_y \end{aligned}\right\} \ldots\ldots (6),$$

and we have

$$\left.\begin{aligned} F_x &= \rho\left[E_x + \frac{1}{c}(v_y H_z - v_z H_y)\right] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_t &= \rho\,(v_x E_x + v_y E_y + v_z E_z) \end{aligned}\right\} \ldots\ldots\ldots (7),$$

where ρ is the density and v the velocity of electricity at the point x, y, z at time t. These quantities are connected with E and H by the usual equations

$$\left.\begin{aligned} \operatorname{curl} H &= \frac{1}{c}\left(\frac{\partial E}{\partial t} + \rho v\right) & \operatorname{div} E &= \rho \\ \operatorname{curl} E &= -\frac{1}{c}\frac{\partial H}{\partial t} & \operatorname{div} H &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (8)$$

of the theory of electrons.

The tensor T_c is of a very interesting type. It possesses, for instance, the properties

$$\left.\begin{aligned} W > 0 \quad S_x^2 + S_y^2 + S_z^2 &\leq c^2 W^2 \\ X_x + Y_y + Z_z + W &= 0 \\ (X_x + W)\,S_x + X_y S_y + X_z S_z &= 0 \\ S_x S_y + c^2 Z_y X_z &= c^2 (X_x + Y_y)\, X_y \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (9),$$

and an interesting question arises as to whether the total tensor T should be required to possess any of these properties.

The tensor T_s has components of the following types

$$\left.\begin{aligned} W &= \frac{2}{c^2}\psi\frac{\partial^2\psi}{\partial t^2}+\Gamma, \quad & S_x &= -2\psi\frac{\partial^2\psi}{\partial x\,\partial t} \\ X_x &= -2\psi\frac{\partial^2\psi}{\partial x^2}+\Gamma, \quad & X_y &= -2\psi\frac{\partial^2\psi}{\partial x\,\partial y} \end{aligned}\right\} \quad\ldots\ldots\ldots(10),$$

where ψ is connected with ρ and v by the equation

$$\Box\psi \equiv \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}$$

and $$\Gamma \equiv \left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2.$$

The potential ψ may be calculated from the distribution of the quantity $\rho\sqrt{\{1-(v^2/c^2)\}}$ by means of the usual formula for a retarded potential.

The tensor T_s does not generally possess the properties mentioned above in (9), but this does not necessarily mean that it has no physical meaning. The force F arising from T_s has components of type

$$F_x^{(s)} = 2\psi\frac{\partial}{\partial x}\left[\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right] \quad\ldots\ldots\ldots(11),$$

and can be used to balance the electrical force derived from T_e in the case of a statical distribution of electricity with radial symmetry. The density ρ is then determined by an equation of type $\psi = b\rho^2$, where b is a constant and two laws of density seem to be possible. This result suggests that there may be two types of discrete particles of electricity in each of which ρ has a value different from zero at the boundary and a numerically greater value at the centre. The percentage difference between the maximum and minimum values of ρ is much greater in one case than in the other.

On account of the discontinuity in ρ at the boundary the tensor $T'' = T_e + T_s$ will not give a stress system which is continuous at the boundary. It is true that ψ, E, and Γ are continuous, but the second derivatives of ψ are discontinuous. The discontinuity in $xX_x + yX_y + zX_z$ is thus the same as that of

$$-2\psi\left(x\frac{\partial^2\psi}{\partial x^2}+y\frac{\partial^2\psi}{\partial y^2}+z\frac{\partial^2\psi}{\partial z^2}\right),$$

and when ψ is a function only of $r=\sqrt{(x^2+y^2+z^2)}$, we find, on writing $\psi=f(r)$, that

$$\frac{\partial\psi}{\partial x}=\frac{x}{r}f'(r),\quad \frac{\partial^2\psi}{\partial x^2}=\frac{x^2}{r^2}f''(r)+\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r),$$

$$\frac{\partial^2\psi}{\partial x\partial y}=\frac{xy}{r^2}f''(r)-\frac{xy}{r^3}f'(r).$$

The discontinuity in

$$x\frac{\partial^2\psi}{\partial x^2}+y\frac{\partial^2\psi}{\partial y^2}+z\frac{\partial^2\psi}{\partial z^2}$$

is thus the same as that of

$$x\left[f''(r)+\frac{2}{r}f'(r)\right]\text{ or of }-x\rho.$$

The quantity $\qquad x(X_x-2\psi\rho)+yX_y+zX_z$

thus has no discontinuity at the boundary.

It is clear from this result that we can make the stress at the boundary continuous if we use the tensor $T''=T_e+T_s+T_c$, where T_c is a tensor with components which are all zero outside the charged particle and which have the following values inside

$$\left.\begin{aligned} X_x=Y_y=Z_z=W=-2\psi_0\rho_0\sqrt{\{1-(v_0^2/c^2)\}}\\ Y_z=Z_x=X_y=S_x=S_y=S_z=0\end{aligned}\right\}\ldots\text{.(12)},$$

where ψ_0, ρ_0, and v_0 are the boundary values of ψ, ρ, and v, and are constant throughout the electric particle. This form of tensor has been chosen so as to satisfy the requirements of the theory of restricted relativity: it will be called a *chaotic tensor* because there is no momentum associated with it.

With the tensor $T''=T_e+T_s+T_c$ the density of energy W is discontinuous at the boundary of the charged particle. To remedy this we add a fourth tensor T_m with components of types

$$\left.\begin{aligned} W=\frac{2\psi_0\rho}{\sqrt{\{1-(v^2/c^2)\}}},\qquad & S_x=\frac{2\psi_0\rho v_x}{\sqrt{\{1-(v^2/c^2)\}}}\\ X_x=-\frac{2\psi_0\rho v_x^2}{c^2\sqrt{\{1-(v^2/c^2)\}}},\quad & X_y=-\frac{2\psi_0\rho v_x v_y}{c^2\sqrt{\{1-(v^2/c^2)\}}}\end{aligned}\right\}\ldots\text{(13)}.$$

Since ρ and v satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0\quad\ldots\ldots(14),$$

the components of the body force $F^{(m)}$ derived from this tensor T_m are of type*

$$-\frac{2\psi_0\rho}{c^2}\frac{d}{dt}\left[\frac{v_x}{\sqrt{\{1-(v^2/c^2)\}}}\right]=F_x^{(m)} \quad\text{.........(15)},$$

while

$$-2\psi_0\rho\frac{d}{dt}\left[\frac{1}{\sqrt{\{1-(v^2/c^2)\}}}\right]=F_t^{(m)} \quad\text{.........(16)},$$

and we clearly have the relation $F_t^{(m)}=(v.F)^{(m)}$, which is of the same form as the relation $F_t^{(e)}=(v.F)^{(e)}$ satisfied by the electromagnetic force $F^{(e)}$.

Assuming that the total body force derived from the tensor

$$T=T_c+T_s+T_c'+T_m$$

vanishes, and that the energy equation is also satisfied, we have the equations of motion

$$\left.\begin{aligned}&\frac{2\rho\psi_0}{c^2}\frac{d}{dt}\left[\frac{v_x}{\sqrt{\{1-(v^2/c^2)\}}}\right]\\&=\rho\left(E_x+\frac{1}{c}v_yH_z-\frac{1}{c}v_zH_y\right)+2\psi\frac{\partial}{\partial x}\left\{\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right\}\\&\quad+2\rho\psi_0\frac{d}{dt}\left[\frac{1}{\sqrt{\{1-(v^2/c^2)\}}}\right]\\&=\rho\left(v_xE_x+v_yE_y+v_zE_z\right)-2\psi\frac{\partial}{\partial t}\left\{\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right\}\end{aligned}\right\}\quad\text{...(17)},$$

and from these we may deduce that

$$2\psi\frac{d}{dt}\left\{\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right\}=0 \quad\text{..............(18)}.$$

This equation is compatible with the Lorentz-Fitzgerald contraction and with the principle that an electron does not alter in size when it passes from one state of motion with velocity v to another.

The above equations of motion apply to each element of the electron, and they should determine the form of the electron, as well as its motion as a whole. To determine the latter we may multiply the equations by $dx\,dy\,dz$ and integrate over the region of space occupied by the electron at an instant of time t. A simple and accurate expression of the result of this integration is not to be expected, but one can get a general idea of the most important terms.

* We write $\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}=\frac{d}{dt}$.

Assuming that ρ has the same sign throughout the electron, we have as a first approximation

$$\frac{2}{c^2}\psi_0\int\rho dxdydz\frac{d}{dt}\left[\frac{v_x}{\sqrt{\{1-(v^2/c^2)\}}}\right]=\frac{2e\psi_0}{c^2}\frac{d}{dt}\left[\frac{v_x}{\sqrt{\{1-(v^2/c^2)\}}}\right] \quad \ldots\ldots(19),$$

where e is a constant representing the electric charge of the electron and where the velocity v on the right-hand side refers to some point of the electron, and is a kind of mean value of the original v. With a good approximation it may be taken to be the velocity of the point at which $|\rho\sqrt{\{1-(v^2/c^2)\}}|$ has its greatest value.

If $\psi\rho\sqrt{\{1-(v^2/c^2)\}}$ is constant over the surface of the electron, we may write

$$\left.\begin{aligned}\int\rho\, dxdydz\left(E_x+\frac{1}{c}v_yH_z-\frac{1}{c}v_zH_y\right)+\int 2\psi\, dxdydz\frac{\partial}{\partial x}\left\{\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right\}\\ =\int\rho\, dxdydz\left[E_x+\frac{1}{c}(v_yH_z-v_yH_y)-2\sqrt{\left(1-\frac{v^2}{c^2}\right)}\frac{\partial\psi}{\partial x}\right]\end{aligned}\right\} \quad \ldots\ldots(20).$$

The portion of this integral, which arises from the external field, is approximately

$$e\left[E_x^*+\frac{1}{c}(v_yH_z^*-v_zH_y^*)-2\sqrt{\left(1-\frac{v^2}{c^2}\right)}\frac{\partial\psi^*}{\partial x}\right] \quad \ldots(21).$$

When the electron is stationary, and the external field is electrostatic, the expression within brackets reduces to $3E_x^*$, and the total force exerted by the external electric charges is exactly three times the usual electrostatic force.

The force exerted by the electron on itself is more difficult to calculate. In the case of a solitary stationary electron with a spherical boundary and a radially symmetric distribution of charge, we have for external points

$$\psi_0=\frac{e}{4\pi r},$$

where r is the distance from the centre. Hence

$$\frac{2e\psi_0}{c^2}=\frac{e^2}{2\pi ac^2},$$

where a is the radius of the boundary of the electron. Regarding this quantity as the *stationary mass* of the electron and denoting it by the symbol m_0, we have an expression for the mass which is just three times the usual expression. Thus in the electrostatic case not only is the force produced by external forces just three times the usual force, but the mass is also three times the usual mass.

It is important to ascertain whether the quantity m_0 is connected with the total energy. When $v=0$ the tensor T_m gives an amount of energy equal to $2e\psi_0$, which is just m_0c^2; hence the sum of the energies contributed by the tensors T_e, T_s, and T_c should be zero. This has been verified in an important case by a long numerical computation. Assuming that an electron has a spherical boundary and a radially symmetric distribution of charge, we have to satisfy the equation $\psi = b\rho^2$, where b is a constant.

Assuming that

$$\rho = A_0 + A_2 r^2 + A_4 r^4 + \ldots,$$

the corresponding expression for ψ is

$$\psi = \tfrac{1}{6}(3a^2 - r^2) A_0 + \tfrac{1}{20}(5a^4 - r^4) A_2 + \tfrac{1}{42}(7a^6 - r^6) A_4 + \ldots,$$

and we find, by equating coefficients of the different powers of r in the equation $\psi = b\rho^2$, that

$$\rho = A_0 \Bigg[1 - \frac{1}{12}\frac{r^2}{A_0 b} - \frac{1}{720}\frac{r^4}{A_0^2 b^2} - \frac{1}{10080}\frac{r^6}{A_0^3 b^3} - \frac{31}{6!\,7!}\frac{r^8}{A_0^4 b^4}$$

$$- \frac{(971)(24)}{6!\,11!}\frac{r^{10}}{A_0^5 b^5} - .000,\,000,\,081,\,758,\,121,\,606\,\frac{r^{12}}{A_0^6 b^6}$$

$$- .000,000,008,592,196\,\frac{r^{14}}{A_0^7 b^7} - .000,000,000,930,706\,\frac{r^{16}}{A_0^8 b^8} - \ldots \Bigg],$$

and that

$$0 = 1 - \tfrac{1}{2}s + \tfrac{1}{48}s^2 + \tfrac{1}{4320}s^3 + \tfrac{1}{80640}s^4$$
$$+ .000,\,000,\,854,\,276,\,895,\,943,\,5s^5$$
$$+ .000,\,000,\,067,\,571,\,103,\,457,\,7s^6$$
$$+ .000,\,000,\,005,\,839,\,865,\,829s^7$$
$$+ .000,\,000,\,000,\,537,\,012,\,283s^8$$
$$+ .000,\,000,\,000,\,051,\,705,\,893s^9 + \ldots,$$

where
$$s = \frac{a^2}{A_0 b}.$$

The equation for s appears to have two positive roots, the smaller of which is approximately

$$s = 2.209012572\,\ldots,$$

while the other is difficult to determine accurately. The calculations will be made with the smaller root.

The total energy inside and outside the electron arising from the tensors T_e, T_s, and T_c is

$$\tfrac{1}{2}\int_0^a 3\ \left(\frac{\partial\psi}{\partial r}\right)^2 4\pi r^2 dr + \tfrac{1}{2}\int_a^\infty 3\ \frac{e^2}{r^4}\cdot 4\pi r^2 dr - 2\psi_0\rho_0\cdot\tfrac{4}{3}\pi r^3,$$

and this is found to be zero to at least seven places of decimals. The total momentum inside and outside a uniformly moving electron may also be found from the tensors associated with the stationary electron by an application of the transformations of the theory of relativity, and it is found that to seven places of decimals the momentum arising from the tensor $T_e + T_s + T_c$ is zero, while the momentum arising from the tensor T_m is

$$\frac{m_0 v}{\sqrt{\{1-(v^2/c^2)\}}}\,.$$

We may thus be justified in regarding the quantity m_0 as the *stationary mass* of the electron.

It thus appears that our equations of motion will lead to dynamical equations which are something like those which are generally adopted, but exact expressions for the forces in accelerated motion are still to be found. It is possible that these will be equal to (or rather three times) the usual expressions only when certain conditions are satisfied. It should be remarked that the assumption that $\psi\rho\sqrt{\{1-(v^2/c^2)\}}$ is constant over the electron may not be true in general.

It is possible that our expression for the tensor T is not yet complete. The continuity of W at the boundary of an electron has been verified only in the electrostatic case,* and the fact that there is a radiation of negative energy in accelerated motion may be regarded as a defect in the theory. It is quite possible that a tensor may be found which will eliminate this negative energy and not materially alter the expressions for the force, total energy and momentum. It is easy to see that a tensor with components of types

$$W = \frac{1}{c^2}\frac{\partial^2\Omega}{\partial t^2} + \Box\Omega, \quad S_x = -\frac{\partial^2\Omega}{\partial x\partial t},$$

$$X_x = -\frac{\partial^2\Omega}{\partial x^2} + \Box\Omega, \quad X_y = -\frac{\partial^2\Omega}{\partial x\partial y},$$

* It is not clear that in uniform motion the *tangential* component of momentum is continuous at the boundary of an electron, though the normal component appears to be. The continuity of the normal component may be really all that is necessary.

will give no force and will also give no total energy if the integral of the normal derivative of Ω over a very large sphere is negligible while $\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+\frac{\partial^2\Omega}{\partial z^2}$ is continuous throughout space.

The case in which $\Omega=\psi^2$ is of special interest and some advantage may be gained by subtracting five-eighths of the corresponding tensor T_a from our tensor $T=T_e+T_s+T_c+T_m$. Indeed, when we consider our stationary electron with a spherical boundary, the addition of T_a does not alter the total energy, it simply alters the distribution. We can say, moreover, that when the electron is moving uniformly, the total energy and momentum can be calculated from the energy and momentum outside the electron by simply multiplying these by 8.

Whether this is true or not for a case of variable motion is difficult to say. The author has calculated the angular momentum outside the Hertzian dipole specified by the potentials

$$\Psi=\Phi=\frac{\partial}{\partial x}\left[\frac{1}{r}f\left(t-\frac{r}{c}\right)\right]+\frac{\partial}{\partial y}\left[\frac{1}{r}g\left(t-\frac{r}{c}\right)\right]+\frac{\partial}{\partial z}\left[\frac{1}{r}h\left(t-\frac{r}{c}\right)\right],$$

$$A_x=-\frac{1}{cr}f'\left(t-\frac{r}{c}\right),\quad A_y=-\frac{1}{cr}g'\left(t-\frac{r}{c}\right),\quad A_z=-\frac{1}{cr}h'\left(t-\frac{r}{c}\right),$$

$$E=-\frac{1}{c}\frac{\partial A}{\partial t}-\text{grad}\,\Phi,\quad H=\text{curl}\,A,$$

and has found it to be identically zero at every point when the tensor T is used, and zero on the average in a periodic motion when the tensor $-\frac{5}{8}T_a$ is added to T. Since the field is that of three electric dipoles, each of which vibrates along a line through the origin, the total angular momentum should be zero. The tensor T thus seems to satisfy conditions better in this case than the tensor $T-\frac{5}{8}T_a$.

There is, perhaps, some doubt as to whether the tensor T_m has been correctly represented for the case of variable motion. Indeed, we may not be entitled to assume that Ψ is constant over the boundary of an electron or positive nucleus which is moving in a variable manner, and if we are not it is difficult to assign a meaning* to Ψ_0. A similar remark applies also to the tensor T_c, since in constructing this we have practically assumed that $\Psi\rho\sqrt{\{1-(v^2/c^2)\}}$ is constant over the boundary

* We can only say that ψ_0 is a constant equal to the boundary value of ψ when the particle is stationary.

of the particle. A combination of the two assumptions implies that both Ψ and $\rho\sqrt{\{1-(v^2/c^2)\}}$ are constant over the boundary.

There is, indeed, some reason for supposing that $\rho\sqrt{\{1-(v^2/c^2)\}}$ may be constant over the boundary. This quantity in fact remains constant during motion, and is constant over the boundary of a stationary electron. There is, however, a possibility that $\rho\sqrt{\{1-(v^2/c^2)\}}$ might be constant over only a portion of the boundary of the electron in variable motion, the electricity associated with the least value of $|\rho\sqrt{\{1-(v/c^2)\}}|$ breaking up into two or more parts just as a thin sheet of water covering a globe might dry up in some places and become thicker in others. This phenomenon is unlikely to occur, but the possibility of it must be borne in mind. It is fairly reasonable, however, to expect that Ψ and $\rho\sqrt{\{1-(v^2/c^2)\}}$ will be constant over the boundary of an electron. If this expectation fails it may be necessary to replace T_c and T_m by tensors which have values different from zero outside the electron, and the external values of these tensors may be just what is needed to eliminate the radiation of negative energy and give a type of radiation in which energy travels directly from atom to atom in the way that Einstein imagines. Even though the tensors T_c and T_m may require modification for the case of variable motion, there is a possibility that our tensor T will still give correct results when considered simply as the basis of a method of calculation and not as a physical picture of the actual processes. It an atom B begins to emit negative energy as soon as the radiant field from an atom A reaches it, and continues to emit negative or positive energy for a short interval of time, the total amount emitted being negative, the effect is practically the same as if light energy in a corpuscular form were to travel with velocity c along a rectilinear path from A to B. If there is no radiation of energy to infinity on the whole, there will be conservation of energy, and the quantum conditions must be regarded as the laws governing the flow of energy across the boundaries of the individual electrons, the energy of an electron being regarded as entirely within the electron and equal to that given by the tensor T_m.

Indeed, if we admit the possibility of the existence of negative energy the phenomena of the photoelectric effect can very likely be explained without any need of a corpuscular hypothesis such as that of Einstein. This hypothesis has many advantages, but among the objections to it we may enumerate (1) the difficulty of explaining interference, (2) the difficulty of understanding how a light quantum knows where to go in order that it may be absorbed, (3) the difficulty of understanding how a light quantum preserves its frequency ν as it travels through space with velocity c.

Really the light quantum $\epsilon = h\nu$ in Einstein's theory is associated with frequencies lying between ν and $\nu + d\nu$. Einstein's original arguments* which were put forward to show that the light quanta were thermodynamically and spacially independent were based partly on Wien's radiation formula and were applied to black body radiation. Wolfke† has recently developed similar arguments in connection with Planck's radiation formula and finds that elements of energy of amounts $\epsilon = nh\nu$ ($n = 1, 2, 3, \ldots$) should be regarded as thermodynamically and spacially independent. It may be remarked also that his analysis may be extended to the fluctuations of energy in a small volume and that the fluctuations may then be accounted for completely as arising from the motion of the quanta without any additional term representing the fluctuations according to the classical wave theory.

This result seems to indicate that the properties of black body radiation can be described in terms of simple light quanta of magnitude $\epsilon = h\nu$ and associated sets of n such quanta. This may be simply a peculiarity of black body radiation arising from the presence of a perfectly reflecting boundary or it may be an indication that the quantum theory is to be regarded as a method of calculation which is complete in itself and quite distinct from the wave theory. Quantum theory may indeed belong naturally to that branch of mathematical physics in which the energy and mass of a particle are regarded as inside the particle instead of outside. The present researches support this view and point very definitely to the conclusion that there is a new type of force, depending on a scalar potential ψ, which under certain circumstances is just twice the ordinary electromagnetic force, the mass having three times its usual value. There may, however, be other forces still to be discovered, and in fact it seems reasonable to adopt the hypothesis that the greatest discoveries have yet to be made and that the universe is far more interesting and complicated than was ever imagined.

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* A. Einstein, *Ann. d. Phys.* (4), Bd. 17 (1905), p. 132.
† M. Wolfke, *Phys. Zeitschr.*, Bd. 22 (1921), p. 375.

AN EXPANSION IN FACTORIALS SIMILAR TO VANDERMONDE'S THEOREM, AND APPLICATIONS.

By *D. Edwardes, B.A.*

If in Vandermonde's product of factors in A.P. we take the initial factor to be, say, $\alpha+\beta+\frac{1}{2}$, instead of $\alpha+\beta$, and seek to determine a series with the same law of progression of the factors in the several terms, we get the result

$$\tfrac{1}{2}\cdot\tfrac{3}{2}\dots\tfrac{1}{2}(2n-1)\,(\alpha+\beta+\tfrac{1}{2})(\alpha+\beta+\tfrac{3}{2})\dots\{\alpha+\beta+\tfrac{1}{2}(2n-1)\}$$
$$=(\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})\dots\{\alpha+\tfrac{1}{2}(2n-1)\}\,.\,(\beta+\tfrac{1}{2})(\beta+\tfrac{3}{2})\dots\{\beta+\tfrac{1}{2}(2n-1)\}$$
$$+k_{n1}\alpha\beta\,.\,(\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})\dots\{\alpha+\tfrac{1}{2}(2n-3)\}\,.\,(\beta+\tfrac{1}{2})(\beta+\tfrac{3}{2})\dots\{\beta+\tfrac{1}{2}(2n-3)\}$$
$$+k_{n2}\alpha(\alpha+1)\beta(\beta+1)\,.\,(\alpha+\tfrac{1}{2})\dots\{\alpha+\tfrac{1}{2}(2n-5)\}\,.\,(\beta+\tfrac{1}{2})\dots\{\beta+\tfrac{1}{2}(2n-5)\}+\dots$$
$$+k_{nn}\alpha(\alpha+1)\dots(\alpha+n-1)\,.\,\beta(\beta+1)\dots(\beta+n-1)\dots\dots\dots(1),$$

where

$$k_{nr}=(2n-4r+1)\,.\,{}^nC_r\,.\,\frac{(2n-2r+3)(2n-2r+5)\dots(2n-1)}{1\,.\,3\,.\,5\dots(2r-1)}$$
$$={}^{2n}C_{2r}-{}^{2n}C_{2r-1}\;\dots\dots\dots\dots\dots\dots(2),$$
$$r=2,\,3,\,\dots,\,n,\;k_{n1}=n(2n-3).$$

The method of construction is given further on.

In particular, by equating the coefficients of the highest power of, say, β on either side, we have

$$(\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})\dots\{\alpha+\tfrac{1}{2}(2n-1)\}+k_{n1}\alpha(\alpha+\tfrac{1}{2})\dots\{\alpha+\tfrac{1}{2}(2n-3)\}$$
$$+k_{n2}\alpha(\alpha+1)\,.\,(\alpha+\tfrac{1}{2})\dots\{\alpha+\tfrac{1}{2}(2n-5)\}+\dots+k_{nn}\alpha(\alpha+1)\dots(\alpha+n-1)$$
$$=\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\dots\tfrac{1}{2}(2n-1)\dots(3).$$

An independent proof follows from writing the series as

$$\Gamma(2\alpha+n+\tfrac{1}{2})\,.\,S/\Gamma(\alpha)\,\Gamma(\alpha+\tfrac{1}{2})\;\dots\dots\dots(4),$$

where

$$S=B(\alpha+n+\tfrac{1}{2},\,\alpha)+k_{n1}B(\alpha+n-\tfrac{1}{2},\,\alpha+1)$$
$$+k_{n2}B(\alpha+n-\tfrac{3}{2},\,\alpha+2)+\dots+k_{nn}B(\alpha+\tfrac{1}{2},\,\alpha+n)$$
$$=\int_0^1 t^{\alpha-\frac{1}{2}}(1-t)^{\alpha-1}\{t^n+k_{n1}t^{n-1}(1-t)+\dots+k_{nn}(1-t)^n\}\,dt,$$

if $\alpha > 0$.* Now if x, y are any numbers and we expand

$$(x+y)^{2n} \pm (x-y)^{2n},$$

multiply the lower expression by y/x and subtract from the upper, we get

$$\{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})^{2n} + (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})^{2n}\}/2\sqrt{x}$$
$$= x^n + k_{n1}x^{n-1}y + k_{n2}x^{n-2}y^2 + \ldots + k_{nn}y^n \ldots (5).$$

Hence, substituting and letting $t=\cos^2\theta$, we have

$$S = \int_0^{\frac{1}{2}\pi} (\sin\theta\cos\theta)^{2\alpha-1}\cos 2\theta\,\{(\cos\theta+\sin\theta)^{2n-1} + (\cos\theta-\sin\theta)^{2n-1}\}.$$

The integrand in the first integral is reproduced with its sign changed on writing $\frac{1}{2}\pi - \theta$ for θ therein, so that

$$S = \int_0^{\frac{1}{2}\pi} (\sin\theta\cos\theta)^{2\alpha-1}\cos 2\theta\,(\cos\theta - \sin\theta)^{2n-1}$$

$$= \frac{1}{2^{2\alpha}} \int_0^{\frac{1}{2}\pi} (\sin 2\theta)^{2\alpha-1} . (1-\sin 2\theta)^{n-\frac{1}{2}} d\sin 2\theta$$

$$= \frac{1}{2^{2\alpha-1}} . \frac{\Gamma(2\alpha)\,\Gamma(n+\frac{1}{2})}{\Gamma(\alpha)\,\Gamma(\alpha+\frac{1}{2})},$$

by the duplication formula for the Gamma function,

$$= \frac{1}{\pi^{\frac{1}{2}}}\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots\tfrac{3}{2}.\tfrac{1}{2}.$$

We will now evaluate a certain double integral by (1). The series on the right-hand side can be written

$$\frac{\Gamma(2\alpha+n+\frac{1}{2})\,\Gamma(2\beta+n+\frac{1}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}\,[B\{\alpha+\tfrac{1}{2}(2n+1),\,\alpha\}\,B\{\beta+\tfrac{1}{2}(2n+1),\,\beta\}$$
$$+ k_{n1}B\{\alpha+\tfrac{1}{2}(2n-1),\,\alpha+1\}\,B\{\beta+\tfrac{1}{2}(2n-1),\,\beta+1\}+\ldots$$
$$+ k_{nn}B(\alpha+\tfrac{1}{2},\,\alpha+n)\,B(\beta+\tfrac{1}{2},\,\beta+n)],$$

and proceeding as before, using (5), taking $\cos^2\theta$, $\cos^2\phi$ as new variables of integration, and omitting a double integral that obviously vanishes, we get

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} (\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1} . \cos(\theta-\phi)\,\{\cos(\theta+\phi)\}^{2n} d\theta d\phi$$
$$= \frac{\pi . 2n!\,\Gamma(2\alpha)\,\Gamma(2\beta)\,\Gamma(\alpha+\beta+n+\frac{1}{2})}{2^{4n+1} . n!\,\Gamma(2\alpha+n+\frac{1}{2})\,\Gamma(2\beta+n+\frac{1}{2})\,\Gamma(\alpha+\beta+\frac{1}{2})} \ldots (6).\dagger$$

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* Since the left-hand side of (3) is of finite degree in α, the equation is true for all values of α if true for $\alpha > 0$.

† $\cos(\theta-\phi)$ may be replaced by either $\cos\theta\cos\phi$ or $\sin\theta\sin\phi$ provided we halve the right-hand member.

If $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, the double and repeated integrals are all equal.

To prove (1), we have

$$\tfrac{1}{2}(\alpha+\beta+\tfrac{1}{2}) = (\alpha+\tfrac{1}{2})(\beta+\tfrac{1}{2}) - \alpha\beta,$$

$$\tfrac{3}{2}(\alpha+\beta+\tfrac{3}{2}) = (\alpha+\tfrac{3}{2})(\beta+\tfrac{3}{2}) - \alpha\beta = 3\{(\alpha+1)(\beta+1) - (\alpha+\tfrac{1}{2})(\beta+\tfrac{1}{2})\}$$

$$= A = B, \text{ say, and thus}$$

$$\tfrac{1}{2}(\alpha+\beta+\tfrac{1}{2})\,\tfrac{3}{2}(\alpha+\beta+\tfrac{3}{2}) = (\alpha+\tfrac{1}{2})(\beta+\tfrac{1}{2})A - \alpha\beta B$$

$$= (\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})(\beta+\tfrac{1}{2})(\beta+\tfrac{3}{2}) + 2\alpha(\alpha+\tfrac{1}{2})\beta(\beta+\tfrac{1}{2}) - 3\alpha(\alpha+1)\beta(\beta+1).$$

Similarly,

$$\tfrac{5}{2}(\alpha+\beta+\tfrac{5}{2}) = (\alpha+\tfrac{5}{2})(\beta+\tfrac{5}{2}) - \alpha\beta$$

$$= 5\{(\alpha+\tfrac{3}{2})(\beta+\tfrac{3}{2}) - (\alpha+1)(\beta+1)\} = -\tfrac{5}{3}\{(\alpha+\tfrac{1}{2})(\beta+\tfrac{1}{2}) - (\alpha+2)(\beta+2)\},$$

and now multiplying and using the appropriate multiplier for each term on the right-hand side, we get

$$\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}(\alpha+\beta+\tfrac{1}{2})(\alpha+\beta+\tfrac{3}{2})(\alpha+\beta+\tfrac{5}{2})$$

$$= (\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})(\alpha+\tfrac{5}{2})(\beta+\tfrac{1}{2})(\beta+\tfrac{3}{2})(\beta+\tfrac{5}{2})$$

$$+ 9\alpha\beta(\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})(\beta+\tfrac{1}{2})(\beta+\tfrac{3}{2})$$

$$-5\alpha(\alpha+1)\beta(\beta+1)(\alpha+\tfrac{1}{2})(\beta+\tfrac{1}{2}) - 5\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2).$$

Assuming then the form of (1), to find the coefficients let $\alpha = -r$, $\beta = -(2n-2r+1)/2$. All the terms vanish except that involving $k_{n,r}$, and we get on reduction the value stated above. Multiplying both sides by $\frac{1}{2}(2n+1)\{\alpha+\beta+\frac{1}{2}(2n+1)\}$, and using the appropriate form of this expression for each term of the series, the coefficient of

$$\alpha(\alpha+1)\ldots(\alpha+r).(\alpha+\tfrac{1}{2})\ldots$$

$$\{\alpha+\tfrac{1}{2}(2n-2r-1)\}.\beta(\beta+1)\ldots(\beta+r).(\beta+\tfrac{1}{2})\ldots\{\beta+\tfrac{1}{2}(2n-2r-1)\}$$

is
$$-\frac{2n+1}{2n-4r+1}k_{n,r} + \frac{2n+1}{2n-4r-3}k_{n,r+1}$$

$$= (2n-4r-1)\frac{(n+1)!}{\overline{r+1}!\,\overline{n-r}!}\,\frac{(2n-2r+3)\ldots(2n+1)}{1.3.5\ldots\overline{2r+1}},$$

that is $k_{n+1,r+1}$, and this holds also for $r=1$. The theorem is therefore established.

In further application of (1), first, if

$$\mu \equiv \alpha+\beta,\; M \equiv \tfrac{1}{2}\pi\,\Gamma(2\alpha+n+\tfrac{1}{2})\,\Gamma(2\beta+n+\tfrac{1}{2}),$$

we have

$$\frac{M}{\Gamma(2\alpha)\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)$$

$$\{\cos(\theta+\phi)\}^{2n}d\theta d\phi=\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots\tfrac{1}{2}(2n-1)(\mu+\tfrac{1}{2})(\mu+\tfrac{3}{2})\ldots\{\mu+\tfrac{1}{2}(2n-1)\}.$$

Also, expanding the cosine and doubling the first of the two equivalent integrals, we have

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)\,d\theta d\phi$$

$$=2^{2\alpha+2\beta-1}\cdot\tfrac{1}{2}B(\alpha,\alpha+\tfrac{1}{2})\cdot\tfrac{1}{2}B(\beta,\beta+\tfrac{1}{2})=\frac{\pi\Gamma(2\alpha)\Gamma(2\beta)}{2\Gamma(2\alpha+\frac{1}{2})\Gamma(2\beta+\frac{1}{2})},$$

so that

$$\frac{2}{\pi}\frac{\Gamma(2\alpha+\frac{1}{2})\Gamma(2\beta+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)d\theta d\phi=1.$$

On substituting then for $n=1, 2, \ldots$ and adding, we have

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)$$

$$F\{2\alpha+\tfrac{1}{2},\,2\beta+\tfrac{1}{2},\,\gamma;\ x\cos^2(\theta+\phi)\}\,d\theta d\phi$$

$$=\pi\frac{\Gamma(2\alpha)\Gamma(2\beta)}{2\Gamma(2\alpha+\frac{1}{2})\Gamma(2\beta+\frac{1}{2})}F(\alpha+\beta+\tfrac{1}{2},\,\tfrac{1}{2},\,\gamma,\,x)\ldots(7)$$

$$(\alpha>\tfrac{1}{2},\ \beta>\tfrac{1}{2}).$$

In the first equation above, change α into $\alpha+\frac{1}{2}$, β into $\beta+\frac{1}{2}$, and n into $n-1$, and multiply throughout by $\gamma.2\alpha.2\beta/2\gamma$. Changing α, β, n, as before and multiplying by

$$\gamma(\gamma+1)\,2\alpha(2\alpha+1)\,2\beta(2\beta+1)/2\gamma(2\gamma+1),$$

and continuing the process, we get

$$\frac{1}{n!}\tfrac{1}{2}\tfrac{3}{2}\ldots\tfrac{1}{2}(2n-1)\{\mu+\tfrac{1}{2}(2n-1)\}\ldots(\mu+\tfrac{3}{2})(\mu+\tfrac{1}{2})$$

$$+\frac{\gamma}{1!\,\overline{n-1}!}\cdot\frac{2\alpha.2\beta}{2\gamma}\times\tfrac{1}{2}\tfrac{3}{2}\ldots\tfrac{1}{2}(2n-3)\{\mu+\tfrac{1}{2}(2n-1)\}\ldots(\mu+\tfrac{3}{2})$$

$$+\frac{\gamma(\gamma+1)}{2!\,\overline{n-2}!}\,\frac{2\alpha(2\alpha+1)\,2\beta(2\beta+1)}{2\gamma(2\gamma+1)}$$

$$\times\tfrac{1}{2}\tfrac{3}{2}\ldots\tfrac{1}{2}(2n-5)\{\mu+\tfrac{1}{2}(2n-1)\}\ldots(\mu+\tfrac{5}{2})+\ldots$$

$$+\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{n!}\cdot\frac{2\alpha(2\alpha+1)\ldots(2\alpha+n-1).2\beta(2\beta+1)\ldots(2\beta+n-1)}{2\gamma(2\gamma+1)\ldots(2\gamma+n-1)}$$

$$=\frac{M}{n!\,\Gamma(2\alpha)\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)\{\cos(\theta+\phi)\}^{2n}$$

$$\times\left\{1+{}^nC_1\frac{\gamma}{2\gamma}\cdot\frac{\sin 2\theta\sin 2\phi}{\cos^2(\theta+\phi)}+{}^nC_2\frac{\gamma(\gamma+1)}{2\gamma(2\gamma+1)}\cdot\frac{(\sin 2\theta\sin 2\phi)^2}{\cos^4(\theta+\phi)}+\ldots\right.$$

$$\left.+\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{2\gamma(2\gamma+1)\ldots(2\gamma+n-1)}\cdot\frac{(\sin 2\theta\sin 2\phi)^n}{\cos^{2n}(\theta+\phi)}\right\}d\theta d\phi$$

$$=\frac{M}{n!\,\Gamma(2\alpha)\Gamma(2\beta)}\cdot\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos(\theta-\phi)\{\cos(\theta+\phi)\}^{2n}$$

$$\times F\left(-n,\gamma,2\gamma;-\frac{\sin 2\theta\sin 2\phi}{\cos^2(\theta+\phi)}\right)d\theta d\phi\ldots(8)$$

$$=\frac{2M}{n!\,\Gamma(2\alpha)\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}\cos\theta\cos\phi\{\cos(\theta+\phi)\}^{2n}$$

$$\times F\left(-n,\gamma,2\gamma;-\frac{\sin 2\theta\sin 2\phi}{\cos^2(\theta+\phi)}\right)d\theta d\phi\ldots(9).$$

Now (Gauss)

$$(1+y)^{2\alpha}F(\alpha,\alpha+\tfrac{1}{2}-\gamma,\gamma+\tfrac{1}{2};y^2)=F\left(\alpha,\gamma,2\gamma,\frac{4y}{(1+y)^2}\right),$$

therefore

$$(1-\tan\theta\tan\phi)^{-2n}F(-n,-n+\tfrac{1}{2}-\gamma,\gamma+\tfrac{1}{2};\tan^2\theta\tan^2\phi)$$

$$=F\left(-n,\gamma,2\gamma,-\frac{\sin 2\theta\sin 2\phi}{\cos^2(\theta+\phi)}\right),$$

and the integral in (9) becomes

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}(\cos\theta\cos\phi)^{2n+1}$$

$$\times F(-n,-n+\tfrac{1}{2}-\gamma,\gamma+\tfrac{1}{2};\tan^2\theta\tan^2\phi)\,d\theta d\phi.$$

Let the whole equation be divided by

$$(\gamma+\tfrac{1}{2})(\gamma+\tfrac{3}{2})\ldots\{\gamma+\tfrac{1}{2}(2n-1)\}.$$

The left-hand member is then seen to be the coefficient of x^n in the series written below, and thus the coefficient of x^n in the series

$$F(\alpha+\beta+\tfrac{1}{2},\tfrac{1}{2},\gamma+\tfrac{1}{2},x)+x\frac{\gamma}{\gamma+\frac{1}{2}}\cdot\frac{2\alpha.2\beta}{2\gamma}F(\alpha+\beta+\tfrac{3}{2},\tfrac{1}{2},\gamma+\tfrac{3}{2},x)$$

$$+x^2\frac{\gamma(\gamma+1)}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})}\cdot\frac{2\alpha(2\alpha+1)2\beta(2\beta+1)}{2!\,2\gamma(2\gamma+1)}+\ldots$$

(uniformly convergent if $\gamma\geq\alpha+\beta$, and $-1+\delta<x<1+\delta$, and therefore expressible as a power series) can be expressed in the form

$$\frac{2M}{n!\,\Gamma(2\alpha)\,\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}(\cos\theta\cos\phi)^{2n+1}$$

$$\times F(-n, -n+\tfrac{1}{2}-\gamma, \gamma+\tfrac{1}{2};\ \tan^2\theta\tan^2\phi)\,d\theta d\phi$$

$$/(\gamma+\tfrac{1}{2})(\gamma+\tfrac{3}{2})\ldots\{\gamma+\tfrac{1}{2}(2n-1)\};$$

the Gaussian function here being simply a polynomial. We shall have to refer presently to this result.

Consider now the product

$$F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \gamma+\tfrac{1}{2}, x)\,.\,F(\alpha, \beta, \gamma+\tfrac{1}{2}, x),\ |x|<1$$

and series absolutely convergent. The coefficient of x^n in the equivalent power series can be expressed at once in the form

$$\frac{1}{n!}\frac{\Gamma(2\alpha+n+\frac{1}{2})\,\Gamma(2\beta+n+\frac{1}{2})}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})}\left[\frac{B\{\alpha, \alpha+\frac{1}{2}(2n+1)\}\,B\{\beta, \beta-\frac{1}{2}(2n+1)\}}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})\ldots\{\gamma+\frac{1}{2}(2n-1)\}}\right.$$

$$+{}^nC_1\frac{B\{\alpha+1, \alpha+\frac{1}{2}(2n-1)\}\,B\{\beta+1, \beta+\frac{1}{2}(2n-1)\}}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})\ldots\{\gamma+\frac{1}{2}(2n-3)\}\,.\,(\gamma+\frac{1}{2})}$$

$$+{}^nC_n\frac{B\{\alpha+2, \alpha+\frac{1}{2}(2n-3)\}\,B\{\beta+2, \beta+\frac{1}{2}(2n-3)\}}{(\gamma+\frac{1}{2})\ldots\{\gamma+\frac{1}{2}(2n-5)\}\,.\,(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})}+\ldots$$

$$\left.+\frac{B(\alpha+n, \alpha+\frac{1}{2})\,B(\beta+n, \beta+\frac{1}{2})}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})\ldots\{\gamma+\frac{1}{2}(2n-1)\}}\right]$$

$$=\frac{2^{2\alpha+2\beta-2}}{\pi n!}\frac{\Gamma(2\alpha+n+\frac{1}{2})\,\Gamma(2\beta+n+\frac{1}{2})}{\Gamma(2\alpha)\,\Gamma(2\beta)\,.\,(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})\ldots\{\gamma+\frac{1}{2}(2n-1)\}}$$

$$\times\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2\alpha-2}(\sin\phi)^{2\beta-2}(\cos\theta)^{2\alpha+2n-1}$$

$$\times(\cos\phi)^{2\beta+2n-1}\left[1+{}^nC_1\frac{\gamma+\frac{1}{2}(2n-1)}{\gamma+\frac{1}{2}}\tan^2\theta\tan^2\phi\right.$$

$$\left.+{}^nC_2\frac{\{\gamma+\frac{1}{2}(2n-1)\}\{\gamma+\frac{1}{2}(2n-3)\}}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})}\tan^4\theta\tan^4\phi+\ldots+\tan^{2n}\theta\tan^{2n}\phi\right]$$

$$\times\sin 2\theta\sin 2\phi\,d\theta d\phi,$$

where for the series in [ ] we may write

$$F(-n, -\gamma-n+\tfrac{1}{2}, \gamma+\tfrac{1}{2};\ \tan^2\theta\tan^2\phi)$$

and reduce the expression to

$$\frac{2M}{n!\,\Gamma(2\alpha)\,\Gamma(2\beta)}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\sin 2\theta)^{2\alpha-1}(\sin 2\phi)^{2\beta-1}$$

$$\times F(-n, -\gamma-n+\tfrac{1}{2}, \gamma+\tfrac{1}{2};\ \tan^2\theta\tan^2\phi)\,d\theta d\phi.$$

Hence, comparing with the former result, we have

$$F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \gamma+\tfrac{1}{2}, x)\,F(\alpha, \beta, \gamma+\tfrac{1}{2}, x) = F(\alpha+\beta+\tfrac{1}{2}, \tfrac{1}{2}, \gamma+\tfrac{1}{2}, x)$$

$$+ x\,\frac{\gamma}{\gamma+\frac{1}{2}}\,\frac{2\alpha.2\beta}{1!\,2\gamma}\,F(\alpha+\beta+\tfrac{3}{2}, \tfrac{1}{2}, \gamma+\tfrac{3}{2}, x)$$

$$+ x^2\,\frac{\gamma(\gamma+1)}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})}\cdot\frac{2\alpha(2\alpha+1)2\beta(2\beta+1)}{2!\,2\gamma(2\gamma+1)}\,F(\alpha+\beta+\tfrac{5}{2}, \tfrac{1}{2}, \gamma+\tfrac{5}{2}, x)+\ldots$$

$$\ldots\ldots(10),$$

provided of course γ is not half an uneven negative integer.

Clausen's power series for the square of a hypergeometric series follows from this result by letting $\gamma=\alpha+\beta$. Each partial series on the right-hand side becomes $(1-x)^{-\frac{1}{2}}$, so that

$$(1-x)^{\frac{1}{2}}F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, x)\,F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2}, x)$$

$$= 1 + \sum_{n=1}^{\infty}\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})\ldots\{\gamma+\frac{1}{2}(2n-1)\}}$$

$$\times\frac{2\alpha(2\alpha+1)\ldots(2\alpha+n-1).2\beta(2\beta+1)\ldots(2\beta+n-1)}{n!\,2\gamma(2\gamma+1)\ldots(2\gamma+n-1)}\,x^n,$$

which may be written

$$\frac{\Gamma(2\alpha+2\beta)\,\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(2\alpha)\,\Gamma(2\beta)\,\Gamma(\alpha+\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\beta+n)\,\Gamma(2\alpha+n)\,\Gamma(2\beta+n)\,x^n}{n!\,\Gamma(2\alpha+2\beta+n)\,\Gamma(\alpha+\beta+n+\frac{1}{2})}.$$

Now (Euler)

$$(1-x)^{\alpha+\beta-\gamma}F(\alpha, \beta, \gamma, x) = F(\gamma-\alpha, \gamma-\beta, \gamma, x),$$

so that

$$(1-x)^{\frac{1}{2}}F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}, x) = F(\beta, \alpha, \alpha+\beta+\tfrac{1}{2}, x)$$

$$= F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}, x),$$

and thus $\{F(\alpha, \beta, \alpha+\beta+\frac{1}{2}, x)\}^2$ = the series above, Clausen's c being here $\gamma+\frac{1}{2}$.*

Again, if we multiply (10) throughout by $(1-x)^{\alpha+\beta-\gamma+\frac{1}{2}}$ and apply Euler's transformation to the first factor on the left and the partial series on the right, then

$$F(\gamma-\alpha, \gamma-\beta, \gamma+\tfrac{1}{2}, x)\,F(\alpha, \beta, \gamma+\tfrac{1}{2}, x) \equiv S$$

$$= \Sigma\,\frac{x^n(\gamma)_{n-1}}{(\gamma+\frac{1}{2})_{n-1}}\,\frac{(2\alpha)_{n-1}(2\beta)_{n-1}}{n!\,(2\gamma)_{n-1}}\,F(\gamma-\alpha-\beta, \gamma+n, \gamma+n+\tfrac{1}{2}, x),$$

* *Vide* Whittaker and Watson, 2nd ed., ex. 11, Misc. Exs., ch. xiv.

where $(\gamma)_{n-1}$ means $\gamma(\gamma+1)\ldots(\gamma+n-1)$, and similarly for $(2\alpha)_{n-1}$, etc.

Assuming that $|x|<1$, and $\gamma>0$, we can replace F by

$$\frac{2(\gamma+\frac{1}{2})_{n-1}\Gamma(\gamma+\frac{1}{2})}{(\gamma)_{n-1}\Gamma(\gamma)\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi}(\sin\alpha)^{2\gamma+2n-1}(1-x\sin^2\theta)^{\alpha+\beta-\gamma}d\theta$$

and write

$$S=\frac{2\Gamma(\gamma+\frac{1}{2})}{\Gamma(\gamma)\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2\gamma-1}(1-x\sin^2\theta)^{\alpha+\beta-\gamma}F(2\alpha,2\beta,2\gamma,x\sin^2\theta)d\theta,$$

whence follows a theorem due to Cayley,* on replacing the integrals

$$\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2\gamma+2n-1}d\theta$$

by their values in

$$(\sin\theta)^{2\gamma-1}(1+Bx\sin^2\theta+Cx^2\sin^4\theta+\ldots)$$

and reducing.

GREEN'S DYADICS IN THE THEORY OF ELASTICITY.

By *C. E. Weatherburn, M.A., D.Sc.*

Introduction.

The close analogy which is known to exist between the problems of Dirichlet and Neumann and the fundamental problems of elasticity,† and which in a recent paper‡ I have further emphasized by the introduction of certain dyadics constructed from Somigliana's integrals of the equations of equilibrium for an elastic isotropic body, suggests a further enquiry as to the existence of other dyadics which, in the theory of elasticity, will play a part similar to that taken by the ordinary Green's functions in the potential problems. In the present paper I show how to construct such dyadics, which I shall call the Green's dyadics for the problems of elasticity corresponding respectively to zero surface displacement and zero surface traction. The former is an ordinary Green's dyadic, and the latter a generalised. Having proved their fundamental properties I show how they may be used for the

* *loc cit.*, Ex 16

† Cf. *e g* Lauricella, *Atti Lincei* (5), t. 15_1 (1906), pp. 426–433 and 610–619; t. 15_2, pp. 75–83; and *Il Nuovo Cimento* (5), t. 13 (1907) Four papers

‡ "On two fundamental problems on the theory of elasticity", *The Philosophical Magazin* vol. xxxii, pp. 15–38 (1916).

solution of various problems related to the differential equation of elastic equilibrium under no bodily forces

$$(1) \qquad L(\mathbf{u}) \equiv \nabla^2\mathbf{u} + k \operatorname{grad} \operatorname{div} \mathbf{u} = 0.$$

In particular I discuss the problems of small vibrations of an isotropic body corresponding to the boundary conditions of zero displacement and zero surface traction; and finally prove bilinear series for the Green's dyadics, corresponding exactly to the ordinary bilinear series for the symmetric kernel of the scalar integral equation. The paper will afford further applications of the theory of vector integral equations, which I have developed in a paper just appearing in the *Transactions of the Cambridge Philosophical Society*; and the important rôle of the conjugo-symmetric kernel there* indicated is here filled by the Green's dyadics to be considered.

Though the greater part of the present investigation is independent of my earlier paper on the problems of elasticity, it will be convenient, for the purpose of identifying the functions found in the first section below, to mention the following results previously obtained. If a denote a constant (say unit) vector, then

$$\mathbf{s}_0(pq) = \frac{\mathbf{a}}{r} - \frac{k}{2(1+k)} \operatorname{grad} \operatorname{div} (\mathbf{a} r)$$

is a particular integral of the equation (1) of Somigliana's type, r being measured from a fixed point p (the pole) to the current point q. To this displacement corresponds the surface traction $\mathbf{F}_0(tp)$ at the boundary point t, given by the ordinary formula. With the notation $\mathbf{F}_1(tp)$, $\mathbf{F}_2(tp)$, $\mathbf{F}_3(tp)$ and $\mathbf{s}_1(pq)$, $\mathbf{s}_2(pq)$, $\mathbf{s}_3(pq)$ for the particular values of $\mathbf{F}_0(tp)$ and $\mathbf{s}_0(pq)$ respectively when $\mathbf{a}$ is put equal to the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in succession, the dyadics†

$$\Psi(tp) = \frac{1}{2\pi}\left[\mathbf{F}_1(tp)\,\mathbf{i} + \mathbf{F}_2(tp)\,\mathbf{j} + \mathbf{F}_3(tp)\,\mathbf{k}\right],$$

$$\Phi(pq) = \frac{1}{2\pi}\left[\mathbf{i}\,\mathbf{s}_1(pq) + \mathbf{j}\,\mathbf{s}_2(pq) + \mathbf{k}\,\mathbf{s}_3(pq)\right]$$

form the basis of displacement functions analogous to the vector potentials of double and simple strata respectively. Regarded as kernels of vector integral equations of Fredholm's type with arbitrary parameter λ, the dyadic $\Psi(ts)$ and its conjugate $X(ts)$ possess resolvent dyadics $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$

* "Vector integral equations and Gibbs' dyadics", §§ 19–24, *Camb. Phil. Trans.*, vol. xxii., pp. 133–155 (1916).

† *Phil. Mag.*, *loc. cit.*, §§ 4–7.

respectively, connected with them by alternative equations of the form *

$$\mathbf{H}(ts)-\Psi(ts)=\lambda\int\mathbf{H}(t\vartheta)\cdot\Psi(\vartheta s)\,d\vartheta=\lambda\int\Psi(t\vartheta)\cdot\mathbf{H}(\vartheta s)\,d\vartheta,\dagger$$

and a similar pair for $\mathbf{H}'(ts)$ and $X(ts)$. Relations of the same form were shown to be true when the resolvents $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$ are extended, the second boundary point s being replaced by a point p not on the boundary; and lastly two other dyadics $\Gamma(pq)$ and $\Gamma'(pq)$ were defined, connected with the preceding by the equations ‡

$$(A)\quad\begin{cases}\Gamma(pq)-\Phi(pq)=\lambda\int\Phi(ps)\cdot\mathbf{H}(sq)ds=\lambda\int\Gamma(ps)\cdot\Psi(sq)ds,\\ \Gamma(pq)-\Phi(pq)=\lambda\int\Phi(ps)\cdot\mathbf{H}'(sq)ds=\lambda\int\Gamma'(ps)\cdot X(sq)ds.\end{cases}$$

In terms of these dyadics the solutions of the first and second boundary problems, which require the determination of displacements $\mathbf{W}(p)$ and $\mathbf{V}(p)$, corresponding respectively to the boundary conditions §

$$(B)\quad\begin{cases}\frac{1}{2}[\mathbf{W}(t^+)-\mathbf{W}(t^-)]-\frac{1}{2}\lambda[\mathbf{W}(t^+)+\mathbf{W}(t^-)]=\mathbf{f}(t),\\ \frac{1}{2}[\mathbf{TV}(t^-)-\mathbf{TV}(t^+)]-\frac{1}{2}\lambda[\mathbf{TV}(t^-)+\mathbf{TV}(t^+)]=-\mathbf{f}(t),\end{cases}$$

in which $\mathbf{TV}(t^+)$ and $\mathbf{TV}(t^-)$ represent the surface tractions for the inner and outer regions respectively when the displacement is $\mathbf{V}(p)$, are given by the expressions ‖

$$(C)\quad\begin{cases}\mathbf{W}(p)=\int\mathbf{f}(t)\cdot\mathbf{H}(tp)\,dt,\\ \mathbf{V}(p)=\int\Gamma'(pt)\cdot\mathbf{f}(t)\,dt,\end{cases}$$

except for certain singular values of the parameter λ.

I.—Determination and Properties of the Green's Dyadics.

§ 1. *Green's vectors and Green's dyadics.* Considering an isotropic body occupying the inner region S, suppose that we require the displacement at any point p when the surface displacement is $\mathbf{s}_1(qt)$, the pole q being a point of the region S occupied by the body. The required displacement is expressible as a double elastic stratum potential $\mathbf{W}_1(p)$ given by

$$\mathbf{W}_1(p)=\int\mathbf{s}_1(qt)\cdot\mathbf{H}_{-1}(tp)\,dt,$$

* *Loc. cit.*, §§ 8–9.

† The differentials $d\vartheta$, dt, ds denote elements of the boundary; while dp, dq will be used to indicate elements of volume.

‡ *Ibid.*, § 9 (33). § *Ibid.*, § 8 (25). ‖ *Ibid.*, § 9 (31).

a result derivable* from (B) and (C) by putting $\lambda = -1$. Hence the vector function

$$\mathbf{G}_1(qp) = \mathbf{s}_1(qp) - \mathbf{W}_1(p)$$
$$= \mathbf{s}_1(qp) - \int \mathbf{s}_1(qt) \cdot \mathbf{H}_{-1}(tp)\, dt$$

is a solution of (1), regular throughout the region S except at the pole q where it becomes infinite like $\mathbf{i}/r$, while over the boundary Σ it vanishes identically. Of a similar nature are the functions

$$\mathbf{G}_2(qp) = \mathbf{s}_2(qp) - \int \mathbf{s}_2(qt) \cdot H_{-1}(tp)\, dt,$$
$$\mathbf{G}_3(qp) = \mathbf{s}_3(qp) - \int \mathbf{s}_3(qt) \cdot \mathbf{H}_{-1}(tp)\, dt.$$

The three vectors $\mathbf{G}_1(qp)$, $\mathbf{G}_2(qp)$, $\mathbf{G}_3(qp)$ may then be called a *set of Green's vectors* for the equation (1) vanishing over the boundary. They represent theoretically the displacements produced in the body when the surface is fixed and a unit force is concentrated at the point q in the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. Forming with these the dyadic

$$\Gamma(qp) = \frac{1}{2\pi}\left[\mathbf{i}\,\mathbf{G}_1(qp) + \mathbf{j}\,\mathbf{G}_2(qp) + \mathbf{k}\,\mathbf{G}_3(qp)\right]$$
$$= \Phi(qp) - \int \Phi(qt) \cdot \mathbf{H}_{-1}(tp)\, dt,$$

we have an expression which is identical with the dyadic $\Gamma_{-1}(qp)$ given by the equations (A) for the parameter value $\lambda = -1$. As dyadics are subject to the ordinary rules of differentiation and integration, we have in $\Gamma(qp)$ one which, regarded as a function of p, vanishes over the boundary and is a regular solution of (1) except at the point q where its consequents become infinite to the same order as $1/2\pi r$. It will therefore be called the *Green's dyadic* of the equation (1) for the inner region satisfying the boundary condition

$$(2) \qquad \Gamma(qt) = 0$$

of zero surface displacement.

Similarly we have in $\Gamma_{-1}'(pq)$ the Green's dyadic for the outer region S' corresponding to zero surface traction. For if q is a point of that region, the displacement with surface traction $\mathbf{F}_1(tq)$ is given by

$$\mathbf{V}_1(p) = \int \Gamma_{-1}'(pt) \cdot \mathbf{F}_1(tq)\, dt,$$

in virtue of (C).† Hence the displacement

* Cf. also *Phil. Mag.*, *loc. cit.*, § 13. † *Ibid.*, § 16.

$$\mathbf{G}_1(pq) = \mathbf{s}_1(pq) - \int \Gamma_{-1}'(pt) \cdot \mathbf{F}_1(tq)\, dt$$

has zero surface traction, and is of the nature of a Green's vector. Forming with this and the other two vectors of the set the dyadic

$$\begin{aligned} \Gamma'(pq) &= \frac{1}{2\pi}\left[\mathbf{i}\,\mathbf{G}_1(pq) + \mathbf{j}\,\mathbf{G}_2(pq) + \mathbf{k}\,\mathbf{G}_3(pq)\right] \\ &= \Phi(pq) - \int \Gamma_{-1}'(pt) \cdot \mathrm{X}(tq)\, dt \\ &= \Gamma_{-1}'(pq), \end{aligned}$$

we have the Green's dyadic for the outer region corresponding to zero surface traction.

§ 2. *The generalised Green's dyadic.* When, however, we try to construct in the above manner a Green's dyadic for the inner region and the condition

$$(2') \qquad \mathrm{T}\Gamma(qt) = 2$$

of zero surface traction, the difficulty of the singular parameter value $\lambda = +1$ confronts us. It is impossible to find a regular displacement satisfying (1) and with surface traction equal to $\mathbf{F}_1(tq)$, for the necessary conditions*

$$\int \mathbf{F}_1(tq)\, dt = 0, \quad \int \rho(t) \times \mathbf{F}_1(tq)\, dt = 0$$

are not satisfied, $\rho(t)$ being the position vector of the point t relative to the c.m. of the body. All that can be done is to find a displacement whose surface traction differs from $\mathbf{F}_1(tq)$ by a function of the form $(\mathsf{a} + \omega \times \rho)$, where a and ω are constant vectors; and thus construct a Green's dyadic corresponding to surface traction of the form $(\mathsf{a} + \omega \times \rho)$. This is hardly what is required. We rather try to construct a generalised Green's dyadic analogous to the generalised Green's function defined by Hilbert† for the self-adjoint partial differential equation of the second order.

The equation (1) admits regular solutions of the form a and $\omega \times \rho$ with zero surface traction. In fact, it is to this contingency that the non-existence of the ordinary Green's vectors and dyadic is due. These solutions may be normalised so that the *volume* integral of the square of each is equal to unity. In this form they may be written $\mathsf{a}/\sqrt{J}$ and $\omega \times \rho/\sqrt{I}$, where a and ω are constant *unit* vectors, J the volume of the body, and I the moment of inertia of the

* *Ibid*, § 15

† "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen", *Zweite Mitt.*, *Gött. Nach.* (1904), S. 219, 238.

volume about the axis through the c.m. parallel to ω. If the principal axes of inertia at the c.m. be taken as the direction of **i**, **j**, **k** respectively, A, B, C being the principal moments of inertia, the six independent normalised solutions of (1) corresponding to zero surface traction are

$$\frac{\mathbf{i}}{\sqrt{J}},\ \frac{\mathbf{j}}{\sqrt{J}},\ \frac{\mathbf{k}}{\sqrt{J}},\ \frac{\mathbf{i}\times\rho}{\sqrt{A}},\ \frac{\mathbf{j}\times\rho}{\sqrt{B}},\ \frac{\mathbf{k}\times\rho}{\sqrt{C}},$$

and these are orthogonal to each other since the products of inertia vanish. Form with these the dyadic

$$\Omega(qp)=\frac{\mathbf{i}\,\mathbf{i}}{J}+\frac{\mathbf{j}\,\mathbf{j}}{J}+\frac{\mathbf{k}\,\mathbf{k}}{J}+\frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A}+\ldots+\ldots$$

$$=\frac{\mathbf{I}}{J}+\frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A}+\frac{\mathbf{j}\times\rho(q)\,\mathbf{j}\times\rho(p)}{B}+\ldots.$$

Now there exists a solution $\mathbf{G}_1(qp)$ of the differential equation

$$(3)\qquad L_p[\mathbf{G}_1(qp)]=\mathbf{i}\cdot\Omega(qp),$$

which becomes infinite at the point $p=q$ in the same manner as an ordinary Green's vector, while it satisfies the boundary condition (2′) of zero surface traction, and is further orthogonal to all the above six orthogonal solutions of (1). This function $\mathbf{G}_1(qp)$ represents theoretically the displacement of the point p of the body when a unit force, concentrated at q, acts in the direction of **i**, while throughout the volume of the body acts a force $-\mathbf{i}\cdot\Omega(qp)$ per unit mass. The displacement due to this system of forces clearly satisfies the differential equation (3) at all points except q; and that the body is in equilibrium under the forces is easily verified* by resolving parallel to **i**, **j**, and **k**, and taking moments about the point q. Further, the displacement $\mathbf{G}_1(qp)$ may be chosen so as to be orthogonal to all the six exceptional solutions; for these six conditions are equivalent to the two

$$(4)\qquad \int\mathbf{G}_1(qp)\,dp=0,\quad \int\mathbf{G}_1(qp)\times\rho(p)\,dp=0,$$

requiring no translation of the body as a whole, and no rotation about the c.m. These conditions can always be satisfied by the addition of a vector of the form $\mathbf{a}+\omega\times\rho$, representing a displacement of the body as a whole.

Two other vectors $\mathbf{G}_2(qp)$ and $\mathbf{G}_3(qp)$ exist, bearing the same relation to the axes of **j** and **k** respectively that $\mathbf{G}_1(qp)$ bears to **i**. These three vectors we shall speak of as a set of

* See note at end of § 2, p. 142.

generalised Green's vectors for the equation (1) and the condition of zero surface traction; while the dyadic

$$\Gamma(qp) = \frac{1}{2\pi}\left[\mathbf{i}\,\mathbf{G}_1(qp) + \mathbf{j}\,\mathbf{G}_2(qp) + \mathbf{k}\,\mathbf{G}_3(qp)\right]$$

formed with them is the corresponding generalised Green's dyadic for the region S. It is clear that $\Gamma(qp)$ satisfies the differential equation

$$(5) \qquad L_p[\Gamma(qp)] = \frac{1}{2\pi}\,\Omega(qp),$$

and, corresponding to (4), the integral relations

$$(6) \qquad \int \Gamma(qp)\,dp = 0, \quad \int \Gamma(qp)\times\rho(p)\,dp = 0.$$

[*Note*.—To prove that the body is in equilibrium under a unit force $\mathbf{i}$ at q and a bodily force $-\mathbf{i}\cdot\Omega(qp)$ per unit mass throughout. First the resultant of the bodily forces is $-\mathbf{i}$. For

$$\int -\mathbf{i}\cdot\Omega(qp)\,dp = -\int \mathbf{i}\cdot\left[\frac{\mathbf{I}}{J} + \frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A} + \ldots + \ldots\right]dp$$
$$= -\mathbf{i},$$

since the centroid of the body is the origin of the position vectors $\rho(p)$, making $\int\rho(p)\,dp = 0$. It only remains then to prove that the resultant moment of all the forces is zero about (say) the point q. But the sum of the moments about q

$$= \int[\rho(p) - \rho(q)] \times \left[\frac{\mathbf{i}}{J} + \frac{\mathbf{j}\times\rho(p)\,\rho(q)\,.\,\mathbf{k}}{B} - \frac{\mathbf{k}\times\rho(p)\,\rho(q)\cdot\mathbf{j}}{C}\right]dp$$

$$= \mathbf{i}\times\rho(q) + \frac{1}{B}\rho(q)\cdot\mathbf{k}\int\rho(p)\times[\mathbf{j}\times\rho(p)]\,dp$$
$$\qquad - \frac{1}{C}\rho(q)\,.\,\mathbf{j}\int\rho(p)\times[\mathbf{k}\times\rho(p)]\,dp$$

$$= \mathbf{i}\times\rho(q) + \frac{1}{B}\rho(q)\cdot\mathbf{k}\left[\tfrac{1}{2}(A+B+C)\,\mathbf{j} - \tfrac{1}{2}(A-B+C)\,\mathbf{j}\right] - \ldots$$

$$= \mathbf{i}\times\rho(q) + \rho(q)\cdot\mathbf{k}\,\mathbf{j} - \rho(q)\,.\,\mathbf{j}\,\mathbf{k} = 0.$$

The argument has depended upon the fact that the products of inertia vanish, since the directions of the principal axes through the centroid are those of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.]

§ 3. *Conjugo-symmetry.* Let $\mathbf{u}$ and $\mathbf{v}$ be two functions of position, regular within the region considered and, when regarded as displacements, having surface tractions $\mathbf{Tu}$ and $\mathbf{Tv}$ respectively. Then Betti's theorem expresses that*

$$(7) \quad \int [L(\mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot L(\mathbf{v})]\, dp = \int [\mathbf{Tu} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{Tv}]\, dt.$$

If in this we put for $\mathbf{u}$ and $\mathbf{v}$ the Green's vectors $\mathbf{G}_1(qp)$ and $\mathbf{G}_1(q'p)$ with poles at q and q' respectively, and possessing either both zero surface displacement or both zero surface traction, we must exclude the poles by (say) small spherical surfaces Z and Z' in the usual manner. The volume integral then vanishes identically; for the ordinary Green's vectors satisfy (1), and though the generalised vectors satisfy (3) yet in virtue of the relations (4) the volume integral in (7) still vanishes. Moreover the surface integral over Σ disappears because of the boundary conditions satisfied by these vectors. The equation then reduces to†

$$\int_{Z+Z'} [\mathbf{G}_1(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_1(sq')]\, ds = 0$$

Similarly if in (7) we substitute in turn the pairs of vectors $\mathbf{G}_1(qp)$, $\mathbf{G}_2(q'p)$ and $\mathbf{G}_1(qp)$, $\mathbf{G}_3(q'p)$ satisfying the same boundary condition as before, we obtain the equations

$$\int_{Z+Z'} [\mathbf{G}_2(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_2(sq')]\, ds = 0,$$

$$\int_{Z+Z'} [\mathbf{G}_3(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_3(sq')]\, ds = 0.$$

Multiplying these equations by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively and adding we find after division by 2π

$$\int_{Z+Z'} [\Gamma(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \Psi(sq')]\, ds = 0,$$

where $\Gamma(qp)$ is the Green's dyadic for the boundary condition considered. Two equations similar to this may be written down by altering the suffix. Then taking the indeterminate products of the three equations by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively as consequents, and adding, we have a result which may be written

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* Cf. Betti, *Il Nuovo Cimento* (1872); *Annali di Mat*, t. 6 (1875); also Lévy, *C.R.*, t. 107 (1888), pp 413 and 453.

† By thus writing the surface traction of $\mathbf{G}_1(qp)$ at Z as $\mathbf{F}_1(sq)$ we are neglecting the traction due to the regular function $\mathbf{W}_1(p)$ of § 1. But when the surface Z becomes vanishingly small the part of the integral due to this also vanishes.

$$\int_{Z+Z'} [\Gamma(q's) \cdot \Psi(sq) - \Psi_c(sq') \cdot \Gamma_c(qs)]\, ds = 0.$$

Now let the radii of the spheres Z and Z' decrease indefinitely; then in the limit we find, exactly as in § 4 of my earlier paper,

$$(8) \qquad \Gamma(q'q) = \Gamma_c(qq').$$

Thus the Green's dyadics are conjugo-symmetric* with properties analogous to those of the symmetric kernel in the ordinary integral equation.

§ 4. *A fundamental formula.* If, however, in Betti's theorem (7) we put for $\mathbf{u}$ any regular solution of the non-homogeneous equation

$$(10) \qquad L[\mathbf{u}(p)] = -2\,\mathbf{f}(p),$$

where $\mathbf{f}(p)$ is continuous, and for $\mathbf{v}$ a Green's vector $\mathbf{G}_1(qp)$ with pole at q, we must isolate this point by a small surface Z in the usual manner. In the case of the ordinary Green's vectors satisfying (1) we thus obtain

$$-2\int \mathbf{G}_1(qp) \cdot \mathbf{f}(p)\, dp$$
$$= \int_{Z+\Sigma} [\mathbf{G}_1(qs) \cdot \mathbf{T}\mathbf{u}(s) - \mathbf{u}(s) \cdot \mathbf{T}\mathbf{G}_1(qs)]\, ds,$$

while two similar equations may be written down in $\mathbf{G}_2(qs)$ and $\mathbf{G}_3(qs)$. Multiplying these in order by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and adding, and then making the surface Z decrease indefinitely, we find in the limit by the same argument as above that

$$(11) \quad \mathbf{u}(q) = \int \Gamma(qp) \cdot \mathbf{f}(p)\, dp$$
$$- \tfrac{1}{2} \int [\mathbf{u}(t) \cdot \mathbf{T}\Gamma(qt) - \Gamma(qt) \cdot \mathbf{T}\mathbf{u}(t)]\, dt.†$$

Now the Green's dyadic is that of zero surface displacement; and therefore if $\mathbf{u}$ is that solution of (10) which vanishes over the boundary, the surface integral in the last formula disappears, leaving us with the result

$$(12) \qquad \mathbf{u}(q) = \int \Gamma(qp) \cdot \mathbf{f}(p)\, dp,$$

which gives a source-representation for the function $\mathbf{u}(q)$. Hence

Theorem 1. Any displacement function $\mathbf{u}(q)$ which is continuous along with its derivatives of the first two orders,

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* Cf "Vector integral equations, etc", §§ 19–24.

† By $\mathbf{T}\Gamma$ is obviously meant the result obtained by performing on Γ the same operation as we perform on $\mathbf{u}$ to find $\mathbf{T}\mathbf{u}$.

and vanishes over the boundary of the region S, is capable of a source-representation as in the last formula.

The solution of (10), which vanishes over the boundary, is thus given uniquely by (12); and we have proved the formula

$$(13) \qquad L\int \Gamma(qp)\cdot \mathbf{f}(p)\,dp = -2\,\mathbf{f}(q).$$

When the boundary condition of zero surface traction is under consideration, we use the generalised Green's vectors satisfying equations of the form (3); and in place of (12) we now find for that solution of (10) which gives zero surface traction

$$(14) \quad \mathbf{u}(q) = \int \Gamma(qp)\cdot \mathbf{f}(p)\,dp + \frac{1}{4\pi}\int \Omega(qp)\cdot \mathbf{u}(p)\,dp.$$

If then $\mathbf{u}(p)$ also satisfies the same relations (4) as the generalised Green's vectors, this formula becomes as before

$$(14') \qquad \mathbf{u}(q) = \int \Gamma(qp)\cdot \mathbf{f}(p)\,dp.$$

As pointed out in § 2, $\mathbf{u}(p)$ can always be chosen so as to satisfy these conditions by the addition of a suitable displacement of the form $(\mathbf{a} + \omega \times \rho)$. Hence

THEOREM 2. Any displacement $\mathbf{u}(p)$, which is continuous along with its derivatives of the first two orders, and produces zero surface traction while it further satisfies the relation (4), is capable of source-representation in terms of the generalised Green's dyadic as in (14′).

It follows from the preceding argument that the Green's dyadics are *closed kernels*. For if there exists a continuous function $\mathbf{f}(p)$ such that

$$\int \Gamma(qp)\cdot \mathbf{f}(p)\,dp = 0,$$

we deduce from (13) that

$$L\int \Gamma(qp)\cdot \mathbf{f}(p)\,dp = -2\,\mathbf{f}(q) = 0.$$

Hence $\mathbf{f}(q)$ vanishes identically, and the Green's dyadic is a closed conjugo-symmetric kernel. It follows therefore that its characteristic numbers are *real* and *infinitely many*.* Further, denoting the normal characteristic functions of $\Gamma(qp)$ by $\phi_i(p)$, we have the following theorems:

THEOREM 3. If $\mathbf{F}(p)$ is a continuous function, and the series

$$\sum_{i=1}^{\infty} \phi_i(p)\int \mathbf{F}(q)\cdot \phi_i(q)\,dq$$

* *Ibid*, § 23, Theorem 3.

converges uniformly, it represents the function $\mathbf{F}(p)$.*

THEOREM 4. Any displacement function $\mathbf{u}(p)$, which is continuous along with its derivatives of the first two orders, which further satisfies the same boundary condition as the Green's dyadic $\Gamma(pq)$, and in the case of the generalised dyadic the further relations (4), can be expanded by the Fourier rule in an absolutely and uniformly convergent series† in terms of the characteristic functions $\phi_i(p)$.

§ 5. *An equation related to* $L(\mathbf{u})=0$. Suppose that we require a regular solution of the equation

$$L(\mathbf{u})+2\lambda\mathbf{u}=0 \tag{16}$$

satisfying the same boundary condition as the Green's dyadic $\Gamma(qp)$, and in the case of the generalised dyadic the relations (4) also; then from the preceding results it is given by the integral equation

$$\mathbf{u}(p)=\lambda\int\Gamma(pq)\cdot\mathbf{u}(q)\,dq, \tag{17}$$

which admits a non-zero solution only when λ is equal to one of the characteristic numbers λ_i of the kernel $\Gamma(pq)$. The corresponding solutions are the characteristic functions $\phi_i(p)$ of this kernel. Hence the differential equation (16) admits the required solution only for these particular values of λ, such solutions being the characteristic functions of the integral equation (17). We may thus speak of these solutions and parameter values as the characteristic functions and parameter values of the differential equation (16). Their number has been proved infinite.

It can be shown, as in the case of the scalar self-adjoint partial differential equation,‡ that the resolvent of $\Gamma(pq)$ is the Green's dyadic for the differential equation (16) satisfying the same boundary condition. This dyadic therefore exists for all values of λ except the characteristic values of $\Gamma(pq)$; but at each of these the Green's dyadic of (16) becomes infinite, having a simple pole, as is the case with the resolvent of any conjugo-symmetric dyadic kernel.§

It is important for our purpose to show that the characteristic numbers λ_i are all positive. Take the form ‖

$$\int\{\Pi[\mathbf{u}(p)]+\mathbf{u}\cdot L(\mathbf{u})\}\,dp=\int\mathbf{u}\cdot\mathbf{T}\mathbf{u}\,dt \tag{18}$$

* Cf Hilbert, "Grundzüge", *Erste Mitt.*, Kapp. IV, Satz 6
† "Vector integral equations, etc.", § 23, Theorem 2.
‡ Hilbert, *loc. cit.*, S. 224, 239
§ "Vector integral equations", § 20, Theorem 4.
‖ Cf. Betti, *loc. cit.*

of Betti's theorem, in which $\Pi(\mathbf{u})$ denotes the potential due to the displacement $\mathbf{u}$. If in this we put for $\mathbf{u}$ a solution of (16), satisfying either of the boundary conditions, the second member vanishes, while the first becomes

$$\int \Pi[\mathbf{u}(p)]\,dp - \lambda \int \mathbf{u}^2 dp.$$

Since the first integral is a positive function representing twice the potential energy of the deformed body, it follows that λ must be positive in order that the whole expression may vanish. Hence the characteristic numbers of the Green's dyadics are all positive.

II.—Equilibrium of an Elastic Isotropic Body.

§ 6. *No bodily forces.* The solution of the *first* boundary problem considered in my earlier paper, requiring the determination of a regular integral $\mathbf{u}(p)$ of (1) when its boundary value $\mathbf{u}(t)$ is known, is given as a particular case of the formula (11). For, taking $\Gamma(qp)$ as the Green's dyadic for zero surface displacement, and putting $\mathbf{f}(p)$ equal to zero, we obtain the required solution in the form

$$(19) \quad \mathbf{u}(p) = -\tfrac{1}{2}\int \mathbf{u}(t)\cdot\mathbf{T}\Gamma(t'p)\,dt = \int \mathbf{u}(t)\cdot\mathbf{H}_{-1}(tp)\,dt,^{*}$$

which is identical with the result previously found.†

In the *second* boundary problem we are given the value of the surface traction. Taking then in (11) $\Gamma(qp)$ as the generalised Green's dyadic for zero surface traction, and putting $\mathbf{f}(p)=0$, we obtain the formula

$$(19') \qquad \mathbf{u}(p) = \tfrac{1}{2}\int \Gamma(pt^{+})\cdot\mathbf{T}\mathbf{u}(t)\,dt,$$

where the displacement $\mathbf{u}(p)$ has been chosen to satisfy the conditions (4), as already explained. This is identical in form with the solution found previously,‡ but the dyadic $\Gamma(pt)$ is not the same as there employed.

A *third* boundary problem for the equation (1) requiring a displacement satisfying the relation

$$\mathbf{T}\mathbf{u}(t) = \lambda\mathbf{u}(t) - \mathbf{U}(t),$$

where $\mathbf{U}(t)$ is a given function of the position of t, may be treated along lines exactly parallel to those followed by Plemelj§ in the corresponding problem for Laplace's equation. The surface traction for a given displacement takes the place of

* Cf. *Phil. Mag., loc. cit.*, formula (50).
† *Ibid.*, § 13. ‡ *Ibid.*, § 15.
§ *Monatshefte fur Math. und Physik*, Bd 18 (1907), S. 180–211

normal derivative, Betti's theorem is used instead of Green's, while in place of the additive constant C there introduced we have a vector of the form $(\mathbf{a}+\omega\times\rho)$, the constant vectors $\mathbf{a}$ and ω being determined so that the solution satisfies the relations (4).

§ 7. *General problem.* We may also write down the solution of the general problem requiring the displacement $\mathbf{u}(p)$ at any point of an isotropic body under the bodily force $\mathbf{P}(p)$ per unit mass, the boundary condition being either one of those already treated, viz.

$$(2)\qquad \begin{cases}(a) & \mathbf{u}(t)=0,\\ (b) & \mathbf{Tu}(t)=0,\end{cases}$$

or one of the non-homogeneous conditions

$$(20)\qquad \begin{cases}(a) & \mathbf{u}(t)=\mathbf{U}(t),\\ (b) & \mathbf{Tu}(t)=\mathbf{V}(t).\end{cases}$$

The differential equation to be satisfied is

$$(21)\qquad L[\mathbf{u}(p)]=-\mathbf{P}(p).$$

From the argument of § 4 it follows that if the displacement satisfies one of the homogeneous boundary relations (2) it is given by

$$\mathbf{u}(p)=\tfrac{1}{2}\int\Gamma(pq)\cdot\mathbf{P}(q)\,dq,$$

with the usual precaution in the case of zero surface traction.

When, however, the boundary displacement has the non-zero value given by (20a), the formula (11) becomes, in terms of the Green's dyadic for zero surface displacement,

$$\mathbf{u}(p)=\tfrac{1}{2}\int\Gamma(pq)\cdot\mathbf{P}(q)\,dq-\tfrac{1}{2}\int\mathbf{U}(t)\cdot\mathbf{T}\Gamma(tp)\,dt.$$

The first integral is the solution of (21) vanishing over the boundary; the second is the integral of (1) with surface value $\mathbf{U}(t)$.

Similarly for the condition (20b) when the surface traction is $\mathbf{V}(t)$, we use the generalised Green's dyadic and (11) becomes

$$\mathbf{u}(p)=\tfrac{1}{2}\int\Gamma(pq)\cdot\mathbf{P}(q)\,dq+\tfrac{1}{2}\int\Gamma(pt)\cdot\mathbf{V}(t)\,dt.$$

For the equilibrium of the body the bodily force and the surface traction must satisfy the relations

$$(22)\qquad \begin{cases}\int\mathbf{P}(p)\,dp+\int\mathbf{V}(t)\,dt=0,\\ \int\rho(p)\times\mathbf{P}(p)\,dp+\int\rho(t)\times\mathbf{V}(t)\,dt=0\end{cases}$$

III.—Small Vibrations of an Elastic Isotropic Body.

§ 8. *General problem of free vibrations.* We shall now consider the small free vibrations of an elastic body occupying the region S, the boundary condition being either that of zero surface displacement or that of zero surface traction. The differential equation satisfied by the displacement $\mathbf{u}(p)$ is, with suitable choice of units,*

$$(23) \qquad L(\mathbf{u}) = \frac{\partial^2}{\partial t^2}\mathbf{u}.$$

For harmonic vibrations we make the ordinary substitution

$$\mathbf{u}(p, t) = \mathbf{v}(p)\begin{cases}\cos kt\\ \sin kt,\end{cases}$$

in which $\mathbf{v}(p)$ is independent of the time, and thus obtain the differential equation

$$L[\mathbf{v}(p)] = -k^2\mathbf{v}(p),$$

while the boundary condition for $\mathbf{v}(p)$ is the same as for $\mathbf{u}(p)$. If $\Gamma(qp)$ be the Green's dyadic for this condition, these are equivalent to the homogeneous integral equation

$$(24) \qquad \mathbf{v}(p) = \lambda\int\Gamma(pq)\cdot\mathbf{v}(q)\,dq \quad (\lambda = k^2/2),$$

the function $\mathbf{v}(p)$ in the case of zero surface traction satisfying the conditions (4). This integral equation admits a non-zero solution only when λ is equal to one of the characteristic numbers λ_i of the Green's dyadic; and as these have been proved all positive, the values of k found from them are all real. If $\mathbf{v}_i(p)$ are the characteristic functions of the equation (24), the solution of the problem thus takes the form

$$(25) \quad \mathbf{u}(p, t) = \sum_{i=1}^{\infty}\mathbf{v}_i(p)\left[A_i\cos\sqrt{(2\lambda_i)}\,t + B_i\sin\sqrt{(2\lambda_i)}\,t\right].$$

It is clear that the constants A_i and B_i are substantially the coefficients in the expansions of the functions $\alpha(p)$ and $\beta(p)$ representing the initial values of the displacement $\mathbf{u}$ and the velocity $\dot{\mathbf{u}}$, in terms of the functions $\mathbf{v}_i(p)$. By theorem 4 (§ 4) these initial functions may be expanded in terms of the normal functions $\mathbf{v}_i(p)$ by the Fourier rule

$$(26) \qquad \begin{cases}\alpha(p) = \sum_{i=1}^{\infty}\mathbf{v}_i(p)\int\alpha(q)\cdot\mathbf{v}_i(q)\,dq,\\ \beta(p) = \sum_{i}\mathbf{v}_i(p)\int\beta(q)\cdot\mathbf{v}_i(q)\,dq.\end{cases}$$

* Cf., *e.g.*, Love, *Mathematical Theory of Elasticity*, vol. i., § 29, 1st ed.

and thus the coefficients A_i and B_i are determined. In the case of zero surface traction, unless the functions $\alpha(p)$ and $\beta(p)$ satisfy the conditions (4), we must determine vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ω_1 and ω_2 in such a way that the functions

$$\begin{cases} \alpha'(p) = \alpha(p) + \mathbf{a}_1 + \omega_1 \times \rho(p), \\ \beta'(p) = \beta(p) + \mathbf{a}_2 + \omega_2 \times \rho(p) \end{cases}$$

do satisfy the conditions (4). Then these functions may be expanded as in (26), giving the required series for $\alpha(p)$ and $\beta(p)$.

§ 9. *Forced vibrations.* In the problem of forced vibrations the natural period of the body is set aside or overpowered by that of the forces controlling the motion. Suppose these forces to be of the harmonic type, their value at any point p being $\mathbf{F}(p)\cos kt$ per unit mass. Then the differential equation of motions is

$$(27) \qquad L(\mathbf{u}) = \frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{F}(p)\cos kt,$$

the boundary condition being as before. Since the vibration of each particle is compelled to keep in step with the force, the required displacement $\mathbf{u}$ will be of the form

$$\mathbf{u}(p, t) = \mathbf{v}(p)\cos kt,$$

$\mathbf{v}(p)$ being independent of t. Making this substitution we find for $\mathbf{v}(p)$ the differential equation

$$L[\mathbf{v}(p)] = -k^2\mathbf{v}(p) - \mathbf{F}(p)$$

with the original boundary condition. If $\Gamma(pq)$ is the appropriate Green's dyadic these are equivalent to the non-homogeneous integral equation

$$\mathbf{v}(p) = \lambda \int \Gamma(pq) \cdot \mathbf{v}(q)\, dq + \mathbf{f}(p),$$

in which $$\mathbf{f}(p) = \tfrac{1}{2}\int \Gamma(pq) \cdot \mathbf{F}(q)\, dq$$

and $$\lambda = \tfrac{1}{2}k^2.$$

The kernel of this equation is conjugo-symmetric, and the solution may be expressed by the extension of Schmidt's formula*

$$\mathbf{v}(p) = \mathbf{f}(p) + \lambda \sum_{n=1}^{\infty} C_n \mathbf{v}_n(p)/(\lambda_n - \lambda),$$

* "Vector integral equations, etc.", § 23, Theorem 4.

where λ_n, $\mathbf{v}_n(p)$ are the characteristic numbers and normal function of the kernel $\Gamma(pq)$, and

$$C_n = \int f(q) \cdot \mathbf{v}_n(q)\, dq.$$

But from the form of $\mathbf{f}(p)$ it may be expanded* in terms of the normal functions in the series

$$\mathbf{f}(p) = \sum_{n=1}^{\infty} C_n \mathbf{v}_n(p).$$

The value of $\mathbf{v}(p)$ then becomes

$$\mathbf{v}(p) = \sum_n C_n \lambda_n \mathbf{v}_n(p) / (\lambda_n - \lambda).$$

giving for the forced vibrations of the system

$$(28) \qquad \mathbf{u}(p, t) = \cos kt \sum_n C_n \lambda_n \mathbf{v}_n(p) / (\lambda_n - \lambda).$$

The *most general vibration* of the body under the conditions of the problem consists of two parts, and may be expressed

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2,$$

where $\mathbf{u}_1$ represents the forced vibrations just found, and $\mathbf{u}_2$ the free vibrations superimposed upon the former. The latter satisfies the homogeneous equation (23). Hence for the most general solution of (27) we have

$$(29) \quad \mathbf{u}(p, t) = \sum_{n=1}^{\infty} \mathbf{v}_n(p) \left[A_n \cos(k_n t) + B_n \sin(k_n t) + \frac{C_n \lambda_n}{\lambda_n - \lambda} \cos kt \right],$$

where, as before, $\qquad k_n^2 = 2\lambda_n$.

Let $\alpha(p)$ and $\beta(p)$ be the initial values of $\mathbf{u}$ and $\dot{\mathbf{u}}$ respectively, with expansions represented by (26). Comparing these with the values of the same quantities obtained from (29) we find

$$\begin{cases} A_n = -C_n \lambda_n / (\lambda_n - \lambda) + \int \alpha(p) \cdot \mathbf{v}_n(p)\, dp, \\ B_n = \dfrac{1}{k_n} \displaystyle\int \beta(p) \cdot \mathbf{v}_n(p)\, dp, \end{cases}$$

$$(30) \quad \mathbf{u}(p, t) = \sum_{n=1}^{\infty} \mathbf{v}_n(p) \left[\frac{C_n \lambda_n}{\lambda_n - \lambda} (\cos kt - \cos k_n t) \right.$$

$$\left. + \cos k_n t \int \alpha(p) \cdot \mathbf{v}_n(p)\, dp + \frac{1}{k_n} \sin k_n t \int \beta(p) \cdot \mathbf{v}_n(p)\, dp \right].$$

* *Ibid*, § 23, Theorem 2.

§ 10. *Damped vibrations.* If the vibrating body is under the influence of dissipative forces such as are represented by the addition of a term* of the form $w \cdot d\mathbf{u}/dt$ to the second member of (23), the vibrations being otherwise free, we have to find a solution of the differential equation.

$$(31) \qquad L(\mathbf{u}) = \frac{\partial^2}{\partial t^2}\mathbf{u} + w\frac{\partial}{\partial t}\mathbf{u}$$

satisfying the same boundary condition (2) as in the previous cases. Assuming w constant for the isotropic body, make the substitution

$$\mathbf{u}(p, t) = \mathbf{v}(p)\, e^{(in - w/2)t}.$$

The equation (31) then becomes

$$(32) \qquad L[\mathbf{v}(p)] = -2\lambda\, \mathbf{v}(p),$$

where we have put $\lambda = (4n^2 + w^2)/8$;

while the boundary condition is unaltered. If $\Gamma(pq)$ is the appropriate Green's dyadic, the function $\mathbf{v}(p)$ is given by

$$\mathbf{v}(p) = \lambda \int \Gamma(pq) \cdot \mathbf{v}(q)\, dq.$$

This integral equation admits solutions $\mathbf{v}_i(p)$ only for the characteristic parameter values λ_i which are all positive, and in terms of which the corresponding values of n are

$$n_i = \tfrac{1}{2}\sqrt{(8\lambda_i - w^2)}.$$

At the most then a finite number of the n's are imaginary. If the dissipative forces are sufficiently small all the n's are real. The solution of the problem is given by

$$(33) \quad \mathbf{u}(p, t) = e^{-wt/2}\sum_{i=1}^{\infty} \mathbf{v}_i(p)\,[A_i \cos n_i t + B_i \sin n_i t],$$

the constants A_i and B_i being determined from the expansions of the functions $\alpha(p)$ and $\beta(p)$ representing the initial condition of the body.

§ 11. *Bilinear dyadic series.* We have already given a physical interpretation to the Green's vectors. The relation of the Green's dyadics to the bilinear series may be seen from

* A resistance proportional to the velocity. It is doubtful if this has any physical significance for a solid body. However, I propose the problem for what it is worth.

the following mechanical considerations,* which at the same time show the physical significance of the characteristic functions.

Let q_n ($n = 1, 2, \ldots, \infty$) be a set of parameters, in the sense of the Lagrangean mechanics, expressing the possible positions of the rigid body, each chosen so as to vanish in the position of equilibrium. On the assumption that the vibrations are simple harmonic we may put

$$(34) \qquad \mathbf{u}(p, t) = \sum_n q_n \mathbf{v}_n(p),$$

the parameters q_n being functions of the time satisfying differential equations of the form

$$\ddot{q}_n + \lambda_n q_n = 0.$$

This necessitates the kinetic energy T being free from product terms $\dot{q}_n \dot{q}_m$ ($m \neq n$), so that

$$T = \tfrac{1}{2} \sum_n \dot{q}_n^{\,2}; \quad V = \tfrac{1}{2} \sum_n \lambda_n q_n^{\,2},$$

V being the potential energy. We might have started with these forms; for "when we have two homogeneous quadratic functions of any number of variables, one of which is essentially positive for all values of the variables, it is known that by a real linear transformation of the variables we may clear both expressions of the terms containing the products of the variables, and also make the coefficients of the squares in the positive function each equal to unity or some given positive constants".† From (34), however, we find for the kinetic energy

$$T = \tfrac{1}{2} \int [\sum_n \dot{q}_n \mathbf{v}_n(p)]^2 dp,$$

dp being the element of mass at p, or the element of volume since the body is of uniform density. Identifying these values of the kinetic energy we find

$$(35) \qquad \int \mathbf{v}_n(p) \cdot \mathbf{v}_m(p)\, dp \begin{array}{ll} = 1, & \text{if} \quad m = n, \\ = 0, & \text{if} \quad m \neq n. \end{array}$$

These are the orthogonal relations satisfied by the normalised characteristic functions.

Suppose now that the body is vibrating under a force-system R, the force on the element dp being $\mathbf{P}(p)\, dp$. Then

* Cf., *e.g.*, Kneser, *Integralgleichungen*, §§ 9, 16. Also Lord Rayleigh, *Theory of Sound*, vol. i., chap. v.

† Routh, *Elementary Rigid Dynamics*, § 459.

the virtual work of these forces, due to a small displacement $\delta \mathbf{u}$, is

$$\int \delta \mathbf{u} \cdot \mathbf{P}(p)\, dp = \sum_n \delta q_n \int \mathbf{v}_n(p) \cdot \mathbf{P}(p)\, dp = \sum_n Q_n \delta q_n,$$

where we have put

$$(36) \qquad Q_n = \int \mathbf{v}_n(p) \cdot \mathbf{P}(p)\, dp.$$

The equation of vibration under the force-system R is then

$$\ddot{q}_n + \lambda_n q_n = Q_n.$$

If, however, we imagine the body in equilibrium under these forces the term $\ddot{q}_n$ is zero, and $q_n = Q_n / \lambda_n$. The equilibrium displacement is then given by

$$(37) \qquad \mathbf{u}(p) = \sum_n Q_n \mathbf{v}_n(p) / \lambda_n.$$

If the boundary condition is that of *zero surface displacement*, the force-system R may, consistently with the equilibrium of the body, be concentrated into a unit force $\mathbf{a}$ at the point q, plus the surface traction necessary to keep the boundary fixed. The latter, however, does not appear in the equation of virtual work. The function Q_n now takes the form $\mathbf{a} \cdot \mathbf{v}_n(q)$, and the displacement $\mathbf{G}(qp)$ under these forces is

$$(38) \quad \mathbf{G}(qp) = \sum_n \mathbf{a} \cdot \mathbf{v}_n(q) \mathbf{v}_n(p) / \lambda_n = \mathbf{a} \cdot \left[\sum_n \frac{\mathbf{v}_n(q)\, \mathbf{v}_n(p)}{\lambda_n} \right]$$
$$= \mathbf{a} \cdot \Theta(qp),$$

where the dyadic $\Theta(qp)$ represents the bilinear series. Suppose we give the unit vector $\mathbf{a}$ the values $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in succession; the corresponding displacements are the Green's vectors $\mathbf{G}_1(qp)$, $G_2(qp)$, $\mathbf{G}_3(qp)$ respectively. Forming with these the Green's dyadic $\Gamma(qp)$, we have

$$(39) \quad \Gamma(qp) = \frac{1}{2\pi} [\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}] \cdot \Theta(qp) = \frac{1}{2\pi} \Theta(qp),$$

that is, the Green's dyadic is represented to a constant factor by the bilinear series.* That this series is convergent follows by Mercer's theorem † from the fact that the characteristic numbers are all positive. Hence the *bilinear formula* (39) is true. On multiplying scalarly both sides of this formula by

* $\Gamma(qp)$ and $\Theta(qp)$ have the same normal functions $\mathbf{v}_n(p)$, but the characteristic numbers of $\Gamma(qp)$ are 2π times those of $\Theta(qp)$.

† *Phil. Trans. Roy. Soc.*, vol. ccix. (A) (1909), pp. 415–446.

$\lambda_n \mathbf{v}_n(q)$, and integrating with respect to q over the region S, we find, in virtue of the relations (35), the integral equation

$$(40) \qquad \mathbf{v}_n(p) = \lambda_n \int \Theta(pq) \cdot \mathbf{v}_n(q)\, dq$$

satisfied by the characteristic functions $\mathbf{v}_n(p)$.

§ 12. In the case of the *zero surface traction*, however, $\lambda_0 = 0$ is a characteristic value to which correspond the six normalised characteristic functions $\mathbf{v}_{0i}(p)$, $(i=1, 2, \ldots, 6)$, equal respectively to

$$\frac{\mathbf{i}}{\sqrt{J}}, \ \frac{\mathbf{j}}{\sqrt{J}}, \ \frac{\mathbf{k}}{\sqrt{J}}, \ \frac{\mathbf{i}\times\rho}{\sqrt{A}}, \ \frac{\mathbf{j}\times\rho}{\sqrt{B}}, \ \frac{\mathbf{k}\times\rho}{\sqrt{C}}.$$

The kinetic energy may be written

$$T = \tfrac{1}{2} \sum_{i=1}^{6} \dot{q}_{0i}^2 + \tfrac{1}{2} \sum_n \dot{q}_n^2,$$

while from the expression for the potential energy the first six terms $\frac{1}{2} \sum_{i=1}^{6} \lambda_0 q_{0i}^2$ disappear. The displacement $\mathbf{u}(p)$ becomes

$$\mathbf{u}(p, t) = \sum_{i=1}^{6} q_{0i} \mathbf{v}_{0i}(p) + \sum_n q_n \mathbf{v}_n(p).$$

Corresponding to the parameter q_{0i} Lagrange's equation of motion under the force system R is

$$\ddot{q}_{0i} = Q_{0i};$$

and therefore when there is equilibrium under these forces

$$O = Q_{0i} = \int \mathbf{P}(p) \cdot \mathbf{v}_{0i}(p)\, dp,$$

a condition which will not in general be satisfied when the system R is concentrated as a unit force at the point q. Suppose however that we introduce at each element of mass dp the additional force

$$-\mathbf{a} \cdot \Omega(qp)\, dp = -\mathbf{a} \cdot \Big[\sum_{i=1}^{6} \mathbf{v}_{0i}(q)\, \mathbf{v}_{0i}(p) \Big] dp,$$

in which $\mathbf{a}$ is a *unit* vector. Then the above condition becomes

$$O = -\int \mathbf{a} \cdot \Omega(qp) \cdot \mathbf{v}_{0i}(p)\, dp + \int \mathbf{P}(p) \cdot \mathbf{v}_{0i}(p)\, dp$$
$$= -\mathbf{a} \cdot \mathbf{v}_{0i}(q) + \int \mathbf{P}(p) \cdot \mathbf{v}_{0i}(p)\, dp,$$

which is satisfied when the force system R is concentrated as a unit force $\mathbf{a}$ at the point q. Then as in the preceding case

Q_n takes the form $\mathbf{a} \cdot \mathbf{v}_n(q)$, and the displacement $G(qp)$ under the combined forces is

$$\mathbf{G}(qp) = \sum_{i=1}^{6} q_{0i}\, \mathbf{v}_{0i}(p) + \mathbf{a} \cdot \left[\sum_n \frac{\mathbf{v}_n(q)\, \mathbf{v}_n(p)}{\lambda_n} \right].$$

We can choose this displacement as explained in § 2, so that all the quantities q_{0i} $(i = 1, 2, \ldots, 6)$ disappear. For the particular unit vectors $\mathbf{a} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively the above displacement then becomes identical with the generalised Green's vectors. Forming with them the generalised dyadic $\Gamma(qp)$ we have again the bilinear formula

$$(39') \qquad \Gamma(qp) = \sum_n \frac{\mathbf{v}_n(q)\, \mathbf{v}_n(p)}{2\pi\lambda_n}.$$

The dyadic series is convergent as before because the values λ_n are all positive, and the bilinear formula is true. Further from the orthogonal relations (25) it is clear that the generalised Green's vectors as just found satisfy the relations

$$\int \mathbf{G}(qp) \cdot \mathbf{v}_{0i}(p)\, dp = 0, \quad (i = 1, 2, \ldots, 6),$$

while on account of the extra bodily force $-\mathbf{a} \cdot \Omega(qp)$ per unit mass, the differential equations satisfied by them are of the form (3). From (39′) it follows as above that the characteristic functions satisfy the integral equation

$$\mathbf{v}_n(p) = 2\pi\lambda_n \int \Gamma(pq) \cdot \mathbf{v}_n(q)\, dq.$$

A PROPERTY OF THE BITANGENTS OF A PLANE QUARTIC CURVE.

By *Prof. Harold Hilton.*

LET $\qquad S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$

$$S' \equiv a'x^2 + \ldots, \quad S'' \equiv a''x^2 + \ldots .$$

Consider any conic with tangential equation

$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0,$$

such that triangles can be found self-conjugate with respect to $\Sigma = 0$ and inscribed in

$$S + 2k_1 S' + k_1^2 S'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and that triangles can be found self-conjugate with respect to (i) and circumscribed to $\Sigma = 0$. Suppose that the same holds for $\Sigma = 0$ and

$$S + 2k_2 S' + k_2^2 S'' = 0, \quad S + 2k_3 S' + k_3^2 S'' = 0 \ldots\ldots \text{(ii)}.$$

Then $\Sigma = 0$ belongs to a fixed tangential net. For, if

$$a_i \equiv a + 2k_i a' + k_i^2 a'', \text{ etc.,}$$

$$a_i A + b_i B + c_i C + 2f_i F + 2g_i G + 2h_i H = 0 \ldots\ldots \text{(iii)},$$

for $$i = 1, \quad i = 2, \quad i = 3,$$

so that there are three linear relations between A, B, C, F, G, H. Then Σ is of the form $\Sigma_1 + \alpha\Sigma_2 + \beta\Sigma_3$, where $\Sigma_1 = 0$, $\Sigma_2 = 0$, $\Sigma_3 = 0$ are fixed conics and α, β are arbitrary constants; which is what is meant by saying that $\Sigma = 0$ belongs to a "tangential net".

We show now that triangles can be found self-conjugate with respect to any conic of the tangential net and inscribed in any conic of the family

$$S + 2kS' + k^2 S'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

It is readily shown that any three given conics can have their equations thrown into the form (i), (ii) and determine a family such as (iv).

We require to prove that, if A, B, C, F, G, H are chosen to satisfy (iii) when $i = 1, 2, 3$, they will also satisfy (iii) when $i = 4$. Now the result of eliminating F, G, H from the equations obtained by putting $i = 1, 2, 3, 4$ in (iii) is

$$\begin{vmatrix} a_1 & f_1 & g_1 & h_1 \\ a_2 & f_2 & g_2 & h_2 \\ a_3 & f_3 & g_3 & h_3 \\ a_4 & f_4 & g_4 & h_4 \end{vmatrix} A + \begin{vmatrix} b_1 & f_1 & g_1 & h_1 \\ b_2 & f_2 & g_2 & h_2 \\ b_3 & f_3 & g_3 & h_3 \\ b_4 & f_4 & g_4 & h_4 \end{vmatrix} B + \begin{vmatrix} c_1 & f_1 & g_1 & h_1 \\ c_2 & f_2 & g_2 & h_2 \\ c_3 & f_3 & g_3 & h_3 \\ c_4 & f_4 & g_4 & h_4 \end{vmatrix} C = 0 \ldots \text{(v)},$$

and we must prove that this relation is true identically.

This follows from the fact that the coefficients of A, B, C in (v) are each of the second degree in k_1 and are zero for $k_1 = k_2, k_3, k_4$.

The equation of the tangential net can be put into its simplest form by supposing it to be formed by the polar conics of the class-cubics

$$\lambda^3 + \mu^3 + \nu^3 + 6m\lambda\mu\nu = 0;$$

when it will be found that $S=0$, $S'=0$, $S''=0$ are any three polar conics of

$$m(x^3+y^3+z^3)-3xyz=0.$$

The Cayleyan of the second cubic is the Hessian of the first.

If (i) is a line-pair, putting $i=1$ in (iii) we get the condition that the line-pair is conjugate with respect to $\Sigma=0$, and if $\Sigma=0$ is a point-pair, we get the condition that the point-pair is conjugate with respect to (i). Now there are six line-pairs belonging to the family (iv), forming a Steiner's complex of bitangents to the quartic $S'^2=SS''$.* We have then: *each pair of a Steiner's complex of bitangents of a plane quartic curve is self-conjugate with respect to every conic of a certain tangential net of conics.*

If a line is divided in involution by three conics of the family (iv), the double points of the involution are conjugate for each of the three conics, and are therefore a point-pair of the tangential net. Hence the line meets every conic of the family (iv) in an involution, if it meets three. In particular, either line of one of the six line-pairs of the family meets each conic of the family in an involution.

ON RATIONAL APPROXIMATIONS TO CYCLICAL CUBIC IRRATIONALITIES.

By *Prof. W. Burnside.*

THE equation

$$ax^3+bx^2+cx+d=0$$

is said to determine a cyclical cubic irrationality if a, b, c, d are rational numbers, and the discriminant of the equation is the square of a real rational number.

If x_1, x_2, x_3 are the roots of the equation, and if

$$\frac{x_1-x_3}{x_2-x_3}=z,$$

then
$$\frac{x_2-x_1}{x_3-x_1}=\frac{z-1}{z},\quad \frac{x_3-x_2}{x_1-x_2}=\frac{1}{1-z},$$

* See Hilton's *Plane Algebraic Curves*, ch. xix, § 3.

and

$$\frac{z^3-3z+1}{3(z^2-z)}=\tfrac{1}{3}\left(z+\frac{z-1}{z}+\frac{1}{1-z}\right)=\tfrac{1}{3}\left(\frac{x_1-x_3}{x_2-x_3}+\frac{x_2-x_1}{x_3-x_1}+\frac{x_3-x_2}{x_1-x_2}\right)$$
$$=k,$$

where, on the assumptions made, k is a rational number. This equation is unaltered by writing $\frac{z-1}{z}$ or $\frac{1}{1-z}$ for z, to that if z is one root, then $\frac{z-1}{z}$ and $\frac{1}{1-z}$ are the other two. Moreover, x_1, x_2, x_3 can clearly be expressed rationally in terms of z, the coefficients being rational numbers.

If $$z-k=y$$

then $$y^3-2(1-k+k^2)y+(1-2k)(1-k+k^2)=0.$$

The method of expansion by Lagrange's theorem gives

$$\frac{1-2k}{3}\sum_0^\infty \frac{3n\,!}{n\,!\,2n+1\,!}\left\{\frac{(1-2k)^2}{3^3(1-k+k^2)}\right\}^n$$

for one root of this equation, the series being convergent for all real, and therefore *à fortiori* for all rational, values of k.

It follows that every cyclical cubic irrationality is a rational quadratic function, with rational numerical coefficients, of

$$\sum_0^\infty \frac{3n\,!}{n\,!\,2n+1\,!}\left\{\frac{(1-2k)^2}{3^3(1-k+k^2)}\right\}^n$$

for some rational value of k.

This series affords a simple illustration of a point in the theory of infinite series. Assuming z real and replacing k and y by their values in terms of z,

$$\frac{(1-2k)^2}{(1-k+k^2)}=\frac{(z+1)^2(2z-1)^2(z-2)^2}{(z^2-z+1)^3},$$

while y is either $z-k$, $\frac{z-1}{z}-k$ or $\frac{1}{1-z}-k$, so that $3y/(1-2k)$ is either

$$3\,.\frac{z^2-z+1}{(z+1)(2-z)},\quad 3\,.\frac{z^2-z+1}{(1-2z)(2-z)}\ \text{or}\ 3\,.\frac{z^2+z+1}{(z+1)(2z-1)}.$$

Hence $$\sum_0^\infty \frac{3n\,!}{n\,!\,2n+1\,!}\left\{\frac{(z+1)^2(2z-1)^2(z-2)^2}{3^3(z^2-z+1)}\right\}^n$$

has one of the last three values.

When $z=1$ nearly, the series is nearly unity and hence, in the neighbourhood of -1, the series is equal to

$$\frac{3(z^2-z+1)}{(1-2z)(2-z)}.$$

When $z=\frac{1}{2}$ nearly, the series is nearly unity and hence, in the neighbourhood of $\frac{1}{2}$, the series is

$$\frac{3(z^2-z+1)}{(z+1)(2-z)}.$$

Similarly in the neighbourhood of 2, the series is

$$\frac{3(z^2-z+1)}{(z+1)(2z-1)}.$$

The first and second of these expressions are equal only when $z=0$, and the second and third are equal only when $z=1$. Finally then

$$\sum_0^\infty \frac{3n!}{n!\,2n+1!}\left\{\frac{(z+1)^2(2z-1)^2(z-2)^2}{3^3(z^2-z+1)}\right\}^n$$

from $z=-\infty$ to $z=0$ is $\dfrac{3(z^2-z+1)}{(1-2z)(2-z)}$,

from $z=0$ to $z=1$ is $\dfrac{3(z^2-z+1)}{(z+1)(2-z)}$,

from $z=1$ to $z=+\infty$ is $\dfrac{3(z^2-z+1)}{(z+1)(2z-1)}$.

It is to be noticed that while for all values of z other than 0, 1, and ∞ the convergence of the series is ultimately the same as that of a geometrical progression, the convergence of the series for the three particular values at which the form of its sum changes is ultimately that of a series whose n^{th} term is $n^{-\frac{3}{2}}$.

ON PELLIAN CHAINS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The Author's acknowledgments are due to Mr. H. J. Woodall, A R.C.Sc, for reading the Proof-sheets and for suggestions].

1. *Introduction.* The object of this Paper is to develop certain *Chain-relations* among the elements (τ', v'), (τ, v) of the Pellian Equations

$$\tau_r'^2 - Dv_r'^2 = -k, \ldots, (1), \qquad \tau_r'^2 - Dv_r'^2 = +k, \ldots, (2),$$

$$\tau_r'^2 - Dv_r'^2 = -1, \ldots, (3), \qquad \tau_r^2 - Dv_r^2 = +1, \ldots, (4),$$

each of which is known to have an infinite number of (integer) solutions (when one such exists), the elements (τ', v'), (τ, v) therein being distinguished by the subscripts $r = 0, 1, 2, 3$, &c.

The only values of k here considered are those which occur as the *middle term* of the partial quotients in the expansion o. $\sqrt{D}$ as a continued fraction. Only one such value (of $\pm k$) occurs for each value of D; so that one (and only one) of the equations (1), (2), (3) exists for each D, and that equation always gives directly the values of τ_1, v_1 in the fundamental "unit-form" $\tau_1^2 - Dv_1^2 = +1$ by the simple formulæ

$$\tau_1 = (\tau_1'^2 + Dv_1'^2) \div k, \quad v_1 = 2\tau_1 v_1 \div k,$$

a property not generally possessed by other values of k.

2. *Numerical Chains.* When a series of *composite* numbers

$$N_1 = L_1 . M_1, \; N_2 = L_2 . M_2, \; N_3 = L_3 . M_3, \; \&c., \ldots,$$

all *formed in the same way*, have their factors (L, M) so related that

$$M_{r-1} = L_r, \; M_r = L_{r-1}, \text{ for all integer values of } r.$$

the Series is styled a *Chain-Series*, and its members $(N_1, N_2, \&c.)$ are said to be *in chain*; the members N_r) are styled *Links* of the Chain, and the factors (L_r, M_r) are styled *Chain-Factors*.

[*Symbol* (:). In numerical Results it is convenient to place a colon (:) between the chain-factors (L_r, M_r) of each Link (N_r): this symbol is both a *multiplication-symbol* and a *separation-symbol*].

Chain-Properties. The salient properties of such Chains are

$$N_r/N_{r-1} = M_r/L_{r-1}, \quad N_r/N_{r+1} = L_r/M_{r+1},$$

$$\frac{N_2 N_4 N_6 \ldots N_{2r}}{N_1 N_3 N_4 \ldots N_{2r-1}} = \frac{M_{2r}}{L_1},$$

$$N_1 N_2 N_3 \ldots N_r = L_1 (L_2 L_3 L_4 \ldots L_r)^2 M_r$$
$$= L_1 (M_1 M_2 M_3 \ldots M_{r-1})^2 M_r.$$

3. *Auxiliary Formulæ.*

For Eq. (1), take $\tau_1' = \sqrt{k}.\tfrac{1}{2}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})$, $v_1' = \sqrt{k}.\tfrac{1}{2}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})/\sqrt{D}$.

These will be found to give—

$$\tau_r' = \sqrt{k}.\tfrac{1}{2}(y^{r-\frac{1}{2}} - y^{-r+\frac{1}{2}}), \quad v_r' = \sqrt{k}\ \tfrac{1}{2}(y^{r-\frac{1}{2}} + y^{-r+\frac{1}{2}})/\sqrt{D}.$$

For Eq. (2), take $\tau_1' = \sqrt{k}.\tfrac{1}{2}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})$, $v_1' = \sqrt{k}.\tfrac{1}{2}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})/\sqrt{D}$.

These will be found to give—

$$\tau_r' = \sqrt{k}.\tfrac{1}{2}(y^{r-\frac{1}{2}} + y^{-r+\frac{1}{2}}), \quad v_r' = \sqrt{k}.\tfrac{1}{2}(y^{r-\frac{1}{2}} - y^{-r+\frac{1}{2}})/\sqrt{D}.$$

For Eq. (3), take $\tau_1' = \tfrac{1}{2}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})$, $v_1' = \tfrac{1}{2}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})/\sqrt{D}$.

These will be found to give—

$$\tau_r' = \tfrac{1}{2}(y^{r-\frac{1}{2}} - y^{-r+\frac{1}{2}}), \quad v_r' = \tfrac{1}{2}(y^{r-\frac{1}{2}} + y^{-r+\frac{1}{2}})/\sqrt{D}.$$

For Eq. (4), take $\tau_1 = \tfrac{1}{2}(y + y^{-1})$, $v_1 = \tfrac{1}{2}(y - y^{-1})/\sqrt{D}$.

These will be found to give—

$$\tau_r = \tfrac{1}{2}(y^r + y^{-r}), \quad v_r = \tfrac{1}{2}(y^r - y^{-r})/\sqrt{D}.$$

These formulæ will be found to be consistent, and to lead to all the well-known relations between τ', v', τ, v: and will be found very helpful in proving new relations.

4. *Chain-Series* of $(t_r^2 \pm Du_s^2)$. From the fundamental forms

$$\tau'^2 - Dv'^2 = -k, \quad \tau^2 + Dv^2 = +k, \quad \tau^2 - Dv^2 = +1,$$

the following *Dimorphs* $N'_{r,s}$, $N_{r,s}$ are seen to exist—

i. $N'_{r,s} = \tau_r'^2 + Dv_s'^2 = \tau_s'^2 + Dv_r'^2 = N'_{s,r}$ [for $-k$, or k].

ii. $N_{r,s} = \tau_r^2 + Dv_s^2 = \tau_s^2 + Dv_r^2 = N_{s,r}$, *always*.

iii*a*. $N''_{r,s} = k\tau_r^2 + Dv_s'^2 = \tau_s'^2 + kDv_r^2 = N''_{s,r}$, [for $+k$].

iii*b*. $N''_{r,s} = k\tau_r^2 = Dv_s'^2 = kDv_r^2 - \tau_s'^2 = N''_{s,r}$, [for $-k$].

And three of them (and only three) co-exist for all values of D, viz. Nos i, ii, iii*a* for $+k$, and i, ii, iii*b* for $-k$. Also, as each of the quantities (N', N) is thus expressed in *two different* *ways* in the *same* 2^{ic} *forms*, it follows that each of them is *composite*. Their resolution, each into two factors (say $N = L, M$), will now be found by aid of the substitutions of Art. 3.

5 *a*. Case i. $N'_{r,s} = \tau_r'^2 + Dv_s'^2$

$$= \tfrac{1}{4}k.(y^{2r-1}+y^{-2r+1}) + \tfrac{1}{4}k.(y^{2s-1}+y^{-2s+1}),$$

therefore $\frac{1}{k}.N'_{r,s} = \tfrac{1}{2}(y^{r-s}+y^{-r+s}).\tfrac{1}{2}(y^{r+s-1}+y^{-r-s+1})$

$$= \tau_{r-s}.\tau_{r+s-1}.$$

Similarly $\frac{1}{k}.N'_{r',s} = \tau_{r'-s}.\tau_{r'+s-1}.$

Here $\tau_{r+s-1} = \tau_{r'-s}$, if $r+s-1 = r'-s$, or $r'-r = 2s-1$.

Similarly $\frac{1}{k}.N'_{r'',s} = \tau_{r''-s}.\tau_{r''+s-1}$,

and $\tau_{r'+s-1} = \tau_{r''-s}$, if $r'+s-1 = r''-s$, or $r''-r' = 2s-1$.

Hereby $\frac{1}{k}.N'_{r,s}$, $\frac{1}{k}.N'_{r',s}$, $\frac{1}{k}.N'_{r'',s}$, &c., will be *in chain*,

if $r'-r = r''-r' = r'''-r'' = \ldots = 2s-1$.

Ex. The most interesting Case is when $s = 1$, whereby

$$r'-r = r''-r' = r'''-r'' = \ldots\ldots = 1,$$

and $\frac{1}{k}.N'_{1,1}$, $\frac{1}{k}.N'_{2,1}$, $\frac{1}{k}.N'_{3,1}$ ......, are *in chain*.

The Table below shows the details of the Chains of $N'_{r,s}$, where $s = 1$ for $D = 2, 3, 7, 33$, which have $v_1' = 1$.

D	$\pm k$	v'_1	r	0	1	2	3	4	5
2	−1	1	τ_r', v_r'	.	1,1	7,5	41,29	239,169	1393,985
			τ_r, v_r	1,0	3,2	17,12	99,70	577,408	3363,2378
			$N'_{r,1}$	1;	1:3;	3:17;	17·97:	97:577;	577:3363:
3	−2	1	τ_r', v_r'	.	1,1	5·3	19,11	71,41	265,153
			τ_r, v_r	1,0	2,1	7,4	26,15	97,56	362,209
			$N'_{r,1}$	1;	1:2;	2:7;	7:26;	26:97;	97·362;
7	+2	1	τ_r', v_r'	.	3,1	45,17	717,271	11427,4319	
			τ_r, v_r	1,0	8,3	127,48	2024,765	32257,12193	
			$N'_{r,1}$	1;	1:8,	8:127;	127:2024;	2024·32257;	
33	+3	1	τ_r', v_r'	.	6,1	270,47	12414,2161		
			τ_r, v_r	1,0	23.4	1057,184	48599,8460		
			$N'_{r,1}$	1;	1:23;	23:1057;	1057:48599;		

5 *b*. Case ii. $N_{r,s} = \tau_r^2 + Dv_s^2$

$$= \tfrac{1}{4}(y^{2r}+y^{-2r}) + \tfrac{1}{4}(y^{2s}+y^{-2s})$$

$$= \tfrac{1}{2}(y^{r-s}+y^{-r+s}).\tfrac{1}{2}(y^{r+s}+y^{-r-s})$$

$$= \tau_{r-s}.\tau_{r+s}.$$

Similarly $N_{r',s} = \tau_{r'-s} . \tau_{r'+s}$.

Here $\tau_{r+s} = \tau_{r'-s}$, if $r+s = r'-s$, or $r'-r = 2s$.

Similarly $N_{r'',s} = \tau_{r''-s} . \tau_{r''+s}$,

and $\tau_{r'+s} = \tau_{r''-s}$, if $r'+s = r''-s$, or $r''-r' = 2s$.

Hereby $N_{r,s}$, $N_{r',s}$, $N_{r'',s}$, ..., will be *in chain*,

if $r'-r = r''-r' = r'''-r'' = ... = 2s$.

Ex. The most interesting Case is when $s = 1$, whereby

$$r'-r = r''-r' = r'''-r'' = ... = 2,$$

and $N_{1,1}$, $N_{3,1}$, $N_{5,1}$, $N_{7,1}$, ..., are *in chain*.

$N_{2,1}$, $N_{4,1}$, $N_{6,1}$, $N_{8,1}$, ..., are *in chain*.

The Table below shows the details of these two Chains of $N_{r,1}$, when $D = 2, 3$.

D	v_1	r	0	1	2	3	4	5	6
2	2	τ_r, v_r	1,0	3,2	17,12	99,70	577,408	3363,2378	19601, &c.
		$N_{r,1}$	.	1:17;	.	17:577;	.	577:19601;	.
		$N_{r,1}$	3	.	3:99;	.	99:3363;	.	3363:114243;
3	1	τ_r, v_r	1,0	2,1	7,4	26,15	97,56	362,209	1351,780
		$N_{r,1}$	.	1:7;	.	7:97	.	97:1351;	.
		$N_{r,1}$	2	.	2:26;	.	26:362;	.	362:5042;

5 c. CASE iii *a*.

$$N''_{r,s} = k . \tau_r^2 + Dv_s'^2$$

$$= \tfrac{1}{4}k(y^{2r} + y^{-2r}) + \tfrac{1}{4}k(y^{2s-1} + y^{-2s+1})$$

$$= \sqrt{k} . \tfrac{1}{2}(y^{r-s+\frac{1}{2}} + y^{-r+s-\frac{1}{2}}) . \sqrt{k} . \tfrac{1}{2}(y^{r+s-\frac{1}{2}} + y^{-r-s+\frac{1}{2}})$$

$$= \tau'_{r-s+1} . \tau'_{r+s}.$$

CASE iii *b*.

$$N''_{r,s} = k\tau_r^2 - Dv_s'^2$$

$$= \tfrac{1}{4}k(y^{2r} + y^{-2r}) - \tfrac{1}{4}k(y^{2s-1} + y^{-2s+1})$$

$$= \sqrt{k} . \tfrac{1}{2}(y^{r-s+\frac{1}{2}} - y^{-r+s-\frac{1}{2}}) . \sqrt{k} . \tfrac{1}{2}(y^{r+s-\frac{1}{2}} - y^{-r-s+\frac{1}{2}})$$

$$= \tau'_{r-s+1} . \tau'_{r+s}.$$

As in both Cases iii *a*, *b*, $N''_{r,s}$ has factors L, M of the same type, they may now be treated together. And, treating in the same way as in Cases i, ii, it is easy to see that it will result that

$N''_{r,s}$, $N''_{r',s}$, $N''_{r'',s}$, &c., ..., will be *in chain*.

if $r'-r = r''-r' = r'''-r'' =$ &c., ..., $= 2s-1$.

Ex. The most interesting Case is when $s=1$, whereby

$$r'-r=r''-r'=r'''-r''=\ldots\ldots 1,$$

and $N''_{1,1}$, $N''_{2,1}$, $N''_{3,1}$, ..., are *in chain.*

The Table below shows the details of the Chains of $N''_{r,1}$, Case iii$a$ when $D=7$, 33, and Case iii$b$ when $D=2$, 3; when $s=1$.

	D	k	υ_1'	r	0	1	2	3	4
Case iii*a*. $+k$	7	2	1	τ_r', υ_r'	.	3,1	45,17	717,271	11427,4319
				τ_r, υ_r	1,0	8,3	127,48	2024 765	32257,12192
				$N''_{r,1}$	3;	3:45;	45:717;	717:11427;	11427: &c.
	33	3	1	τ_r', υ_r'	.	6,1	270,47	12414,2161	570774, &c.
				τ_r, υ_r	1,0	23,4	1057,184	48599,8460	
				$N''_{r,1}$	.	6:270;	270;12414;	12414:570774;	
Case iii*b*. $-k$	2	1	1	τ_r', υ_r'	.	1,1	7,5	41,29	239,169
				τ_r, υ_r	1,0	3,2	17,12	99,70	577,408
				$N''_{r,1}$	1;	1:7;	7:41;	41 239;	239;1393;
	3	2	1	τ_r', υ_r'	.	1,1	5,3	19,11	71,41
				τ_r, υ_r	1,0	2,1	7,4	26,15	97,56
				$N''_{r,1}$	1;	1:5;	5:19;	19:71;	71:265;

6. *Aurifeuillian Chains.* The only Pellians yielding Aurifeuillian Chains are those with determinants $D=2$ and 3, and the Aurifeuillians connected with them are

$D=2$. *Bin-Aurifeuillians*, $N=x^4+4y^4=L.M$,

where $L=(x-y)^2+y^2$, $M=(x+y)^2+y^2$.

$D=3$. *Trin-Aurifeuillians*, $N=x^4-3x^2y^2+9y^4=L.M$,

where $L=x^2-3xy+3y^2$, $M=x^2+3xy+3y^2$.

7 *a*. *Bin-Aurifeuillian Pellians* (N'_r, N_r).

1°. $N_r'=\tau_r'^4+4\upsilon_r'^4=L_r'.M_r'$,

$L_r'=\tau_r'^2+2\upsilon_r'^2-2\tau_r'\upsilon_r'$, $M_r'=\tau_r'^2+2\upsilon_r'^2+2\tau_r'\upsilon_r'$

$=\tau_{2r-1}-\upsilon_{2r-1}$ $\qquad=\tau_{2r-1}+\upsilon_{2r-1}$

$=\upsilon'_{2r-1}$ $\qquad=\upsilon'_{2r}$,

by the usual formulæ connecting υ with τ, υ by aid of the unit-forms

$$\tau'^2-2\upsilon'^2=-1, \quad \tau^2-2\upsilon^2=+1.$$

Similarly $\qquad N_{r-1}=\tau'^4_{r-1}+4\upsilon'^4_{r-1}=L'_{r-1}.M'_{r-1}$,

$L'_{r-1}=\tau_{2r-1}-\upsilon_{2r-1}$, $\qquad M'_{r-1}=\tau_{2r-1}+\upsilon'_{2r-1}$

$=\upsilon'_{2r+1}$ $\qquad=\upsilon'_{2r-2}$.

2°. Also $N_r = \tau_r^4 + 4v_r^4 = L_r \cdot M_r$,

$$L_r = \tau_r^2 + 2v_r^2 - 2\tau_r v_r, \qquad M_r' = \tau_r^2 + 2v_r^2 + 2\tau_r v_r$$
$$= \tau_{2r} - v_{2r} \qquad\qquad = \tau_{2r} + v_{2r}$$
$$= v'_{2r} \qquad\qquad = v'_{2r+1}$$

Similarly $N_{r+1} = \tau_{r+1}^4 + 4v_{r+1}^4 = L_{r+1} \cdot M_{r+1}$,

and $L_{r+1} = v'_{2r+2}$, $M_{r+1} = v'_{2r+3}$.

Hence, comparing the two series of N_r', N_r,

$$M_r' = L_r, \quad M_r = L'_{r+1}, \quad M'_{r+1} = L_{r+1}, \quad M_{r+1} = L'_{r+2}, \text{ \&c.}$$

This shows that, if the two Series (N_r', N_r) be combined into one Series, taking a member from each series alternately, the combined Series form a Chain, wherein the N_r' are the Links in the *odd* places, and the N_r are the Links in the *even* places.

Ex. The Table below shows this clearly—

r	1	2	3	4	5
τ_r', v_r'	1, 1	7, 5	41, 29	239, 169	1393, 985
L_r', M_r'	1:5;	29:169;	985:5741	33461:195025;	1136689: &c.
τ_r, v_r	3, 2	17, 12	99, 70	577, 408	3363, 2378
L_r, M_r	5:29;	169:985;	5741:33461;	195025:1136689;	&c. &c.

Note that by squaring the unit-forms $\tau_r'^2 - \tau v_r'^2 = -1$, $\tau_r^2 - 2v_r^2 = +1$,

$$\tau_r'^4 + 4v_r'^4 = (2\tau_r'v_r')^2 + 1 = \tau_{2r-1}^2 + 1,$$
$$\tau_r^4 + 4v_r^4 = (2\tau_r v_r)^2 + 1 = \tau_{2r}^2 + 1,$$

thus giving the *two* $(a^2 + b^2)$ forms of N_r', N_r at once.

7 *b*. *Trin-Aurifeuillian Pellians* (N_r', N_r).

1°. $$N_r' = \tau_r'^4 - 3\tau_r'^2 v_r'^2 + 9v_r'^4 \qquad = L_r' \cdot M_r',$$

$$L_r' = \tau_r'^2 + 3v_r'^2 - 3\tau_r' v_r', \quad M_r' = \tau_r'^2 + 3v_r'^2 + 3\tau_r' v_r'$$
$$= 2\tau_{2r-1} - 3v_{2r-1} \qquad\qquad = 2\tau_{2r+1} + 3v_{2r-1}$$
$$= \tau_{2r-2} \qquad\qquad = \tau_{2r}$$

by the usual formulæ connecting τ_ρ with $\tau_{\rho-1}$, $v_{\rho-1}$, $\tau_{\rho+1}$, $v_{\rho+1}$ by aid of the unit-forms $\tau'^2 - 3\rho'^2 = -2$, $\tau_r^2 - 3\rho^2 = +1$.

Similarly $N'_{r+1} = \tau'^4_{r+1} - 3\tau'^2_{r+1} v'^2_{r+1} + 9v'^4_{r+1} = L'_{r+1} \cdot M'_{r+1}$,

$$L'_{r+1} = \tau_{2r}, \quad M'_{r+1} = \tau_{2r+2}.$$

Thus $M_r' = L'_{r+1}$, $M'_{r+1} = L'_{r+2}$, and so on.

and N_1', N_2', N_3', ..., are a *Chain-Series*.

2°. Again, $N_r = \tau_r^4 - 3\tau_r^2 v_r^2 + 9v_r^4 = L_r \,.\, M_r,$

$$L_r = \tau_r^2 + 3v_r^2 - 3\tau_r v_r, \quad M_r = \tau_r^2 + 3v_r^2 + 3\tau_r v_r$$

$$= \tau_{2r} - 3 \,.\, \tfrac{1}{2}v_{2r} \qquad = \tau_2 + 3 \,.\, \tfrac{1}{2}v_{2r}$$

$$= \tfrac{1}{2}\tau_{2r-1} \qquad = \tfrac{1}{2}\tau_{2r+1}.$$

Similarly $\qquad N_{r+1} = L_{r+1} \,.\, M_{r+1},$

$$L_{r+1} = \tfrac{1}{2}\tau_{2r+1}, \quad M_{r+1} = \tfrac{1}{2}\tau_{2r+3}.$$

Thus $\qquad M_r = L_{r+1}, \quad M_{r+1} = L_{r+2}$, and so on.

And $\qquad N_1, N_2, N_3, \ldots\ldots$, are a *Chain-Series.*

Ex. The Table below shows this clearly.

r'	0	1	2	3	4	5	6
τ_r', v_r'	.	1,1	5,3	19,11	71,41	2655,153	989,571
$L_r' M_r'$	.	1:7;	7:97;	97:7.193;	7.193:31.607;	31.607:7.37441;	7.37441:&c
τ_r, v_r	1,0	2,1	7,4	36,15	97,56	362,209	1351,780
L_r, M_r	1;	1:13;	13:181;	181:2521;	2521:13.37.73;	13.37.73:489061;	489061:&c

Note that, by squaring the unit-forms $\tau_r'^2 - 3v_r'^2 = -2$, $\tau_r^2 - 3v_r^2 = +1$,

$$(\tau_r'^4 - 3\tau_r'^2 v_r'^2 + 9v_r'^4) - 3\tau_r'^2 v_r'^2 = +4,$$

$$(\tau_r^4 - 3\tau_r^2 v_r^2 + 9v_r^4) - 3\tau_r^2 v_r^2 = +1,$$

whence $\qquad N_r' = 2^2 + 3(\tau_r' v_r')^2, \qquad N_r = 1^2 + 3(\tau_r v_r)^2$

$$= 2^2 + 3(\tfrac{1}{2}\tau_{2r-1})^2 \qquad = 1^2 + 3(\tfrac{1}{2}\tau_{2r})^2,$$

which give the simplest $(A^2 + 3B^2)$ forms of N_r', N_r.

Note further that the Trin-Aurifeuillian N_r is also expressible as a *Sextan*; for (since $\tau_r^2 - 3v_r^2 = +1$),

$$N_r = \tau_r^4 - 3\tau_r^2 v_r^2 + 9v_r^4 = (\tau_r^2 - 3v_r^2)^2 + 3\tau_r^2 v_r^2$$

$$= 1^4 - \tau_r^2 + \tau_r^4 = (\tau_r^6 + 1^6) / (\tau_r^2 + 1^2), \text{ a } \textit{Sextan}.$$

Contrast now the Bin-Aurifeuillian Chain, occurring when $D = 2$, with the Trin-Aurifeuillian Chains occurring when $D = 3$. In the former case the two Series of N_r', N_r have to be combined to form a single Bin-Aurifeuillian Chain; whilst in the latter case the Series of N_r', N_r each yield one Trin-Aurifeuillian Chain.

ON SYMMETRICAL PLANE ALGEBRAIC CURVES.

By *Prof. Harold Hilton.*

§ 1. It is well known that a plane cubic curve, not cuspidal or crunodal, can be projected so as to have the symmetry of the equilateral triangle; and the same is true of a quartic with three real cusps. This suggests the general problem of symmetrical algebraic curves to which this paper is devoted.*

If we write

$$\left.\begin{aligned} u_{nk} &\equiv r^k\left(a_{k0}+a_{k2}r^2+\ldots+a_{k\,n-k}r^{n-k}\right)\cos k\theta \\ &\quad + r^k\left(b_{k0}+b_{k2}r^2+\ldots+b_{k\,n-k}r^{n-k}\right)\sin k\theta \\ \text{or}\quad u_{nk} &\equiv r^k\left(a_{k0}+a_{k2}r^2+\ldots+a_{k\,n-k-1}r^{n-k-1}\right)\cos k\theta \\ &\quad + r^k\left(b_{k0}+b_{k2}r^2+\ldots+b_{k\,n-k-1}r^{n-k-1}\right)\sin k\theta \end{aligned}\right\}\ldots\text{(i)},$$

according as $n-k$ is even or odd, where $n > k$, it is readily shown that the polar equation of any algebraic curve of degree n (an "n-ic") can be put in the form

$$u_{nn}+u_{n\,n-1}+u_{n\,n-2}+\ldots+u_{n0}=0\ \ldots\ldots\ldots\ldots\text{(ii)}$$

in one and only one way, the axes of reference being given.

Suppose that, when we replace θ by $\theta-\alpha$ in (ii), this equation is unaltered, *i.e.* the curve is brought to self-coincidence by a rotation through α about the pole. Then for all values of k and t

$$a_{kt}\cos k\alpha - b_{kt}\sin k\alpha = \lambda a_{kt},\quad a_{kt}\sin k\alpha + b_{kt}\cos k\alpha = \lambda b_{kt}\ldots\text{(iii)},$$

where λ is some real constant.

Now (iii) gives either $a_{kt}=b_{kt}=0$ or else

$$\lambda^2-2\lambda\cos k\alpha+1=0.$$

Since λ is real, the latter alternative is only possible if $\lambda=1$ and $\cos k\alpha=1$ or $\lambda=-1$ and $\cos k\alpha=-1$.

* On the subject of symmetric curves see Carmichael, *Annals of Math.*, II., vol. ix (1908), p. 53, and II., vol. x. (1909), p. 81; Ciani, *Annali Mat. Pura ed Applicata*, III., vol. v. (1901), p. 33.

It follows that, if an n-ic is brought to self-coincidence by a rotation about the pole through an angle $2\pi/k$, where k is a positive integer, its equation is either of the form

$$u_{n0} + u_{nk} + u_{n2k} + \ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

or is of the form $u_{np} + u_{n3p} + u_{n5p} + \ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$

where $p = \frac{1}{2}k$, and k is even.

§ 2. First consider the curve $u_{np} = 0$, which we take in the form

$$(a + br^2 + cr^4 + \ldots + kr^{2l}) \cos p\theta$$
$$+ (A + Br^2 + Cr^4 + \ldots + Kr^{2l}) \sin p\theta = 0 \ldots\text{(i)}.$$

This is the simplest type of curve with symmetry about the pole O. It has "$2p$-al symmetry" about O, *i.e.* it is brought to self-coincidence by a rotation about O through an angle $2\pi/2p$.

If n, m, δ, κ, τ, ι, D are the degree, class, number of nodes, of cusps, of bitangents, of inflexions, and the deficiency of (i), we have in general

$$n = 2l + p, \quad m = 2l(2p+1), \quad \delta = 2l(l-1) + \tfrac{1}{2}p(p-1),$$

$$\kappa = 0, \quad \iota = 3p(4l-1), \quad \tau = 4p(2pl^2 + 2l^2 - 5l + 1) + 2l(l-1),$$

$$D = 2pl - p - l + 1.$$

The only multiple points are the circular points at infinity and a p-ple point (multiple point of order p) at O. The p tangents at O are parallel to the sides of a regular p-sided polygon and each tangent is inflexional. Through either circular point ω pass l linear branches, all touching $O\omega$ at ω. There are p real asymptotes passing through O and parallel to the sides of a regular p-sided polygon.

A general idea of the shape of the curve can be obtained by drawing $S + \epsilon S' = 0$, where ϵ is a small constant and

$$S \equiv (a + br^2 + cr^4 + \ldots) \cos p\theta,$$

$$S' \equiv (A + Br^2 + Cr^4 + \ldots) \sin p\theta.$$

The curve lies close to $S = 0$, crossing it at its intersections with $S' = 0$; and both $S = 0$ and $S' = 0$ consist of circles and straight lines through the pole. The changes of shape as ϵ increases up to infinity will be obvious. The curve consists of p circuits, each passing through O and having O as a centre of symmetry.

The inverse of the curve (i) with respect to O is a curve of the same sort; and so is the locus of a point whose polar conic with respect to (i) is a rectangular hyperbola, namely,

$$\{1(p+1)b+2(p+2)cr^2+3(p+3)dr^4+\ldots\}\cos p\theta$$
$$+\{1(p+1)B+2(p+2)Cr^2+3(p+3)Dr^4+\ldots\}\sin p\theta=0.$$

A curve with k-al symmetry cannot have a degree less than k, unless it is of the type (i).

§ 3. Consider now an n-ic with n-al symmetry about O. If n is even, the curve may be of the type discussed in § 2. If it is not of this type, § 1, (iv) and (v), show that its equation is $u_{n0}+u_{nn}=0$. On turning the curve through a suitable angle about O the equation becomes

$$r^n\cos n\theta=a+br^2+\ldots+jr^{n-3}+kr^{n-1}\;\ldots\ldots\ldots\ldots(\text{i})$$

if n is odd, and

$$r^n\cos n\theta=a+br^2+\ldots+jr^{n-2}+kr^n\;\ldots\ldots\ldots\ldots(\text{ii})$$

if n is even. We may suppose $k\geq 0$.

Since $\theta=0$ is an axis of symmetry, the curve has the symmetry of the regular n-sided polygon.

If n is odd, the asymptotes are real, and form a regular n-sided polygon, each having in general three-point contact at infinity.

If n is even and $k<1$, the asymptotes are real, pass through the pole, and are parallel to the sides of a regular n-sided polygon.

If n is even and $k>1$, the curve is closed.

If n is even and $k=1$, the curve has $\frac{1}{2}n$ biflecnodes at infinity. The asymptotes are real and form a regular n-sided polygon if j is negative.

The locus of a point whose polar conic with respect to (i) or (ii) is a rectangular hyperbola is the concentric circles

$$b+2^2cr^2+3^2dr^4+\ldots=0.$$

If $n=3$, the curve is

$$r^3\cos 3\theta=a+br^2\;\ldots\ldots\ldots\ldots\ldots\ldots(\text{iii}).$$

The asymptotes form an equilateral triangle of altitude b; and the product of the distances of any point on the curve from these asymptotes is $(27a+4b^3)/108$. Hence the curve is identical with

$$(x+y+z)^3+6kxyz=0,$$

referred to an equilateral triangle of reference of altitude b, if

$$k = -18b^3/(27a + 4b^3) \quad\text{.................(iv).}$$

Diagrams of the curve for various values of k are given in Hilton's *Plane Algebraic Curves*, pp. 230–232 (Clarendon Press, 1920).

If $n = 4$, the curve is

$$r^4 \cos 4\theta = a + br^2 + cr^4 \quad\text{..................(v),}$$

where we may suppose $c \geq 0$.

This may be put in the form

$$(\alpha x^2 + \beta y^2 - 1)(\beta x^2 + \alpha y^2 - 1) = \epsilon \quad\text{..........(vi),}$$

where $\quad \alpha + \beta = -4(c+1)/b, \quad \alpha - \beta = 4(2c+2)^{\frac{1}{2}}/b,$

$$\epsilon = 1 - 4a(c+1)/b^2.$$

The shape of the curve may be readily found by considering α, β given and ϵ originally small. The curve then originally approximates to the two conics given by $\epsilon = 0$, and varies its shape continually as ϵ^2 increases. For instance, suppose α and β are positive. If

$$0 > \epsilon > -\alpha\beta(\alpha - \beta)^2/(\alpha^2 + \beta^2)^2,$$

the curve consists of four ovals, each with a bitangent having real points of contact. If

$$-\alpha\beta(\alpha - \beta)^2/(\alpha^2 + \beta^2)^2 > \epsilon > -(\alpha - \beta)^2/4\alpha\beta,$$

the curve consists of four convex ovals. The curve is not real if

$$-(\alpha - \beta)^2/4\alpha\beta > \epsilon.$$

If $1 > \epsilon > 0$, the curve consists of two ovals, one inside the other. If $\epsilon > 1$, there is only one oval. The outer oval (or the only oval) has eight real inflexions if and only if

$$4(\alpha - \beta)^4/(\alpha^2 - 6\alpha\beta + \beta^2)^2 > \epsilon > 0.$$

The twenty-four inflexions lie by eights on three circles with centre at the pole. If r_1, r_2, r_3 are their radii,

$$r_1^2(r_2^2 - r_3^2)^2 + r_2^2(r_3^2 - r_1^2)^2 + r_3^2(r_1^2 - r_2^2)^2 = 0.$$

One at most of the circles is real.

§ 4. Consider now an n-ic with $(n-1)$-al symmetry about the pole O. If n is odd, the curve is either of the type discussed in § 2 or is a cubic with O as centre.

If n is even, it follows from § 1 that its equation can be put in the form

$$r^{n-1}\cos(n-1)\,\theta = a + br^2 + \ldots + jr^{n-2} + kr^n \ldots\ldots \text{(i)},$$

where k is positive.

The curve (i) has the symmetry of the regular $(n-1)$-sided polygon.

It has $\frac{1}{2}n$-point contact with the line at infinity at each circular point, and there are $(n-1)^2$ real finite foci in general.

The locus of a point whose polar conic with respect to (i) is a rectangular hyperbola is the same family of concentric circles as in § 3.

If $n = 4$, the curve (i) is

$$r^3\cos 3\theta = a + br^2 + cr^4 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

The shape of this curve may be found by first supposing c small. The curve then approximates to the cubic of § 2 (iii) in the finite part of the plane.

If c and $(27a + 4b^3)$ have the same sign, the quartic consists of three large convex ovals, with an additional oval surrounding the pole if $k < -\frac{2}{3}$, *i.e.* if a and $(27a+4b^3)$ have opposite signs.

If c and $(27a + 4b^3)$ have opposite signs, the quartic consists of an oval with six inflexions surrounding the pole, with another oval inside it if a and $(27a + 4b^3)$ have opposite signs.

The changes of shape which occur as c^2 increases will now be readily recognized.

§ 5. An n-ic cannot in general be projected so as to have symmetry, if $n > 3$. But if the n-ic satisfies certain geometrical conditions, the projection may be possible. We shall now give a few illustrations of such conditions sufficient to ensure that a curve may be projected so as to have the symmetry of the equilateral triangle.

If an n-ic is unicursal and has three real tangents of n-point contact, its equation can be put in the form

$$x^{1/n} + y^{1/n} + z^{1/n} = 0,$$

and the n-ic can therefore be projected into symmetrical shape.*

Again, if an n-ic has $n + 1$ tangents of n-point contact, n of which are concurrent, while n is odd, the equation of the n-ic can be put in the form

$$x^n + y^n + z^n = 0.\dagger$$

* *Messenger of Mathematics*, vol. xlix. (1920), p. 132.
† *Messenger of Mathematics*, vol. l. (1920), p. 39.

Suppose that ABC is a triangle, that P, Q, R are points on BC, CA, AB such that AP, BQ, CR are concurrent at O, while (BC, PP'), (CA, QQ'), (AB, RR') are harmonic ranges. Then

(1) The equation of an n-ic ($n > 4$), having n-point contact with BC, CA, AB at P', Q', R' and having a multiple point of order $n-2$ at O [an "$(n-2)$-ple point"], can be put in the form

$$\{yz(y-z)(-2x+y+z)^n + zx(z-x)(x-2y+z)^n$$
$$+ xy(x-y)(x+y-2z)^n\} \div (y-z)(z-x)(x-y) = 0 \ldots \text{(i)}.$$

(2) The equation of an n-ic (n even), having n-point contact with BC, CA, AB at P, Q, R and having an $(n-2)$-ple point at O, can be put in the form

$$\{yz(y-z)^{n+1} + zx(z-x)^{n+1} + xy(x-y)^{n+1}\}$$
$$\div (y-z)(z-x)(x-y) = 0 \ldots \text{(ii)},$$

so that both curves can be projected into symmetrical shape.

The equation (ii) may be obtained as follows: A curve having an $(n-2)$-ple point at the origin and having the line at infinity for an n-point tangent at $x = y$ has an equation of the type

$$(x-y)^n + a_0 x^{n-1} + a_1 x^{n-2} y + \ldots$$
$$+ a_{n-1} y^{n-1} + b_0 x^{n-2} + b_1 x^{n-3} y + \ldots + b_{n-2} y^{n-2} = 0.$$

If, moreover, we make the curve have n-point contact with $x = -1$ at $(-1, 0)$ and with $y = -1$ at $(0, -1)$, we obtain $2n$ equations in $a_0, a_1, \ldots, b_0, b_1, \ldots$. These prove to be equivalent to $2n-1$ independent linear equations, which are readily solved and give for the equation of the curve

$$(x-y)^{n+1}(x+1)(y+1) + x^{n+1}(x+1) - y^{n+1}(y+1) = 0.$$

Transferring the origin to $(-1, -1)$, and making the equation homogeneous, we obtain (ii). Similarly we obtain (i).

As another illustration we have: If the sides of a triangle have each n-point contact with a unicursal $2n$-ic at two real points (n odd), and the cross ratios which these points form with the vertices of the triangle are the same for each side, the equation of the curve can be put in the form

$$x^{2/n} + y^{2/n} + z^{2/n} + k(y^{1/n}z^{1/n} + z^{1/n}x^{1/n} + x^{1/n}y^{1/n}) = 0.*$$

* This follows readily from *Proc. Lond. Math. Soc.*, II, vol. xxi. (1921), p. 4.

§ 6. The problem of finding symmetrical algebraic plane curves suggests that of finding symmetrical algebraic surfaces. We conclude by briefly indicating how to find the algebraic surfaces with the symmetry of any one of the 32 crystallographic classes.

First take the "regular system" in which the symmetry-axes are those of the regular tetrahedron or cube. Choose rectangular Cartesian axes of reference so that the trigonal symmetry-axes are the lines $x = \pm y = \pm z$. Suppose that $px^\alpha y^\beta z^\gamma$, $qx^\beta y^\alpha z^\gamma$, $ax^\delta y^\epsilon z^\epsilon$, $bx^\lambda y^\lambda z^\lambda$ are terms in the equation of the surface. Then this equation must evidently involve the terms

$$p\,(x^\alpha y^\beta z^\gamma + x^\gamma y^\alpha z^\beta + x^\beta y^\gamma z^\alpha) + q\,(x^\beta y^\alpha z^\gamma + x^\gamma y^\beta z^\alpha + x^\alpha y^\gamma z^\beta),$$

$$a\,(x^\delta y^\epsilon z^\epsilon + x^\epsilon y^\delta z^\epsilon + x^\epsilon y^\epsilon z^\delta), \quad bx^\lambda y^\lambda z^\lambda.$$

Here p, q, a, b are numbers and α, β, γ, δ, ϵ are zero or positive integers such that $\delta \neq \epsilon$ and no two of α, β, γ are equal.

If the surface has the axes of reference as diagonal axes, so that the symmetry of the surface is that given by the symmetry-axes of the regular tetrahedron, we must have

p and q zero unless α, β, γ are all odd or all even; a zero unless δ, ϵ are both odd or both even.

If, in addition, the surface has a centre of symmetry, we must have

p and q zero unless α, β, γ are all even; a zero unless δ and ϵ are both even, b zero unless λ is even.

If there is no centre of symmetry, but $x = y$, etc., are symmetry-planes, so that the surface has the symmetry of the regular tetrahedron, we must have $p = q$, and both zero unless α, β, γ are all odd or all even; a zero unless δ and ϵ are both odd or both even.

If the surface has the axes of reference as tetragonal axes, so that the symmetry of the surface is that given by the symmetry-axes of the cube, we must have

$a = 0$, unless δ and ϵ are both even; $b = 0$, unless λ is even; and also (i) p and q zero, unless $p = q$ and α, β, γ are all even, (ii) p and q zero, unless $p = -q$ and α, β, γ are all odd.

If the surface has also a centre of symmetry, so that it has the symmetry of the cube, only the first alternative is admissible.

To obtain algebraic surfaces with the symmetry of classes in the hexagonal, tetragonal, ..., crystallographic systems, we note that any algebraic surface of degree n has an equation in cylindrical coordinates of the form

$$f_n + f_{n-1}z + f_{n-2}z^2 + \ldots + f_0 z^n = 0 \text{ ....(i)},$$

where, with the notation of § 1,

$$f_t \equiv u_{tt} + u_{t\,t-1} + u_{t\,t-2} + \ldots + u_{t0}.$$

If the surface is brought to self-coincidence by rotation through $2\pi/k$ about the axis of z, where k is a positive integer, f_t must take the form

$$u_{t0} + u_{tk} + u_{t\,2k} + \ldots \text{ or } u_{tp} + u_{t\,3p} + u_{t\,5p} + \ldots \quad (p = \tfrac{1}{2}k),$$

for each value of t. The surface then has the axis of z as a k-al symmetry-axis.

If the surface has in addition a centre of symmetry, the terms of odd degree in r and z must vanish.

If the surface has a symmetry-plane perpendicular to the axis of z, $f_{n-t} \equiv 0$, when t is odd, etc.

If we apply these considerations to the cubic surface, other than the cubic surface of revolution, we find that a cubic surface can have the symmetry of any one of the classes which in the notation of Hilton's *Finite Groups*, pp. 113–115 (Clarendon Press, 1908) are denoted by

$$C_1, c_1, C_2, c_2, \Gamma_2, D, \delta_2, c_4, d_4, C_4, D_4, \delta_3, \Delta_3, d_6, \theta,$$

or in Hilton's *Mathematical Crystallography* (Clarendon Press, 1903) are denoted by

$$C_1, C_i, C_2, C_s, C_{2h}, Q, C_{2v}, C_4', D_{2d}, C_3, D_3, C_{3v}, D_{3d}, D_{3h}, T_d.$$

For instance, the cubic surface with the symmetry of the regular tetrahedron is

$$xyz + a(x^2 + y^2 + z^2) = 4(a^3 + c^3).$$

The lines on the surface are the lines at infinity in $xyz = 0$, and also the lines

$$z = \pm 2a, \quad (x \pm y)^2 = 4c^3/a;$$

$$z = \pm 2(a^3 + c^3)^{\frac{1}{2}}/a^{\frac{1}{2}}, \quad a^{3/2}(x^2 + y^2) \pm 2(a^3 + c^3)^{\frac{1}{2}} xy = 0, \text{ etc.}$$

They are all real if c/a is positive; otherwise only the lines at infinity are real.

A SUGGESTION FOR A NEW SYMBOLIC TREATMENT OF PROBABILITY.

(Being a Note on Mr. Keynes' *Treatise on Probability.*)

By *Ian Macdonald Horobin.*

MR. KEYNES bases his theory of probability* on the relation a/h where a/h is not necessarily a number†. He works out an algebra for this relation, of which the fundamental result is an addition theorem:

$$a+b/h=a/h+b/h-ab/h.$$

Since he has only defined addition for the single case

$$ab/h+a\bar{b}/h=a/h,$$

and since, even in the form

$$a/h+\bar{a}b/h,$$

the right-hand side is not in this form, it cannot be said that his algebra for non-numerical relations rests on very secure foundations. It seems clear that *something* more must be known about the probability relation before we can construct a satisfactory calculus. A more elaborate investigation of the serial relations of non-numerical probabilities is required. Before we can construct an algebra based on an addition theorem, we must know more of the relation of probable inference to certain inference, for clearly any ordering of probabilities must eventually depend on their relation to the certainty of ordinary inference, which has so hastily been equated to the ratio 1/1 in the ordinary theory. It would seem to the present writer that if the *direct* judgment of *any* probability is a judgment of a ratio then all probabilities are ratios. The point at issue seems to be whether, when we assert some probability to be (say) 1/2, we are not really asserting elliptically that a *certain* inference can be made between an implication of probability and a particular way

* J. M. Keynes, *A Treatise on Probability* (London, 1921).

† If a and b are two propositions *and if a relation of probability holds between them*, Mr Keynes denotes this relation by a/b. He uses h to denote the "general evidence", so that the *a priori* probability of a is a/h and the full statement of the relation between a and b is a/bh. From this material he constructs his logical calculus; explicitly rejecting any assumption about the probability relation, and leading up to numerical probabilities as a special case. The method and aim are clearly desirable; whether he is successful or not is more doubtful.

of stating the evidence.* If this is so it would appear that in order to clear up the question, the first step will have to be to discover if there is any normal or general way of stating the evidence. After this it may be possible to find out whether or not there is *always* theoretically possible the certain inference which takes places when we can find a numerical probability. If I understand Mr. Wittgenstein aright, he would appear to imply that this statement of the evidence can be made in the form of an infinite number of inferences from independent propositions. If the probability relation were ultimately dependent on the numerical relations of two infinite collections, the solution of the problem "are two probability relations always greater, equal, or less than one another?" would appear to be connected with the truth or falsehood of Zermelo's axiom, about which I think nothing is yet known.†

However this may be, it is possible to obtain all Mr. Keynes' results rigidly by a definition which, in the first place, makes the probability relation a ratio. In itself, of course, this does nothing to help on the solution of the major problem, but apart from its intrinsic interest, it is, I think, relevant, because, as a result of what has been said above, I am inclined to believe that the probability relation is not an indefinable relation directly holding between propositions, but

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* To take a very simple instance: I believe there is a fundamental difference between the propositions, "This probability is 1/2" and "This is as likely as not". This would appear to have been overlooked by Mr. Wittgenstein in his *Tractatus Logico-Philosophicus*. If the first were not elliptical all probabilities would be numerical.

† I think the sort of way in which this will happen may be seen from the following illustration.

Suppose there to be a symmetrical die with an infinite number of faces. This is of course impossible in a three-dimensioned space, but the illustration can be easily adapted. Suppose the number of faces unknown. Let some of these faces be

$$\ldots a, b, c, \ldots$$

$$\ldots \alpha, \beta, \gamma, \ldots$$

(both infinite selections). The faces must not, of course, be numbered, otherwise their number will be aleph$_0$.

Suppose we wish to know whether the probability that the face turned up will bear a Greek or Roman letter is the greater. This clearly depends on whether the cardinal number of the class of Greek letters is greater or less than that of the class of Roman letters. But unless Zermelo's axiom is true, we do not know whether these two cardinal numbers are necessarily greater or less than each other, and therefore the probabilities are not necessarily comparable. But I do not know whether this is correct.

A similar problem is suggested by a passage in Mr. Bertrand Russell's little book on *Mathematical Philosophy* (1919) at page 126. Would his millionaire, if he went to the wardrobe in which he kept his purchases, be more likely to pick up a boot or a sock?

a definable one between the sets of values of propositional functions. Mr. Keynes' convenient "h" covers a multitude of logical omissions.

If "a" is any class, let (a) denote the number of its members. We proceed to develop the theory of this ( ) function.

Within it the Boole-Schroder logic of classes applies. The ( ) itself obeys the ordinary laws of arithmetic, being an integer.

The probability relation is defined thus:

$$a/b = \frac{(ab)}{(b)} \text{ Def.}$$

By the definition of number $(a)+(b)=(a+b)$, if a and b have no members in common, *i.e.* if $ab=0$, since there is a one-one relation between their terms

$$(0)=0\text{; hence } (a)\neq 0 \text{ if } a\neq 0,\ (1)=1,$$

for convenience of notation. It would correspond to Mr. Keynes' notation to make $(1)=h$ and to define $a/bh \equiv \frac{(abh)}{(bh)}$, but this introduces needless complications, since our only immediate object is to obtain a formally equivalent algebra, every step of which (unlike Mr. Keynes') has a definite meaning and is not merely a string of letters leading to an interpretable conclusion

$$(1) \quad (a)+(b)-(ab) \equiv (ab+a\bar{b})+(ba+b\bar{a})-(ab)$$

$$\equiv (ab)+(a\bar{b})+(b\bar{a}) \equiv (ab+a\bar{b}+\bar{a}b)$$

$$\equiv (a+b)\text{, the addition theorem.}$$

(2) Every ratio $(a)/(b)$ can be expressed in terms of probability relations:

[If a implies b, it is already a probability relation, since

$$a\bar{b}=0;\ \therefore\ a=ab;\ \therefore\ \frac{(a)}{(b)}=\frac{(ab)}{(b)}\equiv a/b].$$

For
$$\frac{(a)}{(b)} \equiv \frac{(ab)}{(b)} + \frac{\frac{1}{(b)}}{(a+b)} - 1,$$

and, since b implies $a+b$, therefore $\frac{(b)}{(a+b)}$ is a probability relation.

It is interesting to note that this is a type of probability relation never used by Mr. Keynes.

(3) The general probability relation is $\frac{[f(a, b, c, \ldots)]}{[\phi(a, b, c, \ldots)]}$, where f implies ϕ; or, otherwise, $\frac{[f.\phi]}{[\phi]}$.

(4) The general premiss of a process of probable inference is

$$\frac{(f.\phi)}{(\phi)} = \lambda \text{ or } (f.\phi) = \lambda(\phi).$$

(5) By a well-known theorem (taking three primitive classes for simplicity):

$$f(a, b, c) \equiv f_1(b, c).a + f_2(b, c).\bar{a},$$

where f_1, f_2 are new functions.

Similarly we can reduce f_1, f_2 till we arrive finally at

$$\mu_1 c + \mu_2 \bar{c},$$

and the original general function can always be written

$$\equiv \alpha.abc + \Sigma\beta.\bar{a}bc + \Sigma\epsilon.\bar{a}\bar{b}c + \theta.\bar{a}\bar{b}\bar{c},$$

where, as is easily seen (cf. Boole),

$$\alpha = f(1, 1, 1),\ \beta = f(0, 1, 1), \text{ etc., and } \alpha = 0 \text{ or } 1.$$

$$abc,\ \alpha = (1, 1, 1);\quad a\bar{b}c,\ (1, 0, 1) = \gamma;$$

$$\bar{a}bc,\ \beta = (0, 1, 1);\quad ab\bar{c},\ (1, 1, 0) = \delta;$$

$$\bar{a}\bar{b}c,\ (0, 0, 1) = \epsilon;\quad a\bar{b}\bar{c},\ (1, 0, 0) = \eta;$$

$$\bar{a}b\bar{c},\ (0, 1, 0) = \xi;\quad \bar{a}\bar{b}\bar{c},\ (0, 0, 0) = \theta.$$

Since the classes are now exclusive,

$$(\Sigma abc) \equiv \Sigma(abc).$$

(6) Hence every premiss can be written in linear form as $\Sigma\alpha.abc = A$ where α can be written down at sight (see 4) and where A is a number.

Since when $\alpha\ldots$ are all 1, we have

$$\Sigma abc = 1,$$

and since the number of terms in the linear expression of a function of n classes is 2^n, *there are necessary* $2^n - 1$ *premisses* for the complete solution of a problem in n classes.

Example (see p. 187, Keynes):

$$(a_1) = c_1, \quad (ea_1) = p_1 (a)_1, \quad (e\bar{a}_1\bar{a}_2) = 0,$$

$$(a_2) = c_2, \quad (ea_2) = p_2 (a_2), \quad (ea_1a_2) = y,$$

$$(ea_1a_2) + (ea_1\bar{a}_1) = p_1 (a_1),$$

$$(ea_1a_2) + (ea_2\bar{a}_1) = p_2 (a_2).$$

Therefore

$$(e) = (ea_1a_2) + (ea_1\bar{a}_2) + (e\bar{a}_1a_2) + (e\bar{a}_1\bar{a}_2)$$

$$= p_1 (a_1) + p_2 (a_2) - (ea_1a_2) + (e\bar{a}_1\bar{a}_2) = p_1c_1 + p_2c_2 - y. \qquad [Q.E.D.].$$

It will be observed that there are here only six premisses, and the complete solution cannot be possible, as Boole apparently thought, according to Mr. Keynes.

This may be verified thus:

$$(ea_1a_2) = y, \qquad (e\bar{a}_1a_2) = p_2c_2 - y, \qquad (\bar{e}a_1a_2) = \alpha, \qquad (\bar{e}\bar{a}_1a_2) = \gamma,$$

$$(ea_1\bar{a}_2) = p_1c_1 - y, \qquad (e\bar{a}_1\bar{a}_2) = 0, \qquad (\bar{e}a_1\bar{a}_2) = \beta, \qquad (\bar{e}\bar{a}_1\bar{a}_2) = \delta.$$

Adding each of the last four equations to the corresponding one in 'e', we have

$$(a_1a_2) = \alpha + y, \qquad (\bar{a}_1\bar{a}_2) = \delta,$$

$$(a_1\bar{a}_2) = p_1c_1 - y + \beta, \qquad (\bar{a}_1a_2) = p_2c_2 - y + \gamma,$$

and hence we have four equations and five unknowns unless we know one of α, β, γ, δ.

The problem of generalizing the above can be approached in various ways. The first step would certainly be to rewrite it in terms of propositional functions instead of classes. The next step might at first seem to be to make the formulæ applicable to cases where, as with the millionaire and his socks, (a) and (b) in $(a)/(b)$ are infinite. This is not very promising, though a form of the addition theorem

$$(a) + (b\bar{a}) = (a + b)$$

is still unambiguous; it is useless, however, since the various "probabilities" $\frac{(a)}{(c)}, \frac{(b\bar{a})}{(c)}, \frac{(a+b)}{(c)}$ are themselves ambiguous. This line of advance is therefore only a special case of the reconsideration of the meaning of $(\;\;)/(\;\;)$, hinted at as necessary above for complete generality.

Upon this question (namely, "What is it that two groups of propositions have in common when we are able to reason probably between their members?", and further "between what kinds of groups of propositions can this relation hold?") I can throw no light. But both the—as it seems to me—incomplete attempt of Mr. Keynes and the line of thought contained in this note make me feel certain that no solution of the problem can be succesful which endeavours to extract a probability relation from two general propositions taken in isolation.

ON A SPHERICAL CONFIGURATION OF EIGHT POINTS.

By *Prof. W. Burnside.*

SUPPOSE the edges of a cube to be rigid bars freely jointed at the corners. It is proposed to consider those configurations in which the eight corners lie on a sphere.

OA, OB, OC are taken to be three conterminous edges, and O', A', B', C' the corners opposite to O, A, B, C. When the eight points lie on a sphere, each edge subtends the same angle δ at the centre. If the edges are projected on to the surface of the sphere from the centre, the surface is divided into six spherical quadrilaterals, all of whose sides are equal. In a spherical quadrilateral, whose sides are all equal, the opposite angles are equal. This justifies the following notation

$$BOC = BA'C = \alpha;\quad COA = CB'A = \beta;\quad AOB = AC'B = \gamma;$$

where $$\alpha + \beta + \gamma = 2\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i});$$

$$OBA' = OCA' = \alpha';\quad OCB' = OAB' = \beta';\quad OAC' = OBC' = \gamma'.$$

These give

$$B'O'C' = B'AC' = 2\pi - OAB' - OAC' = 2\pi - \beta' - \gamma',$$

$$C'O'A' = C'BA' \qquad = 2\pi - \gamma' - \alpha',$$

$$A'O'B' = A'CB' \qquad = 2\pi - \alpha' - \beta',$$

so that since $B'O'C' + C'O'A' + A'O'B' = 2\pi,$

$$\alpha' + \beta' + \gamma' = 2\pi \text{.....................(ii)}.$$

Each quadrilateral therefore has the same angles as the opposite one.

The spherical quadrilaterals $OBA'C$ or $O'B'AC'$ give

$$\cos\delta = \cot\tfrac{1}{2}\alpha \cot\tfrac{1}{2}\alpha' \quad \text{...............(iii)}.$$

The other two pairs similarly give

$$\cos\delta = \cot\tfrac{1}{2}\beta \cot\tfrac{1}{2}\beta' \quad \text{...............(iv)},$$

$$\cos\delta = \cot\tfrac{1}{2}\gamma \cot\tfrac{1}{2}\gamma' \quad \text{...............(v)}.$$

These equations obviously involve

$$\delta \leq \tfrac{1}{2}\pi.$$

These five equations thus obtained are the only independent ones connecting δ, α, β, γ, α', β', γ'. From them may be deduced

$$\cot^2\delta = \cos\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta \cos\tfrac{1}{2}\gamma \text{...............(vi)},$$

$$\cot^2\delta = \cos\tfrac{1}{2}\alpha' \cos\tfrac{1}{2}\beta' \cos\tfrac{1}{2}\gamma' \text{...........(vii)}.$$

The triangle OBA' gives

$$\cos OA' = \cos^2\delta + \sin^2\delta\cos\alpha', \quad \sin OA' = \frac{\sin\delta\sin\alpha'}{\sin\tfrac{1}{2}\alpha}.$$

The triangle AOA' gives

$$\cos AA' = \cos OA \cos OA' + \sin OA \sin OA' \cos AOA'$$

$$= \cos\delta(\cos^2\delta + \sin^2\delta\cos\alpha') + \frac{\sin^2\delta\sin\alpha'}{\sin\tfrac{1}{2}\alpha}\cos(\beta + \tfrac{1}{2}\alpha)$$

$$= \cos\delta(\cos^2\delta + \sin^2\delta\cos\alpha') + \frac{\sin^2\delta\sin\alpha'}{\sin\alpha}(\cos\beta + \cos\gamma).$$

Now (i) and (vi) give

$$1 + \cos\alpha + \cos\beta + \cos\gamma + 4\cot^2\delta = 0 \quad \text{......(viii)}.$$

Hence

$$\cos AA' = \cos\delta\,(\cos^2\delta + \sin^2\delta\cos\alpha') - \frac{2\sin^2\delta\sin\alpha'}{\sin\alpha}(\cos^2\tfrac{1}{2}\alpha + 2\cot^2\delta).$$

Entering in this the values of $\cos\alpha'$ and $\sin\alpha'$ from (iii),

$$\cos AA' = \cos^3\delta + \frac{\sin^2\delta\cos\delta}{1+\sec^2\delta\cot^2\frac{1}{2}\alpha}$$

$$\times\left[1 - \sec^2\delta\cot^2\tfrac{1}{2}\alpha - 2\sec^2\delta\,(\cot^2\tfrac{1}{2}\alpha + 2\operatorname{cosec}^2\tfrac{1}{2}\alpha\cot^2\delta)\right]$$

$$= \cos^3\delta + \sin^2\delta\cos\delta\left[1 - 4\operatorname{cosec}^2\delta\right]$$

$$= -3\cos\delta.$$

It follows that $\frac{1}{3} \geq \cos\delta \geq 0$, while the angular distance between any pair of opposite corners of the figure is $\cos^{-1}(-3\cos\delta)$. Since the remaining angles are determinate in terms of δ and α, it follows that when δ is given, subject to the above inequality, the spherical figure, apart from displacements as a rigid frame, has just one degree of freedom. When δ and α are given, equation (viii) may be written

$$\cos\tfrac{1}{2}\alpha\cos(\beta + \tfrac{1}{2}\alpha) + \cos^2\tfrac{1}{2}\alpha + 2\cot^2\delta = 0,$$

or

$$4\cos^2\tfrac{1}{2}\alpha\sin^2(\beta+\tfrac{1}{2}\alpha)$$

$$= \{\sqrt{(1-8\cot^2\delta)} - 4\cot^2\delta - \cos\alpha\}\{\sqrt{(1-8\cot^2\delta)} + 4\cot^2\delta + \cos\alpha\}.$$

The greatest value of the right-hand side is $1 - 8\cot^2\delta$, when $\cos\alpha = -4\cot^2\delta$, and this gives

$$\sin^2(\beta + \tfrac{1}{2}\alpha) = \frac{1-8\cot^2\delta}{2-8\cot^2\delta}.$$

It follows that $\cos\alpha$ can take all values between

$$-\sqrt{(1-8\cot^2\delta)} - 4\cot^2\delta \quad\text{and}\quad \sqrt{(1-8\cot^2\delta)} - 4\cot^2\delta,$$

and this determines the amount of play of which the frame work is capable for a given value of δ.

The angle ϕ of an equilateral spherical triangle of side δ is given by

$$\cos\phi = \frac{\cos\delta}{1+\cos\delta}.$$

Hence, as δ increases from $\cos^{-1}(\frac{1}{3})$, no spherical quadrilateral can have a diagonal equal to δ, until

$$\sqrt{(1-8\cot^2\delta)} - 4\cot^2\delta \geq \frac{\cos\delta}{1+\cos\delta},$$

or $$6\cos^3\delta + 11\cos^2\delta - 1 \leq 0.$$

When δ satisfies this relation there is a configuration of the figure in which $\alpha = \phi$. For this configuration $\alpha' = 2\phi$ and

$$\cos\tfrac{1}{2}\phi\cos(\beta + \tfrac{1}{2}\phi) + \cos^2\tfrac{1}{2}\phi + 2\cot^2\delta = 0,$$

$$\cos\phi\ \ \cos(\beta' + \phi) + \cos^2\phi\ \ + 2\cot^2\delta = 0.$$

If θ, θ' are acute angles satisfying

$$\cos\theta = \frac{\cos^2\frac{1}{2}\phi + 2\cot^2\delta}{\cos\frac{1}{2}\phi}, \quad \cos\theta' = \frac{\cos^2\phi + 2\cot^2\delta}{\cos\phi},$$

it is easy to verify that as δ increases from $\cos^{-1}(\frac{1}{3})$ to $\frac{1}{2}\pi$, ϕ, θ, and θ' all increase with δ.

Suppose that $\beta < \gamma$, so that $\beta' > \gamma'$. Then from the above equations

$$\beta + \tfrac{1}{2}\phi = \pi - \theta,\ \gamma + \tfrac{1}{2}\phi = \pi + \theta,\ \beta' + \phi = \pi + \theta',\ \gamma' + \phi = \pi - \theta',$$

so that β and γ' diminish as δ increases. Now β and β' cannot both diminish as δ increases, nor can γ and γ'; so that γ and β' increase as δ increases. Hence that configuration being taken in which BC is δ, as δ increases AC and OC' diminish, while OB' and AB increases. When δ reaches a value for which either AC or OC' is δ, any further increase of δ makes either AC or OC' less than δ. Hence the greatest value of δ which is consistent with no two of the eight points being at a smaller angular distance than δ, is that given by the condition that BC and either AC or OC' are each δ.

The condition $\alpha = \beta = \phi$, where ϕ is necessarily less than $\frac{1}{2}\pi$, is not possible.

If $\alpha = \gamma' = \phi$, then $\alpha' = \gamma = 2\phi$, $\beta = \beta' = 2\pi - 3\phi$.

These give $$\cos\delta = \cot^2\tfrac{1}{2}(3\phi),$$

or $$7\cos^3\delta + 9\cos^2\delta + \cos\delta - 1 = 0;$$

and since $$\tfrac{1}{3} \geq \cos\delta \geq 0,\ \cos\delta = \tfrac{1}{7}(2\sqrt{2} - 1).$$

Hence, if δ is less than $\cos^{-1}\frac{1}{7}(2\sqrt{2} - 1)$, it is possible to mark eight points on a sphere, so that the least arcual distance between any two of them exceeds δ; while if δ is equal to or greater than $\cos^{-1}\frac{1}{7}(2\sqrt{2} - 1)$, this is not possible.

ON THE ZEROS OF AN INTEGRAL FUNCTION REPRESENTED BY FOURIER'S INTEGRAL.

By *G. Pólya.*

We do not possess a general method for discussing the reality of zeros of an integral function represented by Fourier's integral (such a method would be available for Riemann's ξ-function). I present here a special case where the discussion is not quite trivial, but may be carried out with the help of known results.

Consider the function

$$(1) \qquad F_\alpha(z) = \int_0^\infty e^{-t^\alpha} \cos zt\, dt.$$

If $0 < \alpha < 1$, then $F_\alpha(z)$ is defined by this formula only for real values of z. We have

$$F_1(z) = \frac{1}{1+z^2}.$$

For $\alpha > 1$ we get

$$(2) \qquad F_\alpha(z) = \frac{1}{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{2n+1}{\alpha}\right)}{\Gamma(2n+1)} z^{2n}.$$

This development shows that $F_\alpha(z)$ is an integral function of order $\frac{\alpha}{\alpha-1}$. In particular,

$$(3) \qquad F_2(z) = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}z^2}.$$

Following the method employed by G. H. Hardy* to prove that Riemann's $\xi(t)$ has an infinite number of real zeros, F. Bernstein† proved the same thing for $F_4(z)$, $F_6(z)$, $F_8(z)$, ... Now it is easy to go further in the case of $F_\alpha(z)$ [though naturally not in the case of $\xi(t)$], and to prove the following results:

(I) *If* $\alpha = 2$, *then there are no zeros at all.*

(II) *If* $\alpha = 4, 6, 8, \ldots$, *then there are an infinite number of real zeros but no complex zeros.*

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* *Comptes Rendus*, 6 April, 1914.

† *Mathematische Annalen*, vol. lxxix (1919), pp. 265–268.

(III) *If $\alpha > 1$, and is not an even integer, then there are an infinite number of complex zeros and a finite number, not less than $2\left[\frac{1}{2}\alpha\right]$, of real zeros.*

The statement (I) needs no demonstration: compare (3). The proof of (II) is based on the following special case of a theorem of Laguerre: *

If $\Phi(z)$ is an integral function of order less than 2 which assumes real values along the real axis and possesses only real negative zeros, then the zeros of the integral function

$$\Phi(0) + \frac{\Phi(1)}{1!} z + \frac{\Phi(2)}{2!} z^2 + \ldots + \frac{\Phi(n)}{n!} z^n + \ldots$$

are also all real and negative.

Put

$$\Phi(z) = \frac{\Gamma\left(\frac{2z+1}{2k}\right)\Gamma(z+1)}{\Gamma(2z+1)}, \tag{4}$$

where k is a positive integer. The poles of the numerator

$$z = -\tfrac{1}{2},\ -\tfrac{1}{2}(2k+1),\ -\tfrac{1}{2}(4k+1),\ \ldots,\quad z = -\tfrac{2}{2},\ -\tfrac{4}{2},\ -\tfrac{6}{2},\ \ldots,$$

are absorbed by those of the denominator

$$z = -\tfrac{1}{2},\ -\tfrac{2}{2},\ -\tfrac{3}{2},\ -\tfrac{4}{2},\ -\tfrac{5}{2},\ \ldots .$$

Thus $\Phi(z)$ is an integral function satisfying the conditions required by the theorem of Laguerre, and consequently the zeros of

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma\left(\frac{2n+1}{2k}\right)\Gamma(n+1)}{\Gamma(2n+1)} = 2k F_{2k}(i\sqrt{z})$$

are all real and negative; we infer that the zeros of $F_{2k}(z)$ are all real.

The order of the integral function $F_{2k}(z)$ is $\frac{2k}{2k-1}$; if $k = 2, 3, 4, \ldots$, then $1 < \frac{2k}{2k-1} < 2$. Thus $F_{2k}(z)$ is not of integral order and consequently possesses an infinity of zeros; they are all real, and thus (II) is completely proved.

Suppose x is positive. Then we have, by partial integration,

$$x^{a+1} F_a(x) = x^a \int_0^{\infty} \sin xt \,.\, at^{a-1} e^{-t^a} dt.$$

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* *Oeuvres*, vol. i. (Paris, 1898), pp. 200–203.

Introduce the new variable $u = x^a t^a$; then we have

$$(5) \qquad x^{a+1} F_a(x) = \mathfrak{I} \int_0^\infty \exp(iu^{1/a} - ux^{-a})\, du,$$

where $\mathfrak{I} A$ denotes the imaginary part of A. Choose as path of integration, not the positive real axis, but a straight line running from 0 to ∞ in the upper half-plane and making a sufficiently small angle with the positive real axis. With this path we have

$$\lim_{x=+\infty} x^{a+1} F_a(x) = \mathfrak{I} \int_0^\infty e^{iu^{1/a}}\, du.$$

Rotating the path of integration in the positive direction until it reaches the position where $\arg z = \frac{1}{2}\pi\alpha$, we get finally

$$(6) \qquad \lim_{x=+\infty} x^{a+1} F_a(x) = \mathfrak{I} \int_0^{+\infty} e^{-r^{1/a}} e^{i\pi a/2}\, dr$$
$$= \Gamma(\alpha+1) \sin(\pi\alpha/2).$$

If the limit (6) is different from 0, that is, if α is different from 2, 4, 6, ..., then $F_a(z)$ possesses

(*a*) a finite number of real zeros and

(*b*) an infinite number of zeros.

Of these assertions, (*a*) is evident from (6). To prove (*b*) we make use of the theorem that an integral function of finite order having a finite number of zeros is of the form

$$(7) \qquad P(z)\, e^{Q(z)},$$

where $P(z)$, $Q(z)$ are polynomials. Now $F_a(z)$ is certainly not of the form (7), since it converges to 0 when $z \to +\infty$ in the same manner as a negative power of z, as may be seen from (6). The statements (*a*), (*b*) just proved contain the first two parts of (III).

From (1) follows, by Fourier's theorem,

$$\frac{2}{\pi} \int_0^\infty F_a(x) \cos xt\, dx = e^{-t^a} = 1 - \frac{t^a}{1!} + \dots .$$

Differentiating $2m$ times with respect to t, where

$$(8) \qquad 2m < \alpha < 2m+2,$$

and then putting $t = 0$, we get

$$(9) \quad \int_0^\infty F_a(x)\, x^2 dx = \int_0^\infty F_a(x)\, x^4 dx = \dots = \int_0^\infty F_a(x)\, x^{2m} dx = 0$$

The convergence of the integrals (9) is assured by (6) and (8). It follows from (9) that

$$(10) \qquad \int_0^\infty F_a(x)\, x^2 P(x^2)\, dx = 0,$$

where $P(z)$ denotes any polynomial in z of degree not exceeding $m-1$. Assume now, if possible, that $F_a(x)$ changes sign at most $m-1$ times for $x>0$, *e.g.* at the points $x_1, x_2, \ldots, x_{m-1}$, where $0<x_1<x_2<\ldots<x_{m-1}$; and put

$$P(x^2) = (x_1^{\,2}-x^2)(x_2^{\,2}-x^2)\ldots(x^2_{m-1}-x^2).$$

Then the integrand in (10) is never negative and our assumption leads to a contradiction. Thus $F_a(x)$ changes sign at least $m=\left[\tfrac{1}{2}\alpha\right]$ times for $x>0$. We have now proved the whole of Theorem (III).

The results we have obtained may be completed in many respects. If $a \geq 0$ and k is an integer not less than 2, then the zeros of the integral function of z

$$\int_0^\infty e^{at^2-t^{2k}} \cos zt\, dt$$

are all real; the asymptotic distribution of the zeros can be calculated by more laborious and more usual methods; and so on. The function $F_a(z)$ has been considered in connection with questions arising in the theory of errors, especially by Cauchy*, and P. Lévy† proved that $F_a(x) \geq 0$ for $0<\alpha\leq 2$ and for real values of x. More recently W. R. Burwell‡ has discussed the asymptotic expansion of $F_a(z)$ for $\alpha=3, 4, 5, \ldots$, and has shown in particular that, when $\alpha=4, 6, \ldots$, the number of complex zeros is finite. This result is included in Theorem (II) above. Finally we may add that $F_a(z)$ is of much importance in Waring's problem.§

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* *Comptes Rendus*, vol. xxxvii (1853), pp. 202–206, and *passim*.

† *Comptes Rendus*, vol. clxxvi. (1923), pp 1118–1120.

‡ *Proc. Lond. Math. Soc.* (2), vol. xxii. (1923), pp. 57–72.

§ G. H. Hardy and J. E. Littlewood, *Göttinger Nachrichten* (1920), pp 33–54.

THREE N-DIMENSIONALS.

By *F. C. Pitt-Bazett.*

I SHALL begin by defining some n-dimensional terms, that is, I shall enumerate in order after each its analogue in one, two, and three dimensions. The terms are: Close—segment (or finite line), polygon (or closed figure), polyhedron (or solid); Content—length, area, volume; Minimum (close)—segment, triangle, tetrahedron; Power (close)—segment, tetragon (or quadrilateral), hexahedron; and Reciprocal (close)—segment, tetragon, octahedron. I shall represent the adjective n-dimensional by n, and by co-n the general analogue of collinear and co-planar. It is with some of the descriptive properties of the above-mentioned closes that this brief note is concerned, and—in the case of their regular forms—with some of their metrical properties also.

I shall assume the following properties of n-space: that it contains an infinity of $n-1$-spaces, and that by motion to infinite extent in a direction which it does not contain it generates an $n+1$-space: that it is the intersection in general of $r+1$, and never of less than $r+1$ $n+r$-spaces, and can contain $n+1$ and no more arbitrarily situated points.

First, let us investigate the elevation of a close from n- to $n+1$-space. The Minimum is elevated by introducing an $(n+1)$st vertex, non-co-n with the others, and associating it with all the elements of the close. In the case of the Power a second Power is taken nowhere co-n with the first, and corresponding elements associated. The elevation of the Reciprocal will become clearer shortly.

Next, let us consider the number of r's contained in an s in the various cases, and denote this function by $\left\{ \begin{matrix} s \\ r \end{matrix} \right\}$. For the Minimum we have

$$\left\{ \begin{matrix} s \\ r \end{matrix} \right\} = \left\{ \begin{matrix} s-1 \\ r-1 \end{matrix} \right\} + \left\{ \begin{matrix} s-1 \\ r \end{matrix} \right\},$$

and by known results in the lower dimensions we obtain

$$\left\{ \begin{matrix} s \\ r \end{matrix} \right\} = \binom{s+1}{r+1},$$

the binomial coefficient. So for the Power the equation is

$$\begin{Bmatrix} s \\ r \end{Bmatrix} = \begin{Bmatrix} s-1 \\ r-1 \end{Bmatrix} + 2\begin{Bmatrix} s-1 \\ r \end{Bmatrix},$$

and leads in a similar way to

$$\begin{Bmatrix} s \\ r \end{Bmatrix} = 2^{s-r}\binom{s}{r}.$$

We can now define the Reciprocal as having an $n-r$ corresponding to every $r-1$ of the Power of the same order; and, replacing s by n, we have for the Reciprocal

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = 2^{r+1}\binom{n}{r+1};$$

in particular, it has 2^n $n-1$'s and $2n$ vertices. Now the sum-total of elements in a Minimum $n-1$ is 2^n, including the -1 element—the reciprocal of the close itself—and so this identity suggests that the Reciprocal is formed by associating two $n-1$ Minima by means of their reciprocal elements, that is, an $r-2$ of the one to the corresponding $n-r$ of the other, and the consistency of this suggestion will be seen in what follows. The one Minimum may be supposed to be translated and reciprocated into the other, for a Minimum is clearly self-reciprocal.

Now suppose $r > s$, so that $\begin{Bmatrix} s \\ r \end{Bmatrix}$ becomes the number of r's *at* an s, and as this function depends on the order of the complete close of which these are elements, I shall denote it by $\begin{Bmatrix} n \\ r \\ s \end{Bmatrix}$. This gives us

$$\begin{Bmatrix} n \\ r \\ s \end{Bmatrix} = \begin{Bmatrix} n-1 \\ r-1 \\ s-1 \end{Bmatrix} = (\text{eventually}) \begin{Bmatrix} n-s \\ r-s \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} n-s-1 \\ r-s-1 \end{Bmatrix} (\text{Minimum}) = \binom{n-s}{r-s},$$

these equations holding for both the Minimum and the Power. Neither $\begin{Bmatrix} n \\ r \\ s \end{Bmatrix}$ nor, in general, $\begin{Bmatrix} s \\ r \end{Bmatrix}$ can be found for the third close by reciprocation, and so we must resort to another method; $\begin{Bmatrix} s \\ r \end{Bmatrix}$ will hold the same value for the Reciprocal as for the Minimum, except when $s=n$, a case which has already been discussed. It is clear, on reflection, that

$$\begin{Bmatrix} n \\ r \\ s \end{Bmatrix} \Big/ \begin{Bmatrix} n \\ r-1 \\ s \end{Bmatrix} = \begin{Bmatrix} n \\ r \\ r-1 \end{Bmatrix} \Big/ \begin{Bmatrix} r \\ r-1 \\ s \end{Bmatrix} = 2\,\frac{n-r}{r-s},$$

since the last term refers to a Minimum; thus

$$\begin{Bmatrix} n \\ r \\ s \end{Bmatrix} = 2^{r-s}\,\frac{(n-r)_{(r-s)}}{(r-s)^{(r-s)}} \begin{Bmatrix} n \\ s \\ s \end{Bmatrix} = 2^{r-s}\binom{n-s-1}{r-s}.$$

A neater and more satisfactory method comes from using the general relation, true for all closes,

$$\begin{Bmatrix} n \\ r \end{Bmatrix} \begin{Bmatrix} n \\ s \\ r \end{Bmatrix} = \begin{Bmatrix} n \\ r \\ s \end{Bmatrix} \begin{Bmatrix} n \\ s \end{Bmatrix};$$

for $\begin{Bmatrix} n \\ s \\ r \end{Bmatrix}$ in the three cases is simply $\begin{Bmatrix} r \\ s \end{Bmatrix}$ for Minimum, Power, Minimum respectively; this gives the same results as before.

It is interesting to note the relation $\overset{r=n+1}{\underset{r=0}{S}} (-)^r \begin{Bmatrix} n \\ r-1 \end{Bmatrix} = 0$, satisfied by the closes, which has for its analogues: a segment is the join of two points, $n=s$ and $S+F=E+2$; while still more worthy of notice is the -1 element, which persists everywhere, even in the point—the zero-dimensional close—and which reciprocates into the reciprocal close.

By analogy the whole content of a Minimum is the n^{th} power of its edge, here unity, divided by factorial n, and multiplied by the square root of the determinant of the n^{th} order having its principal elements unity and remaining elements equal to $\frac{1}{2}$. If this determinant be denoted by u_n, we have

$$u_n = u_{n-1}/2 + 1/2^n = (n+1)/2^n, \text{ since } u_1 = 1.$$

Thus the whole content is

$$(n+1)^{3/2}/2^{n/2}(n+1)!,$$

and the s content accordingly

$$(s+1)^{3/2}(n+1)!/2^{s/2}(n-s)!\{(s+1)!\}^2.$$

The whole content of the regular Power is by analogy unity, the n^{th} power of its edge, so that its s content is merely $\left\{\begin{matrix} n \\ s \end{matrix}\right\}$. The s content of the Reciprocal will be

$$2^{s/2+1} n!/(n-s+1)!\{(s+1)!\}^2,$$

with an exception, as usual, when $s = n$. The determination of the whole content of a Reciprocal is based on the fact that it has a diametral $n-1$ Reciprocal; this follows by reciprocation from the obvious property of the Power that, corresponding to any pair of opposite $n-1$'s, there exists a single point at infinity at which all the others meet, this property being inherent in the nature of the elevation of the Power. It now appears that every $n-1$ of a Reciprocal belongs to its own n-Minimum, having its opposite vertex at the centre of the Reciprocal. Every edge at this centre will be of length $\sqrt{2}$ and every right angle; thus we obtain the total content

$$2^{n/2}/n!;$$

for example, the length of the Reciprocal segment is $\sqrt{2}$, and not unity.

These results may be conveniently set forth, as in the accompanying table:

Close:	r's in or at an s: $r < s$	$r > s$,	s content
Minimum:	$\binom{s+1}{r+1}$	$\binom{n-s}{r-s}$	$\dfrac{(s+1)^{3/2}(n+1)!}{2^{s/2}(n-s)!\{(s+1)!\}^2}$,
Power:	$2^{s-r}\binom{s}{r}$	$\binom{n-s}{r-s}$	$2^{n-s}\binom{n}{s}$,
Reciprocal: $s \neq n$:	$\binom{s+1}{r+1}$	$2^{r-s}\binom{n-s-1}{r-s}$	$\dfrac{2^{s/2+1}n!}{(n-s-1)!\{(s+1)!\}^2}$
$s = n$:		$2^{r-1}\binom{n}{r+1}$.	

END OF VOL. LII.

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