A.E. BROUWER & A.M. COHEN

THE POINCARÉ SERIES OF THE POLYNOMIALS
INVARIANT UNDER SU_2 IN ITS IRREDUCIBLE
REPRESENTATION OF DEGREE ≤ 17

The Poincaré series of the polynomials invariant under SU_2 in its
irreducible representation of degree ≤ 17

by

Andries E. Brouwer & Arjeh M. Cohen

ABSTRACT

    Recently T.A. Springer established a formula for the Poincaré series
of the invariant polynomials of the compact Lie group SU_2 corresponding to
its irreducible representation of given degree $d+1$. The formula expresses
the Poincaré series as a rational function that is presented as a sum of
several other straightforwardly computable rational functions. By use of a
computer these functions have been determined for all $d \leq 16$.

INTRODUCTION

Springer's formula (cf. [3]) for the Poincaré series $P_d(z)$ of the SU_2-invariant polynomials associated with its $(d+1)$-dimensional irreducible representation reads as follows.

$$P_d(z) = \sum_{0 \le j < \frac{d}{2}} (-1)^j \phi_{d-2j}\left(\frac{(1-z^2)z^{j(j+1)}}{(j,z^2)!(d-j,z^2)!}\right) \quad,$$

where for $n \in \mathbb{N}$ the operator ϕ_n transforms a rational function f in z to the rational function $\phi_n f$ determined by

$$(\phi_n f)(z^n) = \frac{1}{n}\sum_{j=1}^{n} f(e^{2\pi ij/n}z),$$

and where $(n,z)!$ is the polynomial function given by $(n,z)! = (1-z)(1-z^2)...$ $...(1-z^n)$.

About the Calculation

Clearly, if $f = \Sigma_{j=0}^{N} f_j z^j$ is a polynomial function in z, then we have

(1) $$(\phi_n f) = \sum_{j=0}^{\lfloor N/n \rfloor} f_{jn}z^j,$$

where $\lfloor N/n \rfloor$ is the greatest integer $\le N/n$.

Suppose that fg is the product of two rational functions one of which, say g, is a function in z^n, in other words there exists a function h in z such that $h(z^n) = g(z)$. Then

(2) $$\phi_n(fg) = h\phi_n(f)$$

As the denominator of the ϕ_{d-2j}-argument in a summand of $P_d(z)$ is a product of factors of the form $1-z^m$, we may, after suitable multiplication of denominator and numerator by common factors, assume that the denominator is a product of factors having the form $1-z^{(d-2j)m}$. Application of (2) reduces the summand in $P_d(z)$ to the quotient of the ϕ_{d-2j}-image of a polynomial by

a product of terms of the form $1-z^m$.

Using (1), the numerator is readily computed, whereafter we have obtained a summand that is expressed as a rational function with integral coefficients. Addition of these summands (while keeping track of the signs $(-1)^j$ that appear in the formula) establishes the results.

RESULTS. The $P_d(z)$ for $d \leq 16$ are listed in the table below in a so-called representative form (cf. [4]). The interested reader may find the individual summands in the appendix.

TABLE

The Poincaré series of the invariants under SU_2 in its irreducible representation of degree $d+1 \leq 17$.

$$d = 2 \qquad \frac{1}{1-z^2}$$

$$d = 3 \qquad \frac{1}{1-z^4}$$

$$d = 4 \qquad \frac{1}{(1-z^2)(1-z^3)}$$

$$d = 5 \qquad \frac{1+z^{18}}{(1-z^4)(1-z^8)(1-z^{12})}$$

$$d = 6 \qquad \frac{1+z^{15}}{(1-z^2)(1-z^4)(1-z^6)(1-z^{10})}$$

$$d = 7 \qquad \frac{1+2z^8+4z^{12}+4z^{14}+5z^{16}+9z^{18}+6z^{20}+9z^{22}+8z^{24}+9z^{26}+6z^{28}+9z^{30}+5z^{32}+4z^{34}+4z^{36}+2z^{40}+z^{48}}{(1-z^4)(1-z^8)(1-z^{12})(1-z^{12})(1-z^{20})}$$

$$d = 8 \qquad \frac{1+z^8+z^9+z^{10}+z^{18}}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)}$$

$d = 9$

$$\frac{\begin{array}{l}1+z^4+5z^8+4z^{10}+17z^{12}+20z^{14}+47z^{16}+61z^{18}+97z^{20}+120z^{22}+\\165z^{24}+189z^{26}+223z^{28}+241z^{30}+254z^{32}+254z^{34}+241z^{36}+\\223z^{38}+189z^{40}+165z^{42}+120z^{44}+97z^{46}+61z^{48}+47z^{50}+20z^{52}+\\17z^{54}+4z^{56}+5z^{58}+z^{62}+z^{66}\end{array}}{(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})}$$

$d = 10$

$$\frac{\begin{array}{l}1+2z^6+4z^8+4z^9+7z^{10}+8z^{11}+15z^{12}+15z^{13}+20z^{14}+27z^{15}+29z^{16}+\\35z^{17}+40z^{18}+44z^{19}+47z^{20}+55z^{21}+52z^{22}+57z^{23}+56z^{24}+57z^{25}+\\52z^{26}+55z^{27}+47z^{28}+44z^{29}+40z^{30}+35z^{31}+29z^{32}+27z^{33}+20z^{34}+\\15z^{35}+15z^{36}+8z^{37}+7z^{38}+4z^{39}+4z^{40}+2z^{42}+z^{48}\end{array}}{(1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^8)(1-z^9)(1-z^{10})(1-z^{14})}$$

$d = 11$

$$\frac{\begin{array}{l}1+z^4+10z^8+12z^{10}+57z^{12}+95z^{14}+233z^{16}+394z^{18}+745z^{20}+\\1173z^{22}+1925z^{24}+2814z^{26}+4139z^{28}+5669z^{30}+7664z^{32}+\\9821z^{34}+12390z^{36}+14953z^{38}+17692z^{40}+20204z^{42}+22537z^{44}+\\24378z^{46}+25746z^{48}+26413z^{50}+26413z^{52}+25746z^{54}+24378z^{56}+\\22537z^{58}+20204z^{60}+17692z^{62}+14953z^{64}+12390z^{66}+9821z^{68}+\\7664z^{70}+5669z^{72}+4139z^{74}+2814z^{76}+1925z^{78}+1173z^{80}+745z^{82}+\\394z^{84}+233z^{86}+95z^{88}+57z^{90}+12z^{92}+10z^{94}+z^{98}+z^{102}\end{array}}{\begin{array}{l}(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})\\(1-z^{20})\end{array}}$$

$d = 12$

$$\frac{\begin{array}{l}1+z^3+z^4+3z^6+5z^7+7z^8+11z^9+21z^{10}+25z^{11}+40z^{12}+55z^{13}+75z^{14}+\\96z^{15}+130z^{16}+156z^{17}+194z^{18}+228z^{19}+267z^{20}+294z^{21}+330z^{22}+\\345z^{23}+364z^{24}+365z^{25}+364z^{26}+345z^{27}+330z^{28}+294z^{29}+\\267z^{30}+228z^{31}+194z^{32}+156z^{33}+130z^{34}+96z^{35}+75z^{36}+55z^{37}+\\40z^{38}+25z^{39}+21z^{40}+11z^{41}+7z^{42}+5z^{43}+3z^{44}+z^{46}+z^{47}+z^{50}\end{array}}{\begin{array}{l}(1-z^2)(1-z^4)(1-z^5)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)\\(1-z^{11})\end{array}}$$

4

d = 13

$$1+z^4+19z^8+32z^{10}+156z^{12}+340z^{14}+916z^{16}+1913z^{18}+4115z^{20}+$$
$$7741z^{22}+14544z^{24}+25118z^{26}+42489z^{28}+68114z^{30}+106498z^{32}+$$
$$159697z^{34}+233971z^{36}+331412z^{38}+458878z^{40}+617966z^{42}+$$
$$814543z^{44}+1047676z^{46}+1321155z^{48}+1629760z^{50}+1972851z^{52}+$$
$$2341055z^{54}+2728439z^{56}+3120893z^{58}+3509075z^{60}+3876056z^{62}+$$
$$4210210z^{64}+4496206z^{66}+4723183z^{68}+4879836z^{70}+4960463z^{72}+$$
$$4960463z^{74}+4879836z^{76}+4723183z^{78}+4496206z^{80}+4210210z^{82}+$$
$$3876056z^{84}+3509075z^{86}+3120893z^{88}+2728439z^{90}+2341055z^{92}+$$
$$1972851z^{94}+1629760z^{96}+1321155z^{98}+1047676z^{100}+814543z^{102}+$$
$$617966z^{104}+458878z^{106}+331412z^{108}+233971z^{110}+159697z^{112}+$$
$$106498z^{114}+68114z^{116}+42489z^{118}+25118z^{120}+14544z^{122}+$$
$$7741z^{124}+4115z^{126}+1913z^{128}+916z^{130}+340z^{132}+156z^{134}+$$
$$32z^{136}+19z^{138}+z^{142}+z^{146}$$

$$(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})$$
$$(1-z^{20})(1-z^{22})(1-z^{24})$$

d = 14

$$1+z^4+5z^6+3z^7+18z^8+22z^9+55z^{10}+77z^{11}+158z^{12}+227z^{13}+$$
$$396z^{14}+579z^{15}+908z^{16}+1286z^{17}+1904z^{18}+2609z^{19}+3655z^{20}+$$
$$4865z^{21}+6502z^{22}+8386z^{23}+10800z^{24}+13480z^{25}+16765z^{26}+$$
$$20342z^{27}+24480z^{28}+28872z^{29}+33748z^{30}+38709z^{31}+43994z^{32}+$$
$$49174z^{33}+54374z^{34}+59258z^{35}+63866z^{36}+67854z^{37}+71317z^{38}+$$
$$73958z^{39}+75790z^{40}+76724z^{41}+76724z^{42}+75790z^{43}+73958z^{44}+$$
$$71317z^{45}+67854z^{46}+63866z^{47}+59258z^{48}+54374z^{49}+49174z^{50}+$$
$$43994z^{51}+38709z^{52}+33748z^{53}+28872z^{54}+24480z^{55}+20342z^{56}+$$
$$16765z^{57}+13480z^{58}+10800z^{59}+8386z^{60}+6502z^{61}+4865z^{62}+$$
$$3655z^{63}+2609z^{64}+1904z^{65}+1286z^{66}+908z^{67}+579z^{68}+396z^{69}+$$
$$227z^{70}+158z^{71}+77z^{72}+55z^{73}+22z^{74}+18z^{75}+3z^{76}+5z^{77}+z^{79}+$$
$$z^{83}$$

$$(1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)(1-z^{10})$$
$$(1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})$$

$d = 15$

$$1+2z^4+z^6+32z^8+78z^{10}+378z^{12}+1027z^{14}+3124z^{16}+7672z^{18}+$$
$$18676z^{20}+40784z^{22}+86197z^{24}+170002z^{26}+323704z^{28}+$$
$$586665z^{30}+1029080z^{32}+1737391z^{34}+2849501z^{36}+4530000z^{38}+$$
$$7019920z^{40}+10596388z^{42}+15636688z^{44}+22556344z^{46}+$$
$$31885923z^{48}+44179876z^{50}+60104681z^{52}+80312694z^{54}+$$
$$105538484z^{56}+136437227z^{58}+173688963z^{60}+217804048z^{62}+$$
$$269239746z^{64}+328183158z^{66}+394684584z^{68}+468435861z^{70}+$$
$$548927416z^{72}+635239454z^{74}+726230157z^{76}+820357306z^{78}+$$
$$915894836z^{80}+1010799973z^{82}+1102948881z^{84}+1190050285z^{86}+$$
$$1269884526z^{88}+1340258093z^{90}+1399218002z^{92}+1445037255z^{94}+$$
$$1476376889z^{96}+1492284743z^{98}+1492284743z^{100}+$$
$$1476376889z^{102}+1445037255z^{104}+1399218002z^{106}+$$
$$1340258093z^{108}+1269884526z^{110}+1190050285z^{112}+$$
$$1102948881z^{114}+1010799973z^{116}+915894836z^{118}+$$
$$820357306z^{120}+726230157z^{122}+635239454z^{124}+548927416z^{126}+$$
$$468435861z^{128}+394684584z^{130}+328183158z^{132}+269239746z^{134}+$$
$$217804048z^{136}+173688963z^{138}+136437227z^{140}+105538484z^{142}+$$
$$80312694z^{144}+60104681z^{146}+44179876z^{148}+31885923z^{150}+$$
$$22556344z^{152}+15636688z^{154}+10596388z^{156}+7019920z^{158}+$$
$$4530000z^{160}+2849501z^{162}+1737391z^{164}+1029080z^{166}+$$
$$586665z^{168}+323704z^{170}+170002z^{172}+86197z^{174}+40784z^{176}+$$
$$18676z^{178}+7672z^{180}+3124z^{182}+1027z^{184}+378z^{186}+78z^{188}+$$
$$32z^{190}+z^{192}+2z^{194}+z^{198}$$

$$(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})$$
$$(1-z^{20})(1-z^{22})(1-z^{24})(1-z^{26})(1-z^{28})$$

$$d = 16$$

$$\frac{\begin{aligned}&1+z^3+z^4+2z^5+8z^6+11z^7+30z^8+59z^9+112z^{10}+204z^{11}+389z^{12}+\\&655z^{13}+1146z^{14}+1896z^{15}+3067z^{16}+4823z^{17}+7477z^{18}+11167z^{19}+\\&16506z^{20}+23764z^{21}+33593z^{22}+46576z^{23}+63588z^{24}+85049z^{25}+\\&112273z^{26}+145668z^{27}+186298z^{28}+234773z^{29}+292016z^{30}+\\&357883z^{31}+433493z^{32}+518058z^{33}+611720z^{34}+713572z^{35}+\\&822997z^{36}+937742z^{37}+1057289z^{38}+1178403z^{39}+1299472z^{40}+\\&1417560z^{41}+1530508z^{42}+1634576z^{43}+1728646z^{44}+1808808z^{45}+\\&1873856z^{46}+1921520z^{47}+1951002z^{48}+1960390z^{49}+1951002z^{50}+\\&1921520z^{51}+1873856z^{52}+1808808z^{53}+1728646z^{54}+1634576z^{55}+\\&1530508z^{56}+1417560z^{57}+1299472z^{58}+1178403z^{59}+1057289z^{60}+\\&937742z^{61}+822997z^{62}+713572z^{63}+611720z^{64}+\\&518058z^{65}+433493z^{66}+357883z^{67}+292016z^{68}+234773z^{69}+\\&186298z^{70}+145668z^{71}+112273z^{72}+85049z^{73}+63588z^{74}+\\&46576z^{75}+33593z^{76}+23764z^{77}+16506z^{78}+11167z^{79}+7477z^{80}+\\&4823z^{81}+3067z^{82}+1896z^{83}+1146z^{84}+655z^{85}+389z^{86}+204z^{87}+\\&112z^{88}+59z^{89}+30z^{90}+11z^{91}+8z^{92}+2z^{93}+z^{94}+z^{95}+z^{98}\end{aligned}}{\begin{aligned}&(1-z^2)(1-z^4)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^7)(1-z^8)(1-z^9)\\&(1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})(1-z^{15})\end{aligned}}$$

REMARKS.

(i) The series $P_d(z)$ for $d \leq 10$ and $d = 12$ appear in [4]. They agree with the contents of the table. Results for $d \leq 12$ are also available in [1,2], but we have not compared them with ours. The expressions of $P_d(z)$ for $13 \leq d \leq 16$ seem to be new.

(ii) The expression given for $P_d(z)$ in the appendix has the property that the numerator and denominator are not simultaneously divisible by a polynomial of the form $(1-z^m)/(1-z^{\ell})$ (ℓ divisor of m) for m such that $1-z^m$ occurs as an explicitly given factor in the denominator. The representative form however is easily obtained from the given one.

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```
D=2;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)

*****THE POINCARE SERIES FOR D=2*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)



D=3;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z4)(1-Z2)

D=3;     J=1
NUMERATOR:
+1Z2
DENOMINATOR:
(1-Z2)(1-Z4)

*****THE POINCARE SERIES FOR D=3*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z4)



D=4;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z2)

D=4;     J=1
NUMERATOR:
+1Z1
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)

*****THE POINCARE SERIES FOR D=4*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)(1-Z3)



D=5;     J=0
NUMERATOR:
+1Z0+1Z2+4Z4+5Z6+7Z8+4Z10+3Z12
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z2)
```

D=5; J=1
NUMERATOR:
$+2Z_2+2Z_4+3Z_6+1Z_8+1Z_{10}$
DENOMINATOR:
$(1-Z_2)(1-Z_4)(1-Z_2)(1-Z_8)$

D=5; J=2
NUMERATOR:
$+1Z_6$
DENOMINATOR:
$(1-Z_2)(1-Z_4)(1-Z_4)(1-Z_6)$

******THE POINCARE SERIES FOR D=5******:
NUMERATOR:
$+1Z_0-1Z_6+1Z_{12}$
DENOMINATOR:
$(1-Z_4)(1-Z_6)(1-Z_8)$

D=6; J=0
NUMERATOR:
$+1Z_0+1Z_2+2Z_3+2Z_4+1Z_5+2Z_6$
DENOMINATOR:
$(1-Z_2)(1-Z_1)(1-Z_4)(1-Z_5)(1-Z_2)$

D=6; J=1
NUMERATOR:
$+1Z_1+1Z_2+1Z_3+1Z_5$
DENOMINATOR:
$(1-Z_1)(1-Z_1)(1-Z_3)(1-Z_2)(1-Z_5)$

D=6; J=2
NUMERATOR:
$+1Z_3$
DENOMINATOR:
$(1-Z_1)(1-Z_2)(1-Z_2)(1-Z_3)(1-Z_4)$

******THE POINCARE SERIES FOR D=6******:
NUMERATOR:
$+1Z_0-1Z_2-1Z_3+1Z_6+1Z_7-1Z_9$
DENOMINATOR:
$(1-Z_2)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)$

D=7; J=0
NUMERATOR:
$+1Z_0+3Z_2+1Z_4+4Z_6+9Z_8+156Z_{10}+236Z_{12}+288Z_{14}+335Z_{16}+329Z_{18}+304Z_{20}+235Z_{22}+174Z_{24}+100Z_{26}+54Z_{28}+20Z_{30}+6Z_{32}$
DENOMINATOR:
$(1-Z_4)(1-Z_6)(1-Z_8)(1-Z_{10})(1-Z_{12})(1-Z_2)$

D=7; J=1
NUMERATOR:
$+5Z_2+1Z_4+3Z_6+6Z_8+88Z_{10}+93Z_{12}+103Z_{14}+85Z_{16}+69Z_{18}+41Z_{20}+21Z_{22}+5Z_{24}+1Z_{26}$
DENOMINATOR:
$(1-Z_2)(1-Z_4)(1-Z_6)(1-Z_8)(1-Z_{12})$

D=7; J=2
NUMERATOR:

+1Z2+2Z4+9Z6+11Z8+15Z10+15Z12+14Z14+7Z16+5Z18+1Z20
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)

D=7;   J=3
NUMERATOR:
+1Z12
DENOMINATOR:
(1-Z4)(1-Z4)(1-Z5)(1-Z4)(1-Z6)(1-Z8)

*****THE POINCARE SERIES FOR D=7*****:
NUMERATOR:
+1Z0-1Z6+2Z8-1Z10+5Z12+2Z14+6Z15+2Z18+5Z20-1Z22+2Z24-1Z26+1Z32
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)

D=8;   J=0
NUMERATOR:
+1Z0+3Z2+4Z3+7Z4+7Z5+9Z6+8Z7+9Z8+6Z9+5Z10+3Z11+2Z12
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)

D=8;   J=1
NUMERATOR:
+2Z1+3Z2+5Z3+8Z4+10Z5+10Z6+12Z7+10Z8+8Z9+7Z10+4Z11+1Z12+1Z13
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)

D=8;   J=2
NUMERATOR:
+1Z2+1Z3+1Z4+1Z6
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)

D=8;   J=3
NUMERATOR:
+1Z6
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z2)(1-Z4)(1-Z5)

*****THE POINCARE SERIES FOR D=8*****:
NUMFRATOR:
+1Z0-1Z2-1Z3+1Z5+1Z6-1Z11-1Z12+1Z14+1Z15-1Z17
DENOMINATOR:
(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z7)

D=9;   J=0
NUMERATOR:
+1Z0+6Z2+49Z4+160Z6+414Z8+818Z10+1451Z12+2223Z14+3164Z16+4099Z18+5021Z20+560Z22+6105Z24+6092Z26+5773Z28+5077Z30+4235Z32+3244Z34+2341Z36
+1513Z38+993Z40+438Z42+188Z44+53Z46+11Z48
DENOMINATOR:
(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)

D=9;   J=1
NUMERATOR:
+1Z2+67Z4+256Z6+670Z8+1452Z10+2629Z12+4268Z14+6160Z16+8247Z18+10101Z20+11636Z22+12411Z24+12531Z26+11742Z28+10395Z30+8488Z32+6490Z34+449

12

7Z36+2857Z38+1581Z40+766Z42+294Z44+86Z46+13Z48+1Z50
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)

D=9; J=2
NUMERATOR:
+3Z2+20Z4+88Z6+214Z8+447Z10+729Z12+1089Z14+1417Z16+1726Z18+1882Z20+1917Z22+1754Z24+1499Z26+1143Z28+812Z30+486Z32+262Z34+101Z36+32Z38+4Z4
0
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)

D=9; J=3
NUMERATOR:
+1Z4+2Z6+9Z8+11Z10+15Z12+15Z14+14Z16+7Z18+6Z20+1Z22
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)

D=9; J=4
NUMERATOR:
+1Z20
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z10)

*****THE POINCARE SERIES FOR D=9*****:
NUMERATOR:
+1Z0+1Z4-1Z6+5Z8+3Z10+18Z12+15Z14+44Z16+43Z18+82Z20+76Z22+122Z24+107Z26+147Z28+119Z30+147Z32+107Z34+122Z36+76Z38+82Z40+43Z42+44Z44+15Z46
+18Z48+3Z50+5Z52-1Z54+1Z56+1Z60
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

D=10; J=0
NUMERATOR:
+1Z0+1Z1+8Z2+21Z3+50Z4+88Z5+161Z6+244Z7+372Z8+510Z9+678Z10+835Z11+1012Z12+1137Z13+1256Z14+130Z15+1320Z16+1257Z17+1178Z18+1028Z19+881Z20
+704Z21+546Z22+390Z23+277Z24+169Z25+104Z26+53Z27+27Z28+9Z29+4Z30
DENOMINATOR:
(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)

D=10; J=1
NUMERATOR:
+3Z1+6Z2+15Z3+25Z4+39Z5+55Z6+72Z7+83Z8+98Z9+103Z10+104Z11+98Z12+90Z13+73Z14+60Z15+42Z16+29Z17+17Z18+9Z19+2Z20+1Z21
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z2)(1-Z7)(1-Z7)(1-Z2)(1-Z9)

D=10; J=2
NUMERATOR:
+1Z1+2Z2+9Z3+15Z4+26Z5+35Z6+52Z7+60Z8+74Z9+77Z10+80Z11+72Z12+69Z13+52Z14+42Z15+29Z16+19Z17+8Z18+5Z19+1Z20
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)

D=10; J=3
NUMERATOR:
+1Z3+2Z5+2Z6+3Z7+2Z8+2Z9+1Z10+2Z11+1Z12
DENOMINATOR:
(1-Z1)(1-Z4)(1-Z3)(1-Z1)(1-Z2)(1-Z5)(1-Z3)(1-Z7)

D=10; J=4
NUMERATOR:
+1Z1C

DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)

*****THE POINCARE SERIES FOR D=10*****:
NUMERATOR:
+1Z0-1Z2-1Z3+3Z6+2Z8+2Z9+4Z10+5Z12-1Z13+1Z14+3Z16-4Z17+4Z18-3Z19-1Z21+1Z22-6Z23-4Z25-2Z26-2Z27-3Z29+1Z32+1Z33-1Z35
DENOMINATOR:
(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)

D=11;   J=0
NUMERATOR:
+1Z0+1Z2+153Z4+861Z6+3516Z8+11108Z10+29737Z12+69185Z14+145026Z16+277592Z18+493864Z20+829988Z22+1297562Z24+1944356Z26+2786068Z28+3828074
Z30+5063386Z32+6450374Z34+7964750Z36+9506816Z38+11002659Z40+12353603Z42+13472597Z44+14702066Z48+14719445Z50+14330077Z52+1355
827Z54+12467535Z56+11131643Z58+9647262Z60+8103500Z62+6592623Z64+5183387Z66+3933427Z68+2871R89Z70+2013147Z72+1348408Z74+860020Z76+518372
Z78+293572Z80+154219Z82+74416Z84+32197Z86+12274Z88+3912Z90+1004Z92+177Z94+20Z96
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z6)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)

D=11;   J=1
NUMERATOR:
+2Z22+2012Z4+982Z6+316Z8+8046Z10+1704Z12+31863Z14+53525Z16+83060Z18+119965Z20+163632Z22+211257Z24+260380Z26+306277Z28+345814Z30+374338Z
32+390158Z34+390706Z36+377030Z38+349333Z40+311295Z42+265496Z44+216902Z46+166610Z49+124653Z50+86697Z52+56499Z54+33846Z56+18424Z58+8780Z60
+3566Z62+1127Z64+251Z66+27Z68+1Z70
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z4)(1-Z12)(1-Z14)(1-Z16)(1-Z16)(1-Z22)(1-Z20)

D=11;   J=2
NUMERATOR:
+8Z2+9Z4+56Z6+213Z8+607Z10+1420Z12+2896Z14+5224Z16+8611Z18+13061Z20+18535Z22+24779Z24+31340Z26+37669Z28+43235Z30+47351 5Z32+473515232
+49659Z34+49789Z36+47826Z38+43898Z40+38550Z42+32251Z44+25707 5Z46+19385Z48+13804Z50+9168Z52+5650Z54+3163Z56+1591Z58+6887Z60+2
508Z62+685Z64+133Z66+11Z68
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z8)(1-Z10)(1-Z12)(1-Z12)(1-Z16)(1-Z16)(1-Z18)

D=11;   J=3
NUMERATOR:
+9Z4+61Z6+26Z28+770Z10+1863Z12+3709Z14+6596Z16+10459Z18+15366Z20+20812Z22+26538Z24+31595Z26+35635Z28+37787Z30+38081Z32+35148Z34+32587Z36
+27549Z38+22018Z40+16373Z42+11408Z44+7254Z46+4228Z48+2143Z50+946Z52+328Z54+87Z56+12Z58+1Z60
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z6)(1-Z2)(1-Z12)(1-Z14)(1-Z16)

D=11;   J=4
NUMERATOR:
+3Z8+6Z10+19Z12+29Z14+50Z16+67Z18+84Z20+89Z22+97Z24+84Z26+74Z28+54Z30+39Z32+20Z34+12Z36+2Z38+1Z40
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)

D=11;   J=5
NUMERATOR:
+1Z30
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10))(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)

*****THE POINCARE SERIES FOR D=11*****:
NUMERATOR:
+1Z0+1Z4-1Z6+1Z8+11Z10+58Z12+85Z14+222Z16+336Z18+660Z20+951Z22+1589Z24+2154Z26+3188Z28+4080Z30+5510Z32+6633Z34+8310Z36+9443Z38+11059Z40
+1189Z42+1309Z44+1331Z46+1385Z48+1331Z50+1309Z52+1189Z54+1105Z56+9443Z58+8310Z60+6633Z62+5510Z64+4080Z66+3188Z68+2154Z70+1589Z72
+951Z74+660Z76+336Z78+222Z80+85Z82+58Z84+11Z86+10Z88-1Z90+1Z92+1Z96

DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)

D=12;   J=0
NUMERATOR:
+1Z0+1Z1+1Z2+2Z3+5Z4+10Z5+18Z6+27Z7+39Z8+52Z9+64Z10+76Z11+86Z12+93Z13+972Z14+976Z15+943Z16+876Z17+784Z18+667Z19+541Z20+413Z22
1+296Z22+194Z23+117Z24+63Z25+29Z26+11Z27+4Z28
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z1)(1-Z3)(1-Z4)(1-Z7)(1-Z5)(1-Z1)(1-Z2)

D=12;   J=1
NUMERATOR:
+541+2Z2+6Z3+16Z4+37Z25+71Z6+127Z7+213Z8+326Z9+479Z10+67,6Z11+899Z12+1195Z13+1439Z14+1728Z15+2008Z16+2253Z17+2472Z18+2626
8Z19+2709Z20+2717Z21+2648Z22+2509Z23+2308Z24+2062Z25+1785Z26+1497Z27+1215Z28+949Z29+713Z30+514Z31+354Z32+2314Z33+1417Z34+81
4Z35+424Z36+201Z37+81Z38+27Z39+5Z40+1Z41
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z1)(1-Z7)(1-Z8)(1-Z9)(1-Z2)(1-Z11)

D=12;   J=2
NUMERATOR:
+1Z1+7Z2+17Z3+37Z4+68Z5+112Z6+161Z7+221Z8+278Z9+332Z10+369Z11+396Z12+394Z13+382Z14+340Z15+293Z16+234Z17+179Z18+120Z19+80Z20+43Z21+22Z22+
7Z23+3Z24
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z1)(1-Z7)(1-Z2)(1-Z9)(1-Z5)

D=12;   J=3
NUMERATOR:
+1Z2+2Z3+9Z4+15Z5+26Z6+35Z7+52Z8+60Z9+74Z10+77Z11+80Z12+72Z13+69Z14+52Z15+42Z16+29Z17+19Z18+8Z19+6Z20+1Z21
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)(1-Z3)

D=12;   J=4
NUMERATOR:
+1Z5+2Z7+2Z8+3Z9+2Z10+2Z11+1Z12+2Z13+1Z14
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z3)(1-Z7)(1-Z4)

D=12;   J=5
NUMERATOR:
+1Z15
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)

*****THE POINCARE SERIES FOR D=12*****:
NUMERATOR:
+1Z0-1Z2+1Z4+2Z6+4Z7+4Z8+4Z9+10Z10+10Z11+15Z12+20Z13+25Z14+26Z15+35Z16+35Z17+38Z18+37Z19+38Z20+28Z21+26Z22+13Z23+6Z24-6Z25-13Z26-26Z27-2
8Z28-38Z29-37Z30-38Z31-35Z32-35Z33-26Z34-25Z35-20Z36-15Z37-10Z38-10Z39-4Z40-4Z41-4Z42-2Z43-1Z45+1Z47-1Z49
DENOMINATOR:
(1-Z2)(1-Z1)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z11)

D=13;   J=0
NUMERATOR:
+1Z0+23Z2+37Z4+287Z6+1512Z8+6070Z10+20150Z12+57496Z14+145802Z16+335084Z18+709854Z20+1401984Z22+2607171Z24+459763Z26+7736784
5Z28+12482153Z30+19380085Z32+29087026Z34+42270430Z36+596397419Z38+818633011Z40+1094992432Z42+14293919Z44+18231983Z46+2274785102Z
48+277881211Z50+332620051Z52+393870722Z54+449536611Z56+508119388Z58+564012895Z60+61500570026Z62+6589717995Z64+6939720382Z66+7184275
87Z68+7311909549Z70+7316709259Z72+7198328171Z74+696247822Z76+662002460Z78+618665826Z80+568134903Z82+512544690528Z84+45408717174Z86+394

9118422288+336073304332Z90+281953951622Z92+231177108394Z94+185594309796Z96+14575631772Z98+1119631012Z100+8378934422102+6116977229Z104+4344835Z210
6+2996901812108+200259819721I13+12929423921I2+8037993621114+479332202116+27281692211 8+1473648822120+7496772122+356040321124+15900521Z26+6203
50Z128+219297721130+670927132+169152134+3323Z136+4402138+332140
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z2)

D=13; J=1
NUMERATOR:
+4Z22+6412+4+48402 6+240407 8+911092 10+282918212+756662214+179400421 6+386126621 8+765954022 0+14180466 722+2472242 4+24+408971972 6+645598012 28
+977184073 0+1423569642 32+200254644732+2722702924236+462769288240+578640027242+705296968744+8399627 1246+974894735248+1107244
946Z50+123604090725 2+1338384289 54+1425518690Z56+1486402715258+15185989912160+15200280792 62+1490053557264+1431884069266+134720681126 6+12
41135432270+1119911313022+987183163274+8513 288221276+7172340702 78+589748993282+369031893284+280046452 86+206220207 768+147052
45Z90+10127077 5292+671526192294+427080387298+25935191298+149495917100+81233622 102+412186721104+193102207 106+82176072108+311266211 0+101965211
2+27437Z114+56677Z115+793211 8+54Z120+1Z122
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)

D=13; J=2
NUMERATOR:
+192Z+323Z4+245026+11080Z 8+37352Z10+100760Z12+232740214+474244216+877302218+1496115220+2387579Z22+3595701224+5153833Z26+706429722 8+930 56
79Z30+1181256623 2+1449375823 4+1721680236+1983885123 8+2219506240+24139617 42+255306 9744+2631384748+2564853950250+2432196825
2+224362027 54+2015136256+1774722 58+1479540620 60+121052 8752+957138926 4+72988602 66+53479392 68+375201627 0+2504759272+15813167 74+93392527
6+51039127 8+2530522 80+111544282+4197878 84+1292328 86+291728 6+661288+7429 0+2629 2
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z16)(1-Z14)(1-Z16)(1-Z20)(1-Z20)(1-Z22)

D=13; J=3
NUMERATOR:
+1Z2+53Z4+499Z6+2788Z8+10959Z10+34202Z12+89031Z14+202328Z16+410514Z18+760221Z20+1301267222+208501972 24+3150499228+6200838230+8
15008232+1025722Z34+125425902 36+147481642 38+1677188572 40+19686236244+20346716246+2039338 1248+1981348250+1866726752+170353
13Z54+1505434057 56+12862036Z58+106161107 60+84455902 62+64648027 64+47446967 66+33282787 68+2218624770+139772472 72+8242811 74+4508607 76+225129 77
8+1009880Z80+39527728Z2+1309828Z84+3436786Z86+661288+7429 0+26292
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)

D=13; J=4
NUMERATOR:
+1Z4+13Z6+113Z8+500210+1726212+46982 14+111022 16+228542 18+428142 Z20+732442 22+1169482 24+174808 226+247443228+332087230+425609232+521008234+6
12010236+68965923 8+747865240+779577242+782912244+756126246+7033657 48+6285825 50+5400832 52+4444972 54+3504817 56+2630927 58+18354260+127225Z
62+8098720 64+47864266 6+2619225 68+1289027 0+560272+211927 74+665576+150278+25280+1282
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)

D=13; J=5
NUMERATOR:
+1Z1C+2Z12+13Z14+28Z16+58Z18+10Z20+174Z22+245Z24+347226+438228+532230+591232+646234+638236+620238+552240+473242+369244+288246+191248+12
4Z50+71Z52+38Z54+13Z56+7258+1Z60
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)

D=13; J=6
NUMERATOR:
+1Z42
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)

*****THE POINCARE SERIES FOR D=13*****:
NUMERATOR:
+1Z0-1Z2+2Z4-2Z6+2Z8+1Z10+145Z14+195Z14+72Z16+192Z18+2923720+4187Z22+972622 24+1539Z226+270972228+4101Z730+65481Z32+94216Z234+13975Z236+
19657Z38+26722Z740+35074Z42+4639798Z44+58378746+73727774 8+89248Z750+10803652 52+126068Z754+146772Z56+16531417Z58+18559326Z60+20201227Z62+

16

2190088Z64+2306118Z66+2417065Z68+2462771Z70+2497692Z72+2452771Z74+2417065Z76+2306118Z78+2190088Z80+2020122Z82+1855934Z84+1653141Z86+1467
752Z88+1260687Z90+1080368Z92+892483Z94+737277Z96+583878Z98+463798Z100+350476Z102+267221Z104+191657Z106+139755Z108+94216Z110+65481Z112+41
017Z114+27097Z116+15392Z118+9726Z120+4481Z122+2923Z124+1192Z126+721Z128+195Z130+145Z132+112Z134+21Z136-22Z138+21Z140-1Z142+1Z144
DENOMINATOR:
(1-Z2)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)

D=14; J=0
NUMERATOR:
+1Z0+3Z1+28Z2+113Z3+390Z4+1078Z5+2717Z6+6047Z7+12570Z8+24150Z9+43995Z10+75811Z11+125325Z12+198553Z13+304184Z14+450521Z15+648585Z16+90783
8Z17+1240034Z18+1653185Z19+2156821Z20+2754202Z21+3448978Z22+4236090Z23+5110360Z24+6057225Z25+7057225Z26+8087789Z27+9123203Z28+10129010Z29+11029010Z29+11029
9+110764172Z30+119286039Z31+126588848Z32+132352473Z33+136359373Z34+138593734Z35+138680Z36+136805Z37+133050Z38+127501Z39+120421Z40+112
2037Z41+102696852Z42+92668538Z43+82369946Z44+72024632Z45+61967Z46+52407Z47+44355Z48+35529Z49+28480Z50+22358Z51+17198Z52+12993
490Z53+950952Z54+681331Z55+476497Z56+322161Z57+211652Z58+134005Z59+81788Z60+47603Z61+26467Z62+13805Z63+6793Z64+3034Z65+1258Z66+444Z67+14
1Z68+33Z69+7Z70
DENOMINATOR:
(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z1)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z2)

D=14; J=1
NUMERATOR:
+7Z1+28Z2+94Z3+243Z4+529Z5+1020Z6+1782Z7+2865Z8+4304Z9+6098Z10+8223Z11+10621Z12+13220Z13+15916Z14+18607Z15+21163Z16+23453Z17+25339Z18+26
703Z19+27438Z20+27497Z21+26866Z22+25608Z23+23798Z24+21567Z25+19045Z26+16370Z27+13663Z28+11044Z29+8605Z30+6431Z31+4578Z32+3081Z33+1937Z34+
+1128Z35+595Z36+278Z37+112Z38+37Z39+7Z40+1Z41
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z1)(1-Z4)(1-Z5)(1-Z1)(1-Z1)(1-Z7)(1-Z2)(1-Z13)

D=14; J=2
NUMERATOR:
+3Z1+20Z2+102Z3+312Z4+845Z5+1901Z6+3926Z7+7310Z8+12836Z9+21045Z10+32964Z11+49133Z12+70572Z13+97463Z14+130525Z15+169144Z16+213385Z17+2616
61Z18+313191Z19+365390Z20+416937Z21+464601Z22+506927Z23+540826Z24+565378Z25+578240Z26+579723Z27+566682Z28+546755Z29+514225Z30+473792Z31+
426634Z32+376034Z33+323460Z34+271919Z35+226605Z36+177718Z37+137675Z38+103694Z39+75410Z40+53054741+35765742+23168743+14186744+8261745+444
7Z46+2236Z47+989Z48+397Z49+122Z50+33Z51+4Z52
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z1)(1-Z6)(1-Z1)(1-Z7)(1-Z8)(1-Z9)(1-Z11)(1-Z12)

D=14; J=3
NUMERATOR:
+3Z2+12Z3+377Z4+96Z5+194Z6+356Z7+593Z8+917Z9+1314Z10+1903Z11+2332Z12+2907Z13+3466Z14+3997Z15+4423Z16+4764Z17+4931Z18+4959Z19+4809Z20+4534
7Z21+410Z22+361Z23+304Z24+248Z25+192Z26+143Z27+100Z28+67Z29+40Z30+23Z31+11Z32+5Z33+15Z34+5Z35
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z3)(1-Z5)(1-Z1)(1-Z3)(1-Z5)(1-Z7)(1-Z9)(1-Z11)

D=14; J=4
NUMERATOR:
+3Z4+6Z5+20Z6+32Z7+59Z8+98Z9+161Z10+204Z11+292Z12+340Z13+432Z14+460Z15+535Z16+529Z17+564Z18+514Z19+507Z20+425Z21+387Z22+293Z23+246Z24+16
5Z25+125Z26+70Z27+48Z28+21Z29+12Z30+2Z31+1Z32
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z1)(1-Z5)(1-Z2)(1-Z5)(1-Z7)(1-Z8)(1-Z3)(1-Z10)

D=14; J=5
NUMERATOR:
+1Z8+2Z9+2Z10+2Z11+5Z12+5Z13+7Z14+7Z15+7Z15+5Z17+6Z18+4Z19+5Z20+3Z21+2Z22+1Z24
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z5)(1-Z1)(1-Z2)(1-Z3)(1-Z5)(1-Z7)(1-Z9)

D=14; J=6
NUMERATOR:
+1Z21
DENOMINATOR:

$(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)$

*****THE POINCARE SERIES FOR D=14*****:
NUMERATOR:
+1Z0+1Z4-1Z5+5Z7+18Z8+21Z9+56Z10+72Z11+155Z12+209Z13+375Z14+523Z15+836Z16+1131Z17+1695Z18+2234Z19+3132Z20+4029Z21+5371Z22+6691Z23+85
66Z24+10348Z25+12736Z26+14971Z27+17789Z28+20306Z29+23460Z30+25973Z31+29023Z32+31385Z33+34068Z34+35858Z35+37893Z36+38831Z37+39932Z38+3989
0Z39+39932Z40+38831Z41+37893Z42+35858Z43+34068Z44+31385Z45+29023Z46+25973Z47+23460Z48+20306Z49+17789Z50+14971Z51+12736Z52+10348Z53+8566Z
54+6691Z55+5371Z56+4029Z57+3132Z58+2234Z59+1695Z60+1131Z61+836Z62+523Z63+375Z64+209Z65+155Z66+72Z67+56Z68+21Z69+18Z70+5Z71+5Z72-1Z73+1Z77
+1Z78
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)$

D=15;   J=0
NUMERATOR:
+1Z0+3Z2+7B1Z4+684Z6+39203Z8+153746Z10+548461Z12+1541113Z14+3776418Z16+8263874Z18+15480492Z20+30410166Z22+52520437Z24+85693905Z26+1331
20549228+19804653Z30+28361359B2Z32+39252060Z34+52686026Z36+68774836Z38+86774836Z38+86774836Z40+10877550682Z42+13229493470Z44+15747714302Z46+18362E+69
9Z34Z48+21059783Z16Z50+23637280B2Z52+26174074Z0Z54+28467Z38782Z56+31885832602Z58+33509449164Z60+33509449164Z62+33509449164Z64+33581871455Z66+3330
017998Z766+31971606Z70+30472Z699192Z72+28566731942Z74+26332731832Z76+23853351278Z78+21235Z75042Z80+18562621692Z82+15208566428Z84+13381436228Z86
+110276747Z682+888506113Z290+699496301Z292+5367454222Z94+400674476296Z96+290114572298Z100+136636382Z102+883803132Z104+543502626Z106+31
602454Z108+17208650Z7110+867638772Z112+3990534Z114+1643369Z116+589934Z118+189222Z120+43342Z122+78477Z124+907Z126+547Z128
DENOMINATOR:
$(1-Z4)(1-Z2)(1-Z8)(1-Z2)(1-Z4)(1-Z6)(1-Z2)(1-Z4)(1-Z6)(1-Z26)(1-Z2)(1-Z28)(1-Z2)(1-Z13)$

D=15;   J=1
NUMERATOR:
+1772+1Z66Z4+166876+10648Z6+50504Z10+192887Z12+6247203Z14+17744574Z16+4592581718+105731137Z20+228818638722+46389139024+6841661072
6+161806625728+281810306730+47148423082Z32+7605563172Z34+11871573921Z36+17972843588Z38+25455901838Z40+37940223493Z42+531035918755Z44+726
4031985574Z46+9725326061Z248+1275793832Z19250+1641514394Z24Z52+2073410211Z254+25730167350Z256+313920660166Z58+376775891173Z260+445111804407Z
62+5178277405907Z64+5934932320266Z66+6703789778446Z68+744517181198270+81977423314Z272+88794810090837Z74+948867977037Z76+1005076870357B+1041
0868643492780+1069172159324282+1083751285539294+1084295176842786+107678261764288+10437068299517B90+100404213967192+9531945073B9294+89291
84744347Z96+825272794798+75222726772Z98+75222726772Z100+67444290647106+45056589045Z108+318500235512Z110+3185416
4350Z112+261424136933Z114+210943308396Z116+167234546143Z118+13016490068072120+9937570615972122+7434482226072124+5443923137726+389669717
3Z128+27Z24481007Z130+18533315340Z132+12269277555Z134+78804793612138+29359326072140+1691205005142+931931647144+448861329
3Z146+24213120172148+112487892Z153+4648730002152+191392771542+67982233156+212229587156+563615Z160+121205Z162+1948071642+203972166+997168+121717
0
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)(1-Z28)$

D=15;   J=2
NUMERATOR:
+4122+106624+1159026+760842Z8+3637537210+1379222712+143959902Z14+12205126216+303167417218+686317332Z20+1435802062Z0722+28120860462Z24+51911261010726+
91011730112Z28+1524180025Z730+244959604372Z32+379295585123Z34+56767331922Z36+82350592823B238+11606377290740+1502459993642742+2130723496644+27843484
461246+35580570026824B+445125427672Z50+545693287Z2702652+774231525B+102180707614260+114388696219262+125993743
0763576B4+136391172822660+13727564162278B4+1269364452257286+115471929392788+10330682682742290+908608828492292+78528325577794Z2966692273479266566693273479Z96+555351797
4298+453618325942100+363443828328326B2B+284968519624104+218527710532106+163683199477108+1195793292221108+11957932922110850583995932112+587938749872114+39399
0561221162552772650Z118+1593954219712Z0+95541976212Z122+5470072912124+29801683321262B+153122694Z128+73627335723Z130+327593932132+1331593427134+44
8479571362+143073Z138+138397140+8886Z142+139327144+137927146+567148
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)$

D=15;   J=3
NUMERATOR:
+522+22024+24624Z6+148368+622877210+203060Z12+52988214+130019Z216+272138021B+518444Z5Z20+912028822Z24+150027Z12Z24+232779242Z26+3432759 5Z28+4
8369794Z30+654Z29688732+852Z43577Z34+107Z2777737Z36+13067714623B+154365862Z42+1770554862Z42+19742330642Z4+226130822482Z46+232511504Z
50+232859742922Z52+22712307Z7254+2157383128Z56+199479169Z58+1779463272Z60+156956218Z262+1333216864Z64+109824372Z66+8759864526B+6749944127Z0+50121
2B2272+3573528Z74+2436505Z76+157954543Z78+967050526Z80+55395352B2+29396416290+719822Z92+1760529416290+719822Z92+17605294Z3053962Z96+294
Z98+97Z100

DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z10)(1-Z4)(1-Z2)(1-Z16)(1-Z4)(1-Z2)(1-Z20)(1-Z22)(1-Z8)

D=15;   J=4
NUMERATOR:
+1024+187Z6+166Z18+9137Z10+380Z12+127672Z14+365451Z16+917298Z18+207137Z20+427249Z22+8165350Z24+1459735Z26+2462648Z28+3946428Z30+6
940695Z32+8870199Z34+1254457O4Z36+1713229657038+2265441957040+2905970612742+3622343428744+4439497967046+5191370537048+598120187Z50+677249705
252+7383098542 54+7918779812 56+83604661 58+850403981260+8517698182 62+8340556112 64+79802175266+74802175268+6825075927 70+6091279272 72+53
0548308274+4 76666214776+3729671802 78+300412503280+235169296282+17864794284+1314249 0396+93421234 88+6397004790+4205468792+2642177929 294
+1578485929 6+6944938298+4708429710 0+230829710 2+107244Z104+419581Z106+14988Z108+4566Z110+1149Z12+274Z114+15Z118
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z8)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z16)(1-Z18)(1-Z20)(1-Z22)

D=15;   J=5
NUMERATOR:
+1Z6+13Z8+113Z10+500Z12+1726Z14+4698Z16+11102Z18+22854Z20+42814Z22+73244Z24+116948Z26+174808Z28+247443Z30+332087Z32+425609Z34+521003Z36+
6120Z38+68926O9Z40+7478652 42+7795772 44+78291Z246+75615Z248+703365Z50+628585Z52+5406O3Z54+444497256+350481Z58+263309260+1883542662+127225
2Z64+8098Z766+4766Z268+2619Z270+1289Z272+568Z074+211Z276+665Z78+150Z80+25Z82+1Z84
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z2)(1-Z6)(1-Z4)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z4)

D=15;   J=6
NUMERATOR:
+1Z14+2Z16+13Z18+28Z20+58Z22+101Z24+174Z26+2457 28+347Z30+436Z32+532Z34+59 Z36+646Z38+638Z40+620Z42+552Z44+447Z46+369Z48+288Z50+191Z52+12
4Z54+71Z56+38Z58+13Z260+7Z62+1Z64
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z8)(1-Z14)(1-Z16)(1-Z5)

D=15;   J=7
NUMERATOR:
+1Z56
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z4)(1-Z2)(1-Z12)(1-Z14)(1-Z16)(1-Z10)(1-Z24)(1-Z14)(1-Z16)

*****THE POINCARE SERIES FOR D=15******

NUMERATOR:
+1Z0-1Z2+3Z4-2Z6+3Z8+44Z10+334Z12+593Z14+2431Z16+524Z18+13435Z20+27349Z22+58848Z24+111154Z26+212550Z28+37415Z30+654965Z32+1082426Z34+
1767075Z36+2762925Z38+4259995Z40+633993Z74+929729Z44+1325904Z45+1862687Z48+2555300Z50+345516Z79Z52+457761015Z54+59777469Z56+76659758Z
58+9702920Z60+12077843Z62+14864903Z64+17971825Z66+214966329Z68+25346953Z70+29545784Z72+33978157Z74+38644858Z76+43390871Z78+47844819
100+73210808O7102+71292917Z5104+68628887106+65396926Z108+61915260110+5741350257112+52881395711 4+481986117Z116+43906791Z118+386448
5877120+339731577Z122+29545788Z124+24966502Z128+179718Z52130+148469032132+120774843134+970292052136+766597582138+5977
746941440+45761015Z42+34551679144+25553002146+18628742148+132590492150+92979512152+639393Z154+42569792156+27629525Z158+176707572160+10
82426Z162+654965Z164+374115Z166+212550Z168+1115547170+58848Z172+27349Z174+13435Z176+5241Z178+2431Z180+693Z182+334Z184+447Z186+34Z188-2Z19
0+37Z192-1Z194+1Z196
DENOMINATOR:
(1-Z2)(1-Z8)(1-Z10)(1-Z10)(1-Z12)(1-Z12)(1-Z4)(1-Z4)(1-Z14)(1-Z16)(1-Z22)(1-Z22)(1-Z24)(1-Z26)(1-Z28)

D=16;   J=0
NUMERATOR:
+1Z0+4Z1+36Z2+153Z3+52Z4+150Z5+380Z6+860Z7+1789Z078+345Z29+6262Z10+10756Z6Z11+17613Z12+27651Z313+41790Z5214+61023Z715+86377Z5Z16+1188
295Z17+159245Z718+208274Z8Z19+266319Z220+333859Z221+409010Z22+492705Z23+584926Z24+677004Z225+773956Z226+870660Z227+964336Z228+105191949
Z29+1130722973 0+1197744173 1+1250669123 2+128749642 33+13068579Z34+130800929Z36+129089993236+12505186423 8+113978912 39+10623842242+
9756019Z41+8826080 Z42+7851059Z24 3+689068Z6Z44+5940938Z45+5034920Z46+419 1230Z247+34239312 48+27420762 49+21503890 50+16488872 51+12343072 52+9002
68253+63393900 54+438915255+29177Z56+18679Z57+1147007O58+671Z259+3732326O+194827261+948Z762+4230Z63+1705264+60226526+1827664+1267+8Z68
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z7)(1-Z9)(1-Z1)(1-Z13)(1-Z1)(1-Z27)(1-Z15)(1-Z2)

D=16;   J=1
NUMERATOR:
+1Z1+79Z2+402Z3+1517Z4+4773Z5+1300Z6+3179Z7+7095Z8+14718Z9+28648Z10+5282812Z11+928867Z12+156654Z13+254527011Z14+399932Z15+60954543Z1
6+490359712Z17+1305635121Z18+1842461412Z19+254319812820+348873512Z21+4560041012Z22+595969612Z23+759596912Z24+955846892Z5+118383340Z26+14440334827Z7+1
7357594828+205708867Z29+240468763Z30+273863382Z31+315849844Z32+3551289226733+394381597Z34+432692894Z35+469097175Z36+502627550Z37+532343430
9Z38+557385413Z39+577700247Z40+590597620Z41+59774564442+59822626247243+592020018Z44+579259392Z45+56053508Z46+53626434Z47+5071226924Z48+
47408657849249+480321992Z59+39993503Z51+360758159Z52+321430662Z53+28280186Z54+245624887Z55+210522607Z56+17991787Z57+148380521758+12190Z03
9987599804467Z60+78573468761+615608747Z62+53843599Z64+255849882Z65+193179532Z66+137344042Z67+9538097268+645880426Z69+42502977Z70+
2720477271+16826777Z72+1003433273+574154Z74+313678Z75+16243862Z76+791217Z77+35809278+14982Z79+5541Z70+1810281+491282+104Z83+13Z84+1Z85
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z2)(1-Z15)

D=16;   J=2
NUMERATOR:
+4Z1+36Z2+143Z3+425Z4+1069Z5+2328Z6+4554Z7+8168Z8+13599Z9+21271Z10+31506Z11+4448Z12+6019Z13+78438Z14+9876Z15+12056Z16+14301Z17+1651
79Z18+1860111Z19+204451Z20+21946Z21+230189Z22+23596Z23+23594Z24+23150Z25+22157Z26+2072Z127+189281Z28+16877Z29+14756730+12428Z431+
1023204432+3168973+63071Z34+46915Z35+33472Z36+22799Z37+14714Z38+8931Z39+5052Z40+2621Z41+1225Z42+503Z43+173Z44+43Z45+8Z46
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z2)(1-Z2)(1-Z5)(1-Z7)(1-Z2)(1-Z13)(1-Z13)(1-Z77)

D=16;   J=3
NUMERATOR:
+9Z2+62Z3+283Z4+49Z5+270Z6+557Z7+144Z8+2899Z9+54280Z10+95604Z11+16000Z12+255686Z13+392668Z14+581551Z15+83377016+116013617+15708
0718+207298020Z19+267155120+335620721+41525872Z22+5019504Z23+591935Z24+592485Z25+591342Z26+886586027+9808874Z28+10647655Z29+11370122
230+119458147Z31+123514382Z32+12568707Z33+12588805Z34+12410181Z35+120414602Z36+1149762137+10800245738+998432539+9075630740+6110100Z41+712
2278242+014309024Z43+5201421Z44+4319875Z45+351653926+280277Z47+2185046Z48+1668064Z49+123531250+89273051+6266035Z42+25881Z53+27951Z54+
176283Z55+1064182Z56+610512Z57+305225Z58+166802Z59+777026Z60+324426Z61+1198762+36726Z63+8826Z64+12765+1Z66
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z3)(1-Z3)(1-Z4)(1-Z71)(1-Z6)(1-Z7)(1-Z2)(1-Z9)(1-Z11)(1-Z12)(1-Z12)(1-Z13)

D=16;   J=4
NUMERATOR:
+3Z3+12Z4+37Z5+96Z6+194Z7+356Z8+593Z9+917Z10+1314Z11+1803Z12+2332Z13+29C7Z14+3466Z15+3997Z16+442Z17+4764Z18+4931Z19+4959Z20+4809Z21+453
4Z22+4102Z23+3612Z24+3042Z25+2480Z26+1924Z27+1438Z28+1091Z29+670Z30+467Z31+230Z32+111Z33+52Z34+15Z35+5Z36
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z5)(1-Z7)(1-Z1)(1-Z11)(1-Z1)(1-Z23)

D=16;   J=5
NUMERATOR:
+1Z5+2Z6+13Z7+28Z8+62Z9+115Z10+209Z11+329Z12+520Z13+738Z14+1035Z15+1363Z16+1757Z17+2150Z18+2594Z19+2981Z20+3368Z21+3661Z22+390Z23+4009Z
24+405Z25+3938Z26+3763Z27+346Z28+312Z29+2707Z30+2313Z31+187Z32+149Z33+1132Z34+838Z35+576Z36+395Z37+241Z38+144Z39+76Z40+39Z41+13Z42+
7Z43+1Z44
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z4)(1-Z5)(1-Z5)(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z78)(1-Z3)(1-Z10)(1-Z11)

D=16;   J=6
NUMERATOR:
+1Z11+2Z12+2Z13+2Z14+5Z15+5Z16+7Z17+7Z18+7Z19+5Z20+6Z21+4Z22+5Z23+3Z24+2Z25+1Z27
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z3)(1-Z1)(1-Z2)(1-Z2)(1-Z5)(1-Z3)(1-Z4)(1-Z9)(1-Z5)

D=16;   J=7
NUMERATOR:
+1Z2E
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)

*****THE POINCARE SERIES FOR D=15******:
NUMERATOR:
+1Z0+1Z3+1Z4+2Z.5+8Z6+11Z7+30Z8+59Z9+112Z10+204Z11+389Z12+655Z13+1146Z14+1896Z15+3067Z16+4823Z17+7477Z18+11167Z19+16506Z20+23764Z21+33593

$$Z22+46576Z23+63588Z24+85049Z25+112273Z26+145668Z27+186298Z28+234773Z29+292016Z30+357883Z31+433493Z32+518058Z33+611720Z34+713572Z35+82299$$
$$Z36+937742Z37+1057289Z38+1178403Z39+1299472Z40+1417560Z41+1530508Z42+1634576Z43+1728646Z44+1808808Z45+1873856Z46+1921520Z47+1951002Z48+$$
$$1960390Z49+1951002Z50+1921520Z51+1873856Z52+1808808Z53+1728646Z54+1634576Z55+1530508Z56+1417560Z57+1299472Z58+1178403Z59+1057289Z60+9377$$
$$42Z61+822997Z62+713572Z63+611720Z64+518058Z65+433493Z66+357883Z67+292016Z68+234773Z69+186298Z70+145668Z71+112273Z72+85049Z73+63588Z74+46$$
$$576Z75+33593Z76+23764Z77+16505Z78+11167Z79+7477Z80+4823Z81+3067Z82+1896Z83+1145Z84+655Z85+389Z86+204Z87+112Z88+59Z89+30Z90+11Z91+8Z92+2Z$$
$$93+1Z94+1Z95+1Z98$$

DENOMINATOR:

$$(1-Z2)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z15)$$