Problemen

| Problem Section

Problem 18 (Alex Heinis)

Let a, k be positive numbers. Count the number of maps $f \colon \mathbf{N} \to \mathbf{N}$ such that $f^k(n) = n + a$ for all $n \in \mathbf{N}$.

Problem 19 (R. Ceulemans, open problem)

A billiard table T has the shape of a regular pentagon, with sides of length of 1 meter. You can put a billiard ball anywhere on the table and shoot it in any direction over a distance of 10 meters. Give the maximal number of times the ball can hit a side.

Problem 20 (A.F. Tiggelaar)

Suppose that ABC and DEF are two triangles in $\mathbf{R}^3$ and that V is the plane that contains ABC. Suppose that AD, BE and CF are perpendicular to V and that $AD = BC$, $BE = AC$, $CF = AB$. Construct the point in V that is equidistant to D, E and F.

Solutions to volume 1, number 4 (December 2000)

Problem 10

Let a, b, c be integers such that the symmetric matrix

$$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

has three integer eigenvalues. Prove, or give a counterexample to, the following statement: either $abc = 0$ or $(a^2 - b^2)(a^2 - c^2)(b^2 - c^2) = 0$.

This open problem has been solved by Ronald van Luijk and by Raimundas Vidunas. The solution will appear in one of the next issues of Nieuw Archief.

Problem 11

An abc-triple is a triple of pairwise coprime positive integers a, b, c with $a + b = c$ for which the product r of the distinct prime numbers dividing abc satisfies $r < c$. Prove that there are infinitely many abc-triples.

Solution by Raimundas Vidunas. For each positive integer k, we have that $9^k = 8N + 1$ for some integer N. This gives an abc-triple $1, 8N, 9^k$ since $r \le 6N < 9^k$.

Problem 12

Prove that there are infinitely many abc-triples for which a is equal to a given positive integer.

Solution by Hendrik Lenstra. Fix any a, and let p be an odd prime number not dividing a. By the Chinese remainder theorem we can choose a primitive root g modulo p^2 with $\gcd(g, a) = 1$. Let m be such that $p^{m-1} > ag$. Then g is a primitive root modulo p^m, so there exist infinitely many n with $g^n \equiv a \bmod p^m$ and $g^n > a$. For any such n, we can take $b = g^n - a$, $c = g^n$. These numbers are pairwise coprime, and since b is divisible by p^m we have $r \le a \cdot (b/p^{m-1}) \cdot g < b < c$.

Problem 13

Let m be a positive integer. Prove that there is an abc-triple with the property that any odd prime number dividing abc exceeds m.

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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Solution by Hendrik Lenstra. We may assume $m \geq 2$. Let k be the product of all prime numbers that are at most m. Denote by φ the Euler-φ-function. All primes dividing $\varphi(k/2)$ are at most m, so they divide k. Hence $\gcd\big((k+1)^2, \varphi(k/2)\big) = 1$, so there exists n with $n \cdot (k+1)^2 \equiv 1 \bmod \varphi(k/2)$. Put $c = (k+2)^{n\cdot(k+1)^2}$, $a = 1$, $b = c - 1$. Modulo the odd number $k/2$, we have $c \equiv 2^{n\cdot(k+1)^2} = 2^{1 \bmod \varphi(k/2)} \equiv 2^1 = 2$, so $b \equiv 1$ and $abc \equiv 2$. Hence abc is not divisible by any odd prime that is at most m. The residue class $k+2 \bmod (k+1)^3$ belongs to the set of units modulo $(k+1)^3$ that are $1 \bmod k+1$. That set is a group of order $(k+1)^2$, so $(k+2)^{(k+1)^2} \equiv 1 \bmod (k+1)^3$ and therefore $c \equiv 1 \bmod (k+1)^3$ and $b \equiv 0 \bmod (k+1)^3$. Therefore $r \leq \big(b/(k+1)^2\big)\cdot(k+2) < b < c$.

Problem 14

Let n be a positive integer. Prove that there exist n different abc-triples with the same value of c.

Solution by Hendrik Lenstra. Choose any $h > 1$ (for example, $h = 2$), and let P be any set of n prime numbers not dividing h. Choose n_0 such that for all $n \geq n_0$ and all $p \in P$ one has $p^{2n-1} > h \cdot (1 + hp^n)$. Choose m such that $h^m \equiv 1 \bmod \prod_{p\in P} p^{n_0}$. For each $p \in P$, let n_p be the number of factors p in $h^m - 1$; then we have $n_p \geq n_0$, and we can write $h^m = 1 + k_p \cdot p^{n_p}$ where p does not divide k_p. Choose l such that for all $p \in P$ one has $h^{ml} \equiv 1 + h \cdot p^{n_p} \bmod p^{2n_p}$; this can be done, since by $h^{ml} = (1 + k_p \cdot p^{n_p})^l \equiv 1 + l \cdot k_p \cdot p^{n_p} \bmod p^{2n_p}$ it suffices to solve $l_p \cdot k_p \equiv h \bmod p^{n_p}$ and to choose l such that for each $p \in P$ one has $l \equiv l_p \bmod p^{n_p}$. Now put $c = h^{ml}$ and $a_p = 1 + h \cdot p^{n_p}$, for $p \in P$, so that $c \equiv a_p \bmod p^{2n_p}$. We have $a_p \equiv 1 \bmod h$, so $\gcd(a_p, c) = 1$ and $a_p \neq c$. From

$$a_p = 1 + h \cdot p^{n_p} < h \cdot (1 + h \cdot p^{n_p}) < p^{2n_p - 1} < p^{2n_p}$$

it follows that a_p is the least number in its residue class mod p^{2n_p}. Since c lies in the same residue class, one has $c > a_p$, so we can write $a_p + b_p = c$. The number b_p is divisible by p^{2n_p}, so we have $r \leq a_p \cdot (b_p/p^{2n_p-1}) \cdot h = b_p \cdot h \cdot (1 + hp^{n_p})/p^{2n_p-1} < b_p < c$. Therefore a_p, b_p, c constitute an abc-triple, for each $p \in P$. Because p is the unique prime number dividing $(a_p - 1)/h$, no two of these triples coincide.

Solutions to some problems of yore

Below are the solutions to the problems 972–979, which belong to the Problem Section of the fourth series of Nieuw Archief.

Problem 972 (H. Alzer)

For any pair x, y of distinct positive real numbers, we define

$$A(x,y) = \frac{x+y}{2}, G(x,y) = \sqrt{xy}, H(x,y) = \frac{2}{1/x + 1/y}.$$

Prove that if $x, y \geq e$ then

$$\begin{aligned} A(x,y)^{H(x,y)} &< \left(\frac{G(x,y) + A(x,y)}{2}\right)^{H(\sqrt{x},\sqrt{y})^2} \\ &< G(x,y)^{G(x,y)} \\ &< H(\sqrt{x},\sqrt{y})^{G(x,y)+A(x,y)} \\ &< H(x,y)^{A(x,y)} \end{aligned}$$

and if $x, y \leq e$ then we have the reverse inequalities.

Solutions by G.W. Veltkamp and H.J. Seiffert. Below is the solution by H.J. Seiffert which is remarkably short compared to the problem. Abbreviate $G = G(x,y)$ and $A = A(x,y)$. Since $G < A$ we have

$$\frac{G^2}{A} < \frac{2G^2}{G+A} < G < \frac{G+A}{2} < A.$$

The function $F(t) = t^{1/t}$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$. Apply f to the inequality and raise to the power G^2, and observe that $H = G^2/A$ and $H(\sqrt{x}, \sqrt{y})^2 = 2G^2/(G + A)$.

Problem 973 (W. Bencze)
Prove for $n = 2, 3, \ldots$ the inequalities

$$\frac{1}{2} + \sqrt{n - \frac{1}{2}} < \sqrt{n + \sqrt{n - 1 + \sqrt{n - 2 + \ldots + \sqrt{2 + \sqrt{1}}}}} < \frac{1}{2} + \sqrt{n + \frac{1}{4}}.$$

Solutions by J.H. van Geldrop, W. van der Meiden, G.W. Veltkamp, C. Jonkers, A.A. Jagers, H.J. Seiffert, R.H. Jeurissen. There was a misprint in the original problem where the final quotient $\frac{1}{4}$ was omitted. Some readers assumed the quotient to be $\frac{1}{2}$ which enables a crisp solution. For $n + \frac{1}{4}$ the problem is only slightly harder and essentially all solutions are the same. Denote the inequality by $\frac{1}{2} + \sqrt{n - \frac{1}{2}} < w_n < \frac{1}{2} + \sqrt{n + \frac{1}{4}}$. Apply induction. Clearly the inequality holds for $n = 2$. Assume it holds for $n - 1$. Squaring the inequality gives the equivalent form

$$\frac{1}{4} + \sqrt{n - \frac{1}{2}} < a_{n-1} < \frac{1}{2} + \sqrt{n + \frac{1}{4}},$$

which follows from the induction hypothesis and the inequality

$$-\frac{1}{4} + \sqrt{n - \frac{1}{2}} \leq \frac{1}{2} + \sqrt{n - \frac{3}{2}}.$$

Problem 974 (M.L.J. Hautus)
Let $f\colon \mathbf{R} \to \mathbf{R}$ be a C^1-function satisfying $|f'(x)/f(x)| \to \infty$ as $x \to \infty$. Show that for every $n \in \mathbb{N}$ and every n-tuple of distinct pairs (a_i, b_i) with $a_i > 0$ and $b_i \in \mathbb{R}$, the functions $g_1, \ldots, g_n$ defined by $g_i(x) = f(a_i x + b_i)$ are linearly independent.

Solution by A.A. Jagers. By the condition on f the zeroes of f' are bounded away from ∞ and hence so are all zeroes of f by a mean value theorem. Four possible cases arise, for x large enough. We only consider the case $f, f' > 0$, the other cases being similar, and use induction on n. Suppose that $\lambda_1 g_1 + \ldots \lambda_n g_n = 0$. Then for x large enough

$$\lambda_n = -\lambda_1 g_1(x)/g_n(x) - \ldots - \lambda_{n-1} g_{n-1}(x)/g_n(x)$$

and for $i < n$ we have that $g_i(x)/g_n(x) \to 0$ as $x \to \infty$. Indeed, for x large enough there exists an intermediate $\xi \in \mathbb{R}$ with $a_i x + b_i < \xi < a_n x + b_n$ such that

$$\log\{f(a_n x + b_n)/f(a_i + b_i x)\} = \log f(a_n x + b_n) - \log f(a_i + b_i x) = f'(\xi)/f(\xi)$$

for $x \to \infty$ by the condition on f. It follows that $\lambda_n = 0$.

Problem 975
No response.

Problem 976 (C. Notari)
Find all pairs (n, m) of positive integers such that $n^2 + n + 1 = m^3$.

Solutions by A.A. Jagers and H.J. Seiffert. Both remark that the solution is classical, $(n, m) = (18, 7)$, and refer to (L.J. Mordell, Diophantine Equations, (1969), p. 208-209). Jagers even includes a relatively recent paper on this problem (N. Tzanakis, *The Diophantine equation $x^3 - 3xy^2 - y^3 = 1$ and related equations*, J. Number Theory **18**, 192-205 (1984)) and he notes that a solution depends on the unique factorization in the ring $\mathbb{Z}[\rho]$ where $\rho = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$.

Problem 977 (J. Ponstein)

For a given $k \in \mathbf{N}$ we define the sequence $a(m)$ by $a(1) = k$ and

$$a(m) = k^m - \sum_{1 \le i < m,\ i|m} a(i).$$

Show that m divides $a(m)$ for all $m \in \mathbf{N}$.

Solutions by R.H. Jeurissen, A.A. Jagers, H.J. Seiffert. All solutions depend on Möbius inversion, which gives that $a(m) = \sum_{i|m} k^i \mu(m/i)$ where μ denotes the Möbius function. Seiffert now concludes by invoking Gauss' generalization of the Fermat Little Theorem: $\sum_{i|m} k^i \mu(m/i) \equiv 0 \mod m$ and the desired result follows.

Problem 978 (F. Rothe)

On a square lattice we consider *lattice triangles*, i.e, triangles with all three vertices lattice points. There exist lattice triangles with $i = 1, 2, 3, 4$ or 9 lattice points in their interior. For $i = 1, 2, 4, 9$ the centre of gravity can be a lattice point. Show that for $i = 3$ the centre of gravity cannot be a lattice point.

Solution by C.B.J. Jonkers, which is surprisingly short compared to the original four page proof by Rothe. Denote the integer lattice by R and denote the middle of lattice points P and Q by $m(PQ)$. We start with two observations. First that for every set of five points $P_i \in R$ there is at least one pair such that $m(P_iP_j) \in R$. Second that if ABC is a lattice triangle with $m(AB)$ and $m(AC)$ in R, then so is $m(BC)$.
Consider a triangle ABC with three interior points in R. Suppose that $Z \in R$ is the center of gravity of ABC. Observe that if $m(AB) \in R$ then $m(CZ) \in R$. There are two cases.

CASE 1. Suppose that $m(AZ), m(BZ), m(CZ)$ are not in R. Then $m(AB)$ cannot be a lattice point, since this would imply that $m(CZ) \in R$. Let P be a second interior point of ABC. By our first observation, one of the middles of A, B, C, Z, P is in R. There are four possibilities $m(AP), m(BP), m(CP)$ or $m(PZ)$, which are all interior points. The first three possibilities are equivalent and to fix ideas assume that $Q = m(PA) \in R$. Apply the first observation to B, C, P, Q, Z to find a fourth interior point, contradicting that the triangle only has three interior points. If $Q = m(PZ)$ then apply the argument to A, B, C, P, Q.

CASE 2. Suppose that $P = m(AZ) \in R$. This is equivalent to $m(BC) \in R$ and our second observation implies that $m(AB), m(AC)$ are not in R. Equivalently, $m(BZ)$ and $m(CZ)$ are not in R. There has to be a third interior point Q. One of the middles of A, B, C, P, Q is in R. Verify that this has to be an interior point, which contradicts that there are only three interior points.

Problem 979 (F.W. Steutel)

Prove that the function $g(x) = \int_0^\infty e^{-t^2} cos(tx + ct^2)\, dt$ is nonnegative for all $c \in \mathbb{R}$ if and only if $c = 0$.

Solutions by K.W. Lau, A.A. Jagers, G.W. Veltkamp. Solution by A.A. Jagers. Let $\varphi = \arg(1 + ic)$ or, equivalently, $c = \tan\varphi$ with $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. If $c = 0$ then $\varphi = 0$ and $g(x) = g(-x) = \frac{1}{2}\sqrt{\pi} > 0$. If $c \neq 0$ then $\varphi \neq 0$ and

$$\begin{aligned}(g(x) + g(-x))/2 &= Re\left(\int_0^\infty e^{-x^2/(4-ic)} \cos(tx)dt\right)\\ &= \frac{\sqrt{\pi}}{2}\, Re\left(\frac{e^{-x^2/(4-4ic)}}{\sqrt{1-ic}}\right)\\ &= \frac{1}{2}\sqrt{\pi\cos\varphi}\, e^{-(x\cos\varphi)^2/4} \cos\Big(\frac{\varphi}{2} - \frac{\sin 2\varphi}{8}x^2\Big),\end{aligned}$$

which is not nonnegative for all $x \in \mathbf{R}$, but changes sign infinitely often.