On the number of specific spanning subgraphs of the graphs $G \times P_n$

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Abstract

This paper investigates the number of spanning subgraphs of the product of an arbitrary graph G with the path graphs P_n on n vertices that meet certain properties: connectivity, acyclicity, Hamiltonicity, and restrictions on degree. A general method is presented for constructing a recurrence equation $R(n)$ for the graphs $G \times P_n$, giving the number of spanning subgraphs that satisfy a given combination of the properties. The primary result is that all constructed recurrence equations are homogeneous linear recurrence equations with integer coefficients. A second result is that the property "having a spanning tree with degree restricted to 1 and 3" is a comparatively strong property, just like the property "having a Hamilton cycle", which has been studied extensively in literature.

1 Introduction

The graph $G'(V', E')$ is a spanning subgraph of the graph $G(V, E)$ when $V' = V$ and $E' \subset E$. The vertices of G are considered to be labeled. For each graph G there exist 2^n possible spanning subgraphs where n is the number of edges of G.

$S^P(G)$ is defined as the set of all spanning subgraphs of a graph G having the property P and $C^P(G)$ is defined to be equal to its cardinality: $|S^P(G)|$.

The basic properties to be studied are: *restrictions on degree* (denoted by D, where D is the set of allowed degrees), *connectivity* (denoted by C), and *acyclicity* (denoted by AC). Special cases or combinations of these properties are known as: *domino tiling* (DT: $D = \{1\}$), *2-factors* ($2F$: $D = \{2\}$), *Hamilton cycles* (HC: $C \wedge D = \{2\}$), *Hamilton paths* (HP: $C \wedge AC \wedge D = \{1, 2\}$), and *spanning trees* (ST: $C \wedge AC$). A property P is one the basic properties or any combination of these. The number of Hamilton cycles of G is usually defined as being equal to $C^{HC}(G)$.

Let $G(V, E)$ and $G'(V', E')$ be graphs. The product graph $G \times G'$ is defined as the graph having the Cartesian product $V \times V'$ as its vertices, while its edge set contains exactly all the pairs $\{(i, j), (i, k)\}$ where $\{j, k\} \in E'$ and all the pairs $\{(i, j), (k, j)\}$ where $\{i, k\} \in E$. Hence $G \times G'$ contains two types of edges. These will be call G'-edges and G-edges, respectively.

The notation $C^P_G(n)$ will abbreviate $C^P(G \times P_n)$, likewise $S^P_G(n)$ abbreviates $S^P(G \times P_n)$.

In [3] and [4], $C^{HC}_G(n)$ is studied where G is one of K_2, P_3, P_4 or K_3. Upper and lower bounds of $C^{HC}(P_n \times P_m)$ are also presented. In [12] and [6, 7], C^{HC}_G is studied where G is P_4 and P_5 respectively. In [1], $C^{HP}_{P_n}$ is studied for $n < 6$. Other related results can be found in [5, 8, 11]. In all of these cases, homogeneous linear recurrence equations with integer coefficients for C^{HC}_G were found.

In the following section, we describe our method for finding the recurrence equation of $C^P_G(n)$. Our method is related to a method which is more generally known as the *transfer-matrix* method, which is discussed in more detail in [10]. First, we apply this method to restrictions on degree, because it is the least complicated property. Second, we apply the method to the connectivity and acyclicity properties. Third, we show how the recurrence equation can be found for any combination of the basic properties. Finally, we give a table of recurrence equations we have found by applying the method to some combinations of properties and graphs.

2 Main method

Let S' be some graph in $S_G^P(n)$. The vertex set of S' consists of all (i,j) where $i \in \{1,\ldots,m\}$, m is the number of vertices in G, and $j \in \{1,\ldots,n\}$. We assume that the vertices of G are labeled with the numbers 1 to m in some arbitrary way.[1]

The vertices of S' can be partitions into n groups where the j-th group contains all (i,j) vertices. The G-edges of S' are between vertices of the same group. The P_n-edges of S' are between vertices of two successive groups.

Which G-edges of $G \times P_n$ are in S' can be described by n spanning subgraphs of G, one for each group. Let $G_1,\ldots,G_n$ be such that $\{i,k\} \in E(G_j)$ iff edge $\{(i,j),(k,j)\}$ is in S'. Which P_n-edges of $G \times P_n$ are in S' can be described by $n-1$ vectors of m elements on $\{1,0\}$, one for each set of edges between two successive groups. Let $A_1,\ldots,A_{n-1}$ be such that $A_j[i]=1$ iff the edge $\{(i,j),(i,j+1)\}$ is in S' (where $A[i]$ denotes the i-th element of the vector A).

The sequence $G_1A_1G_2\ldots G_{n-1}A_{n-1}G_n$ describes the graph S' from 'left' to 'right'. When looking at the sequences of all the graphs in $S_G^P(n)$ for all possible values of n, one might think that there exists a system of states and state transitions by which all the sequences can be generated. Many of the results discovered so far have been found by constructing such systems. We will show that it is indeed possible to construct a system of states and state transitions for each combination of a property P and a graph G.

We take the vectors on m elements as states and the spanning subgraphs of G as state transitions. There also will be a single begin state and a single end state. For some properties there will be more than one state with certain vectors. To make the vectors unique we will use vectors on natural numbers instead of on $\{0,1\}$. For this reason, we redefine the existence of the P_n-edges based on the vectors in the following way: $\{(i,j),(i,j+1)\}$ is in S' iff $A_j[i] > 0$ (instead of $A_j[i]=1$).

In some systems, more than one state transition between two states may exist, but these state transitions will always be labeled by different spanning subgraphs.

We will use directed multi-graphs for representing the state transition systems. The vertices of the directed multi-graphs will represent the states and we will label them with vectors of m elements. The directed edges will represent the state transitions and they will be labeled with spanning subgraphs of G. The vertices of the begin and end states will be denoted by V_b and V_e.

We will use (v_1, v_2, G') to denote the edge from v_1 to v_2 that is labeled with the graph G'.

The graph represented by the walk $v_0e_1v_1\ldots v_{n-1}e_nv_n$ is the graph represented by the sequence $G_{e_1}A_{v_1}\ldots A_{v_{n-1}}G_{e_n}$. In the remaining of the paper we only consider walks for which v_0 equals V_b. But in many cases v_n will not be equal to V_e, as we also have to consider incomplete walks to V_e.

For each property P, we will present a method to construct (for each graph G) a directed multi-graph M_G^P, such that each walk from V_b to V_e over n edges represents a graph in $S_G^P(n)$, and such that for each graph S' in $S_G^P(n)$ there is exactly one walk from V_b to V_e over n edges that represents S'. We define $W_G^P(n)$ as the set of all walks from V_b to V_e over n edges in M_G^P. From this it follows that $|W_G^P(n)| = C_G^P(n)$.

Furthermore, it is known that the number of directed walks over n edges can be described by a homogeneous linear recurrence equation in n with only integer coefficients.

Assuming that the vertices of M_P^G have been numbered, let N be the adjacency matrix of M_G^P such that the value of $N_{i,j}$ equals the number of directed edges from vertex i to vertex j in M_G^P. Let b be the number of vertex V_b and e be the number of vertex V_e. Then $C_G^P(n)$ equals $(N^n\bar{e}_b)[e]$, where $\bar{e}_b$ is the vector from the standard basis whose elements are all equal to 0 except the b-th element which is equal to 1. The Cayley-Hamilton Theorem (see, for example, [9]) from the field of linear algebra states that $\sum_{j=0}^{\ell} p_jN^j$ equals the zero matrix when $\sum_{j=0}^{\ell} p_jx^j$ is the characteristic polynomial of the matrix N, which can be found by calculating the determinant of $N - xI$. If we

[1] This assumption will be used in the remaining part of the paper without further notice.

assume that $p_\ell = 1$, then $N^n = -\sum_{j=1}^{\ell} p_{\ell-j} N^{n-j}$ holds for all $n \geq \ell$. From this equation the recurrence equation $C_G^P(n) = -\sum_{j=0}^{\ell} p_{\ell-j} C_G^P(n-j)$ for all $n \geq \ell$ can be derived.

3 Restrictions on degree

Given a graph $G(V, E)$ and a set of allowed degrees D we present the construction of M_G^D. The vertex set of M_G^D consists of 2^m ($m = |V(G)|$) vertices which are uniquely labeled with all the possible vectors of m-elements on $\{0, 1\}$. The special vertices V_b and V_e of M_G^D are equal to the vertex that is labeled with $\bar{0}$. The edge set of M_G^D consists of all the edges (v_1, v_2, G') where G' is a spanning subgraph of G, such that $d_{G'}(i) + A_{v_1}[i] + A_{v_2}[i] \in D$ for all $1 \leq i \leq m$. $d_{G'}(i)$ denotes the degree of the vertex labeled with i in G'.

Due to the way the restriction is formulated, one can easily see that if an edge (v_1, v_2, G') exists in a graph M_G^D, an edge (v_2, v_1, G') will also exist. Thus, in this case it is also possible to use an undirected multi-graph to represent the state transition system correctly.

3.1 Correctness of M_G^D

To prove that the construction of M_G^D is correct, it is sufficient to prove that for each S' in $S_G^D(n)$ there exists exactly one walk in $W_G^D(n)$, and that each walk in $W_G^D(n)$ represents a graph in $S_G^D(n)$.

Theorem 1 *There is a one-to-one correspondence between the walks in $W_G^D(n)$ and the graphs in $S_G^D(n)$.*

Proof: ($\Rightarrow$) For each walk $V_b e_1 v_1 \ldots v_{n-1} e_n V_e$ in M_G^D, let S' be the graph represented by $G_1, \ldots, G_n$ (the labels with the edges $e_1, \ldots, e_n$) and $A_1, \ldots, A_{n-1}$ (the labels with the edges $v_1, \ldots, v_{n-1}$). Let $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ be the vertices of the graph S'. It is clear that the degree of each vertex (i, j) equals $d_{G_j}(i) + A_j[i] + A_{j+1}[i]$, where $A_0 = A_n = \bar{0}$. According to the definition of the edge set of M_G^D we know that the degree of all the vertices is an element of D. From this we conclude that S' is in $S_G^D(n)$.

($\Leftarrow$) For each graph S' in $S_G^D(n)$ there exists a unique set of $G_1, \ldots, G_n$ and $A_1, \ldots, A_{n-1}$ which represent the graph, where A_i are vectors over $\{0, 1\}$. There exists vertices $v_1, \ldots, v_{n-1}$ in M_G^D labeled with $A_1, \ldots, A_{n-1}$. For each A_i there exists exactly one vertex v_i labeled with A_i by definition.

Let $A_0 = A_n = \bar{0}$. It is clear that for each $1 \leq j \leq n$, $d_{G_j}(i) + A_j[i] + A_{j+1}[i]$ is element of D for all $1 \leq i \leq m$, which implies there must exist an edge (v_j, v_{j+1}, G_j) in M_G^D where $v_0 = V_b$ and $v_n = V_e$.

By definition there exists only one edge between v_j and v_{j+1} that is labeled with G_j. From this we conclude there is exactly one walk $V_b e_1 v_1 \ldots v_{n-1} e_n V_e$ in M_G^D for each graph S' in $S_G^D(n)$. □

Corollary 1 *$C_G^D(n)$ is equal to the number of walks in $W_G^D(n)$ for all n.*

4 Acycliness and connectedness

The properties acyclicity and connectedness are related in the sense that they both depend on the (non-)existence of certain paths in a graph. A graph G is connected iff there exists a path between any two vertices. A graph G is acyclic iff there does not exist a path (of length greater than 0) from a vertex to itself.

In M_G^C and M_G^{AC} we should only allow walks from V_b to V_e for which some global properties about vertices being connected hold in the graphs represented by these walks. However, we can only express these as local conditions, e.g., by putting restrictions on the edges between the vertices of M_G^C and M_G^{AC}.

Consider, for example, the walk $V_b v_1 \ldots v_{n-1} V_e$ in M_G^{AC} that represents a certain graph S' in S_G^{AC}. Let $v_0 = V_b$ and $v_n = V_e$. The existence of the edge (v_{i-1}, v_i, G_i) for some $1 < i \leq n$

should somehow depend on what vertices are connected in the graph S'' represented by the walk $V_b \dots v_{i-1}$. As a consequence, v_{i-1} must be labeled with some kind of connection coding which tells which of the vertices $(\ell, i-1)$ and $(k, i-1)$ are connected in S'' and which are not. We could use a vector on natural numbers with m elements (where m is the number of vertices in G), such that $A_{v_{i-1}}[\ell] = A_{v_{i-1}}[k]$ iff the vertices $(\ell, i-1)$ and $(k, i-1)$ are connected in S''. Remember, however, that $A_{v_{i-1}}$ also tells which of the edges $\{(\ell, i-1), (\ell, i)\}$ appear in S'. Because some $A_{v_{i-1}}[\ell]$ might be equal to 0, only those $A_{v_{i-1}}[\ell]$ unequal to 0 say something about which of the vertices in S'' are connected. For this reason, we will call these vectors *partial connection codings*. We would like to make the partial connection codings unique. This we can do by restricting the natural numbers that can be used, as is done by the following definition.

Definition 1 (partial connection coding) *Given a graph $G(V, E)$ and an ordered subset $(v_1, \dots, v_m)$ of the vertices V, we will call the vector $(a_1, \dots, a_m)$ on natural numbers a partial connection coding of $(v_1, \dots v_m)$ if:*

- *for all $a_i > 0$ and $a_j > 0$ it is true that $a_i = a_j$ iff v_i and v_j are connected in G.*
- *(to make the coding unique) $a_1 \in \{0, 1\}$ and for all $i > 1$, $a_i \leq max(\{a_1, \dots, a_{i-1}\}) + 1$.*

For each vector A on m-elements of natural numbers, we define $N(A)$ as a vector of m-elements of $\{0, 1\}$ such that for all $1 \leq i \leq m$, $N(A)[i]$ is equal to 0 if $A[i] = 0$ and equal to 1 otherwise. Note that if A_1 and A_2 are partial connection codings for a subset $(v_1, \dots v_m)$ of the vertices of a graph, then $N(A_1) = N(A_2)$ implies $A_1 = A_2$.

Returning to our example, if $A_{v_{i-1}}$ and G_i are known, then we should also be able to determine that vertices (ℓ, i) and (k, i) are connected in the graph represented by the walk $V_b \dots v_i$. This can be found by adding edges to the graph G_i for each two vertices that are connected according to partial connection coding $A_{v_{i-1}}$. We will use the notation $G_i[A_{v_{i-1}}]$ for this way of adding edges; it is defined more precisely by the following definition:

Definition 2 *For a graph G where the vertices are labeled with the numbers 1 to m, let A be some vector with m-elements of natural numbers. $G[A]$ is defined as the graph on the vertices of G, where the set of edges is the union of the edges of G and all edges $\{i, j\}$ for which $A[i] = A[j] \neq 0$.*

The following lemma states that if (as in our example) v_i is labeled with a correct partial connection coding for the graph $G_i[A_{v_{i-1}}]$, then this partial connection coding is also correct for the vertices $((1, i), \dots (m, i))$ of the graph represented by the walk $V_b \dots v_i$.

Lemma 1 *Suppose that the spanning subgraph S' of $G \times P_n$ is represented by $G_1, \dots, G_n$ and $A_1, \dots, A_{n-1}$. Further suppose that A_{n-1} is a correct partial connection coding for the vertices $((1, n-1), \dots, (m, n-1))$ of the graph represented by $G_1, \dots, G_{n-1}$ and $A_1, \dots, A_{n-2}$, where m is the number of vertices of G. The vertices (p, n) and (q, n) are connected in S' iff the vertices p and q are connected in $G_n[A_{n-1}]$.*

Proof: ($\Rightarrow$) If (p, n) and (q, n) are connected in S', there must exist a walk from (p, n) to (q, n); let $(a_1, b_1), \dots (a_k, b_k)$ be this walk where $(a_1, b_1) = (p, n)$ and $(a_k, b_k) = (q, n)$. For all $1 \leq i < k$ where $b_i = b_{i+1} = n$, there exists an edge $\{a_i, a_{i+1}\}$ in G_n and of course also in $G_n[A_{n-1}]$. For all $1 < i < j < k$ where $b_{i-1} = b_{j+1} = n$, and $b_\ell < n$ for all $i \leq \ell \leq j$, it is the case that $a_{i-1} = a_i$, $a_j = a_{j+1}$ and $b_i = b_j = n-1$. Because the vertices (a_i, b_i) and (a_j, b_j) are connected in the graph represented by $G_1, \dots, G_{n-1}$ and $A_1, \dots, A_{n-2}$, we can conclude that $A_{n-1}[a_{i-1}] = A_{n-1}[a_{j+1}] \neq 0$ and that $\{a_{i-1}, a_{j+1}\}$ is an edge in $G_n[A_{n-1}]$. Thus there exists a walk in $G_n[A_{n-1}]$ over the vertices a_i for which $b_i = n$, where $a_1 = p$ and $a_k = q$. Hence p and q are connected in $G_n[A_{n-1}]$.

($\Leftarrow$) If the vertices p and q are connected in $G_n[A_{n-1}]$, there must exist a walk on the vertices $a_1, \dots, a_k$ where $a_1 = p$ and $a_k = q$. By definition of $G_n[A_{n-1}]$, each edge $\{a_i, a_{i+1}\}$ for $1 \leq i < k$ is an edge of G_n, or otherwise $A_{n-1}[a_i] = A_{n-1}[a_{i+1}] \neq 0$. If $\{a_i, a_{i+1}\}$ is an edge of G_n, then $\{(a_i, n), (a_{i+1}, n)\}$ is an edge of S'. If $A_{n-1}[a_i] = A_{n-1}[a_{i+1}] \neq 0$, then S' contains the edges

$\{(a_i, n), (a_i, n-1)\}$ and $\{(a_{i+1}, n-1), (a_{i+1}, n)\}$, and there exists a walk from the edge $(a_i, n-1)$ to the edge $(a_{i+1}, n-1)$ in the graph represented by $G_1, \ldots, G_{n-1}$ and $A_1, \ldots, A_{n-2}$. From this we can conclude that there exists a walk through the vertices $(a_1, n), \ldots, (a_k, n)$ in S', and that (p, n) and (q, n) are connected in S'. □

M_G^C and M_G^{AC} will be constructed in such a way that for each directed walk with arbitrary length n from V_b to a vertex v, the label A with v will be a correct partial connection coding for the vertices $((1, n), \ldots, (m, n))$ of the spanning subgraph of $G \times P_n$ which is represented by this walk.

The following theorem states that this is true if for all edges (v_1, v_2, G') in M_G^P, A_{v_2} is a correct partial connection coding of $G'[A_{v_1}]$.

Theorem 2 *Suppose that the vertices of M_G^P are labeled with possible partial connection codings $(a_1, \ldots, a_m)$, where m is equal to the number of vertices in G. Further suppose that for all the edges (v_1, v_2, G') of M_G^P, v_2 is labeled with a correct connection coding for the vertices $(1, \ldots, m)$ of the graph $G'[A_{v_2}]$, and vertex V_b is labeled with $\bar{0}$. Then for each walk of arbitrary length n from V_b to v in this M_G^P, v will be labeled with a correct partial connection coding for the vertices $((1, n), \ldots, (m, n))$ of the graph represented by this walk.*

Proof: We prove this by induction on the length of the walk.

(Initial step:) For each walk $V_b e_1 v_1$, the graph $G'[A_{V_b}]$ is equal to the graph G', with which the edge e_1 is labeled. From this we conclude that the vertices $(p, 1)$ and $(q, 1)$ are connected in the graph represented by the walk $V_b e_1 v_1$ iff the vertices p and q are connected in G'. If v_1 is labeled with a correct partial connection coding for the vertices $(1, \ldots, n)$ of $G'[A_{V_b}]$ then it is labeled with a correct partial connection coding for the vertices $((1, n), \ldots, (m, n))$ of the graph represented by the walk $V_b e_1 v_1$.

(Induction step:) For each walk $V_b e_1 v_1 \ldots v_{n-1} e_n v_n$ (with $n > 1$), suppose that v_{n-1} is labeled with the correct partial connection coding $A_{v_{n-1}}$ of the vertices $((1, n-1), \ldots, (m, n-1))$ of the graph represented by the walk $V_b e_1 v_1 \ldots v_{n-2} e_{n-1} v_{n-1}$ in M_G^P. From Lemma 1 we know that the vertices (p, n) and (q, n) are connected iff p and q are connected in $G_{e_n}[A_{v_{n-1}}]$. If v_n is labeled with a correct partial connection coding for the vertices $(1, \ldots, n)$ of $G_{e_n}[A_{v_{n-1}}]$, then it is labeled with a correct partial connection coding for the vertices $((1, n), \ldots, (m, n))$ of the graph represented by the walk $V_b e_1 v_1 \ldots v_{n-1} e_n v_n$ in M_G^P. □

4.1 Connectedness

In this section we present the construction of M_G^C and we prove its correctness. The vertex set of M_G^C consists of the vertices which are uniquely labeled with all the possible partial connection codings of m-elements joined with a unique vertex V_e ($m = |V(G)|$). The number of vertices exceeds the number of possible partial connection codings by one. The special vertices V_b and V_e of M_G^C are two separate vertices, both labeled with $\bar{0}$. The edge set of M_G^C consists of all the edges (v_1, v_2, G'), such that:

- A_{v_2} is a correct partial connection coding of the vertices $(1, \ldots, m)$ of the graph $G'[A_{v_1}]$.
- $v_2 \neq V_b$.
- either $v_2 = V_e$ and $G'[A_{v_1}]$ is connected, or $v_2 \neq V_e$ and each vertex i in $G'[A_{v_1}]$ is connected (or equal) to a vertex j for which $A_{v_2}[j] \neq 0$.

The following theorem states the correctness of M_G^C constructed according the above method.

Theorem 3 *There is a one-to-one correspondence between the walks in $W_G^C(n)$ and the graphs in $S_G^C(n)$.*

Proof: ($\Rightarrow$) For each walk $V_b e_1 v_1 \ldots v_{n-1} e_n V_e$ in M_G^C, let S' be the graph represented by this walk. Suppose that S' is not connected. Then S' must have at least two components. Let (i, k)

be a vertex with maximal k that is not connected with the vertex $(1, n)$. Such a vertex (i, k) must exist. There are two cases: $k = n$ or $k < n$. In the case $k = n$, the vertices 1 and i of $G_{e_n}[A_{v_{n-1}}]$ are not connected according to Lemma 1. This means that by definition the edge (v_{n-1}, V_e, G_{e_n}) does not exist in M_G^C, which is a contradiction. In the case $k < n$, we know that because k is maximal there does not exist an edge $\{(j,k),(j,k+1)\}$ in S', such that (j,k) and (i,k) are connected in the graph represented by the walk $V_b e_1 v_1 \ldots v_{k-1} e_k v_k$. This means there does not exist a j, such that the vertices i and j are connected in $G_{e_k}[A_{v_{k-1}}]$, and $A_{v_k}[j] \neq 0$, by application of Lemma 1. This means that, by definition, the edge (v_{k-1}, v_k, G_{e_k}) does not exist in M_G^C, which is a contradiction.

($\Leftarrow$) For each graph S' in $S_G^C(n)$, let $G_1, \ldots, G_n$ and $A_1, \ldots, A_{n-1}$ represent this graph, such that for all $1 \leq i < n$, A_i is a correct partial connection coding for the vertices $(1,i), \ldots, (m,i)$ of the graph represented by $G_1, \ldots, G_i$ and $A_1, \ldots, A_{i-1}$. This completely determines $A_1, \ldots, A_{n-1}$. Let $v_1, \ldots, v_{n-1}$ be the vertices that are labeled with $A_1, \ldots, A_{n-1}$. By definition there is only one v_i labeled with A_i for $1 \leq i < n$. Let $v_0 = V_b$ and $A_0 = \bar{0}$.

We have to prove that the edges (v_{i-1}, v_i, G_i) exist for $1 \leq i < n$, and that the edge (v_{n-1}, V_e, G_n) exists. Of course A_i is a correct partial connection coding of $G_i[A_{i-1}]$ for $1 \leq i < n$, and $\bar{0}$ a correct partial connection coding of $G_n[A_{n-1}]$. If an edge (v_{i-1}, v_i, G_i) for $i < n$ does not exist, this means that there exists a vertex p for which there does not exist a vertex q, such that p and q are connected in $G_i[A_{i-1}]$, and such that $A_i[q] \neq 0$. By applying Lemma 1, this means that (p,i) is not connected with any $(q, i+1)$. Hence S' is not connected. Which is a contradiction. By application of Lemma 1, we know that $G_n[A_{n-1}]$ must be connected. Thus the edge (v_{n-1}, V_e, G_i) does exist. Because $v_1, \ldots, v_{n-1}$ are uniquely defined, and because there does not exist more than one edge between two vertices labeled with a certain graph, there can only be one walk V_b and V_e representing the graph S'. □

Corollary 2 *$C_G^C(n)$ is equal to the number of walks in $W_G^C(n)$ for each n.*

4.2 Acycliness

In the section we will describe the construction of M_G^{AC} and prove it correctness. The vertex set of M_G^{AC} consists of the vertices which are uniquely labeled with all the possible partial connection codings of m-elements ($m = |V(G)|$). The number of vertices is equal to the number of possible partial connection codings. The edge set of M_G^{AC} consists of all the edges (v_1, v_2, G'), such that:

- A_{v_2} is a correct partial connection coding of the vertices $(1, \ldots, m)$ of the graph $G'[A_{v_1}]$.
- G' is acyclic.
- there does not exist a sequence $a_1 b_1 \ldots a_k b_k$ of distinct numbers taken from $1, \ldots, m$, such that for all $1 \leq \ell \leq k$, the vertices labeled with a_ℓ and b_ℓ are connected in G', and such that $A_{v_1}[b_k] = A_{v_1}[a_1] \neq 0$, and $A_{v_1}[b_\ell] = A_{v_1}[a_{\ell+1}] \neq 0$ for all $1 \leq \ell < k$,

The special vertices V_b and V_e of M_G^{AC} are equal to the vertex labeled with $\bar{0}$.

The following theorem states the correctness of this construction.

Theorem 4 *There is a one-to-one correspondence between the walks in $W_G^{AC}(n)$ and the graphs in $S_G^{AC}(n)$.*

Proof: ($\Rightarrow$) Let S' be the graph represented by the walk $V_e e_1 v_1 \ldots v_{n-1} e_n V_b$ in M_G^{AC}, and suppose that S' contains a cycle. Let j be maximal such that (i,j) is a vertex of this cycle. Let V_j be the set of vertices $\{(1,j), \ldots, (m,j)\}$. Either all the vertices of the cycle are in V_j or not. In the first case this implies that G_{e_j} contains a cycle, which is a contradiction with the definition of the edge set of M_G^{AC}.

In the other case the cycle must contain alternating parts in and out of V_j. Let $a_1 b_1 \ldots a_k b_k$ be the boundaries of these parts, such that only the vertices (a_ℓ, j), (b_ℓ, j) and those in between are in V_j for all $1 \leq \ell \leq k$. This means that all the vertices a_ℓ and b_ℓ are connected in G_{v_j}. It also means that the edges $\{(a_\ell, j-1), (a_\ell, j)\}$ and $\{(b_\ell, j-1), (b_\ell, j)\}$ exist in S', and that there exists walks between each pair of vertices $(b_\ell, j-1)$ and $(a_{\ell+1}, j-1)$, and between the pair $(b_k, j-1)$ and $(a_1, j-1)$ in the graph represented by the walk $V_b e_1 v_1 \ldots e_{j-1} v_{j-1}$ in M_G^{AC}. From this follows that $A_{v_{j-1}}[b_\ell] = A_{v_{j-1}}[a_{\ell+1}] \neq 0$ for all $1 \leq \ell < k$, and $A_{v_{j-1}}[b_k] = A_{v_{j-1}}[a_1] \neq 0$. From the definition of M_G^{AC} we know that the edge (v_{j-1}, v_j, G_{e_j}) does not exist in M_G^{AC}. Thus S' must be acyclic.

($\Leftarrow$) For each graph S' in $S_G^{AC}(n)$, let $G_1, \ldots, G_n$ and $A_1, \ldots, A_{n-1}$ represent this graph such that for all $1 \leq i < n$, A_i is a correct partial connection coding for the vertices $((1,i), \ldots, (m,i))$ of the graph represented by $G_1, \ldots, G_i$ and $A_1, \ldots, A_{i-1}$. This completely determines $A_1, \ldots, A_{n-1}$.

Let $v_1, \ldots, v_{n-1}$ be the vertices that are labeled with $A_1, \ldots, A_{n-1}$. By definition there is only one v_i labeled with A_i for $1 \leq i < n$. Let $v_0 = V_b$, $v_n = V_e$ and $A_0 = A_n = \bar{0}$. We have to prove that the edges (v_{i-1}, v_i, G_i) exist for $1 \leq i \leq n$. Of course A_i is a partial connection coding of $G_i[A_{i-1}]$ for $1 \leq i \leq n$.

Suppose that the edge (v_{i-1}, v_i, G_i) does not exist, then this means that G_i must contain a cycle, or there exists a sequence $a_1 b_1 \ldots a_k b_k$ of distinct numbers taken from $\{1, \ldots, m\}$, such that for all $1 \leq \ell \leq k$, the vertices labeled with a_ℓ and b_ℓ are connected in G', and such that $A_{i-1}[b_k] = A_{i-1}[a_1] \neq 0$, and $A_{i-1}[b_\ell] = A_{i-1}[a_{\ell+1}] \neq 0$ for all $0 \leq \ell < k$. If G_i contains a cycle, then S' must also contain a cycle, which is a contradiction. If a sequence $a_1 b_1 \ldots a_k b_k$ with the above properties exists, then there must exist a cycle through the vertices $(a_1, i), \ldots, (b_1, i), (b_1, i-1), \ldots$ $\ldots, (a_k, i-1), (a_k, i), \ldots, (b_k, i), (b_k, i-1), \ldots, (a_1, i-1), (a_1, i)$ in S'. Which is a contradiction. Thus the edges (v_{i-1}, v_i, G_i) for all $1 \leq i \leq n$ must exist.

Because $v_1, \ldots, v_{n-1}$ are uniquely defined, and because there does not exist more than one edge between two vertices labeled with a certain graph, there can only be one walk from V_b to V_e representing the graph S'. □

Corollary 3 *$C_G^{AC}(n)$ is equal to the number of walks in $W_G^{AC}(n)$ for each n.*

5 Joining two (or more) properties

For each graph G and properties P and P', the directed multi-graph $M_G^{P \wedge P'}$ can be constructed from M_G^P and $M_G^{P'}$. The vertices of $M_G^{P \wedge P'}$ are the subset of the Cartesian product of the vertices of M_G^P and $M_G^{P'}$, where only the pairs (v, v') are taken for which $N(A_v) = N(A_{v'})$. Note that $N(A_v) = N(A_{v'})$ does not imply $A_v = A_{v'}$ in all cases.

The labeling of the vertices of $M_G^{P \wedge P'}$ should be such that $N(A_{(v,v')}) = N(A_v) = N(A_{v'})$.

The edge set of $M_G^{P \wedge P'}$ consists of all the edges $((v_1, v_1'), (v_2, v_2'), G')$ for which (v_1, v_2, G') and (v_1', v_2', G') are edges in resp. M_G^P and $M_G^{P'}$.

The special vertices V_b and V_e of $M_G^{P \wedge P'}$ are the vertices (v_1, v_1') and (v_2, v_2') resp., where v_1 is V_b of M_G^P, v_1' is V_b of $M_G^{P'}$, v_2 is V_e of M_G^P and v_2' is V_b of $M_G^{P'}$.

Theorem 5 *If M_G^P and $M_G^{P'}$ are correct for $S_G^P(n)$ and $S_G^{P'}(n)$, then the $M_G^{P \wedge P'}$ as constructed according the above described method is correct for $S_G^{P \wedge P'}(n)$.*

Proof: To prove that $M_G^{P \wedge P'}$ is correct with respect to $S_G^{P \wedge P'}$, we have to prove that each walk in $M_G^{P \wedge P'}$ represents a graph in $S_G^{P \wedge P'}$, and that for each graph S' in $S_G^{P \wedge P'}$ there is exactly one walk in $M_G^{P \wedge P'}$.

($\Rightarrow$) For each walk $(v_0, v_0') \ldots (v_n, v_n')$ in $M_G^{P \wedge P'}$, such that (v_0, v_0') and (v_n, v_n') are resp. V_b and V_e of $M_G^{P \wedge P'}$, we know that $N(A_{v_i}) = N(A_{v_i'}) = N(A_{(v_i, v_i')})$, and that the existence of $((v_{i-1}, v_{i-1}'), (v_i, v_i'), G_i)$ implies the existence of (v_{i-1}, v_i, G_i) and (v_{i-1}', v_i', G_i) in resp. M_G^P and $M_G^{P'}$ for all possible i. Also v_0 is V_b of M_G^P, v_0' is V_b of $M_G^{P'}$, v_n is V_e of M_G^P and v_n' is V_e of $M_G^{P'}$. From this we conclude that in M_G^P and $M_G^{P'}$ there exists resp. the walks $v_0 v_1 \ldots v_{n-1} v_n$ and $v_0' v_1' \ldots v_{n-1}' v_n'$ which represent the same graph as the walk $(v_0, v_0') \ldots (v_n, v_n')$, where v_1 is V_b of M_G^P, v_1' is V_b of $M_G^{P'}$, v_2 is V_e of M_G^P and v_2' is V_b of $M_G^{P'}$. Because M_G^P and $M_G^{P'}$ are correct, it means that this graph is both in $S_G^P(n)$ and $S_G^{P'}(n)$, hence also in $S_G^{P \wedge P'}(n)$.

($\Leftarrow$) For all S' in $S_G^{P\wedge P'}(n)$, S' is in $S_G^P(n)$ and in $S_G^{P'}(n)$. Because M_G^P and $M_G^{P'}$ are correct, there exist exactly one walk $v_0 v_1 \ldots v_{n-1} v_n$ in M_G^P, and exactly one walk $v_0' v_1' \ldots v_{n-1}' v_n'$ in $M_G^{P'}$ which represent the graph S', such that v_0 and v_n are resp. V_b and V_e of M_G^P and that v_0' and v_n' are resp. V_b and V_e of $M_G^{P'}$. It is clear that $N(A_{v_i}) = N(A_{v_i'})$ and that the edges between the vertices v_{i-1} and v_i, and between v_{i-1}' and v_i' are labeled with the same spanning subgraph of G for all possible i. From this we conclude that $(v_0, v_0') \ldots (v_n, v_n')$ is a walk in $W_G^{P\wedge P'}$ which represents S'. Suppose there exist two different walks from V_b to V_e in $W_G^{P\wedge P'}(n)$ that represent the same graph S'. Let these walks be $V_b(v_1, v_1') \ldots (v_{n-1}, v_{n-1}')V_e$ and $V_b(w_1, w_1') \ldots (w_{n-1}, w_{n-1}')V_e$. Let i be minimal such that (v_{i-1}, v_{i-1}') is not equal to (w_{i-1}, w_{i-1}'). Such i must exist, otherwise the walks are not different. This means that at least $v_i \neq w_i$ or $v_i' \neq w_i'$. In the case $v_i \neq w_i$ it means that there exists two walks in M_G^P that represent the graph S', which is a contradiction. In the case $v_i' \neq w_i'$ it means that there exists two walks in $M_G^{P'}$ that represent the graph S', which is also a contradiction. Of course there only exists at most one edge between each two vertices in $M_G^{P\wedge P'}$ labeled with the same spanning subgraph of G. □

At first sight it may look like an $M_G^{C\wedge AC}$ constructed according the above method will contain more edges than M_G^{AC} or M_G^C. But this is not true. $M_G^{C\wedge AC}$ will generally contain more vertices than M_G^{AC} and M_G^C, but many of these vertices, especial all those (v, v') for which $A_v \neq A_{v'}$, will not have any edges connected to them. An alternative way to construct a correct $M_G^{C\wedge AC}$ is by taking a correct M_G^C, and remove all edges (v_1, v_2, G') for which there does not exist an edge (v_1', v_2', G') in M_G^{AC} such that $A_{v_1} = A_{v_1'}$ and $A_{v_2} = A_{v_2'}$.

When P is one of C, AC or $C \wedge AC$, an alternative and correct $M_G^{P\wedge D}$ for some D can be constructed by taking M_G^P and remove all edges (v_1, v_2, G') for which there does not exist an edge (v_1', v_2', G') in M_G^D such that $N(A_{v_1}) = A_{v_1'}$ and $N(A_{v_2}) = A_{v_2'}$. Proof of the correctness of these construction methods is left to the reader.

6 On reducing M_G^P

It is obvious that some M_G^P get very large. From a theoretical view point this is of course no problem at all. But the calculation of the characteristic polynomial of a matrix becomes increasingly more complex if the size of the matrix becomes larger.

A certain M_G^P can be reduced in size in a number of ways. The simplest way is to remove all vertices (and adjacent edges) for which there does not exist a directed walk from V_b to the vertex or a walk from the vertex to V_e. These vertices will never be included in any walk from V_b to V_e in M_G^P.

If the graph G has a non-trivial automorphism, then it is possible to reduce M_G^P even further. Each automorphism can be represented by a permutation of the numbers 1 to m, where m is the number of vertices of G.

Now it is possible to join all the vertices in M_G^P that have the same labeling according to these permutations, and select one labeling for the joined vertices. The edges, which were labeled with spanning subgraphs of G, should now also be labeled with one of the permutations to indicate which mapping has been performed. Note that it is now possible that there is more than one edge between two vertices labeled with the same spanning subgraph of G, but they will always have different permutations.

If an M_G^P has been reduced according to these methods, the characteristic polynomial can possibly have less terms, which implies that also the recurrence equation will have less terms.

7 Results

In this section we will present all the results that were found by applying the above mentioned methods, by means of a computer program, to a number of problems.

7.1 The program

The computer program used to find the results is written in C, and is available through the author.

The program roughly consists of two parts. The first part constructs the directed multi-graph for a specific problem (a combination of a graph G and a property P). The second part of the program determines the recurrence equation based on the adjacency matrix of the directed multi-graph. Actually, the edges of the directed multi-graph are not stored, but instead the adjacency matrix is filled in the first part.

The algorithm to construct the directed multi-graph is an incremental algorithm, that starts with a set of vertices only containing V_b and an empty set of edges. In each step of this algorithm, another vertex from the set of vertices is processed. For this vertex, the set of outgoing edges is determined. The edges that are found are added to the set of edges, and any new vertices, to which the edges lead, are added to the set of vertices. When all vertices have been processed, the directed multi-graph is constructed, except for the vertices (and adjacent edges) that cannot be reached from V_b.

To find the outgoing edges with a certain vertex v, we try all spanning subgraphs G' of G that are possible with A_v depending on the property P. For all valid combinations of A_v and G we try to find all valid A' such that edge (v, v', G'), where v' is labeled with A', is correct according the property P. When property P is a combination of the basic properties D, C and AC, only the edges for which all required basic properties hold are added, following the suggestion made at the end of Section 5. Both the process of trying all G' and trying all A', are implemented by a back-tracking algorithm over, respectively, the edges in G, and the elements of the vector A'. For the basic properties C and AC, we first determine which element of the vector A' will be zero and which non-zero. Only after this has been determined for all elements, we calculate the partial connection coding.

We also applied the optimization of joining vertices in case G has a non-trivial automorphism, as described in Section 6.

The second part of the program—which is the most time consuming for the more complex problems—tries to find the recurrence equation. Several alternative algorithms have been employed.

The most straight forward approach for determining the recurrence equation is by calculates the characteristic polynomial of the adjacency matrix, which consists of calculating the determinant of $N - xI$. Although these matrices are rather sparse, this soon turns out to take too much time, due to the complexity of this algorithm, which is roughly of exponential order to the size of the matrix.

The two other approaches both try to find possible recurrence equations. If the recurrence equation is of order k and the matrix is of size ℓ, we can check whether the recurrence equation holds for $C(k+1)$ to $C(\ell)$. If this is the case, we can assume that the recurrence equation found is indeed correct. It is possible that the recurrence equations found in this way are different from the those found by calculating the characteristic polynomial. In all these cases we need to multiply the characteristic equation of the recurrence relation by another polynomial in order to get the characteristic polynomial of the multi-graph under investigation.

The second approach consists of finding the recurrence equations by solving the equation $\sum_{j=0}^{k} p_j N^j = 0$, for k ranging from 2 to ℓ. Note that because N is a ℓ by ℓ matrix this equation consists of ℓ^2 linear equation with p_1 to p_k as free variables. This set of equations can be reduced with a row sweeping technique to a set of independed equations. We set p_k equal to 1, and calculate the values for p_j for $0 < j < k$. Those p_j that do not appear in the equations are set to 0. The recurrence equations found by this approach are checked for $C(k+1)$ to $C(\ell)$.

A third approach is to find the recurrence equation from the values of $C(n)$. For this we try to solve the equations $\sum_{j=0}^{k} p_j C(i+j) = 0$ for $0 \leq j < k$, where k is ranging from 2 to ℓ. From here on this approach is the same as the second.

The complexity of the last two approaches is far more acceptable, although the computations require operations on integers of arbitrary size. For more complex problems the integers soon become very large. For instance, the integers used in the computation of $C_{P_8}^{HC}$ have more than 40000 digits.

Only the first and the last approach have been implemented completely. A limited form of the second approach has been implemented as well. All solutions presented in the sections below are found with the third approach, and most of them are also found with the first approach. In case the solutions differ, between the first and the last approach, or when only the third approach lead to a solution, we have indicated this.

7.2 The problems

The path graphs on n vertices will be denoted with P_n, as we have defined previously. The cycle graphs on n vertices will be denoted by C_n. K_n will denote the complete graphs on n points.

In the presentation of our results we will use $C(n)$ as an abbreviation of $C_G^P(n)$ whenever P and G are determined by the context. $F \circ G$ is defined as follows: take a copy of F and one of G, and draw a line from each vertex of F to each vertex of G.

We define the property $ST_{1,3}$ (the spanning subgraphs with degree 1 or 3) as $C \wedge AC \wedge D = \{1, 3\}$. In each of the following subsections we will present the results with a certain graph for the properties DT, $2F$, HC, HP, ST, and $ST_{1,3}$ as far as they were found. In all cases where solutions were already given in the literature, we found the same solutions. The solution for $C_{P_6}^{HC}$ and the numbers for $C_{P_7}^{HC}$ agree with results known to Y.H.H. Kwong. It is known that $C^{HC}(G) < C^{2F}(G)$, $C^{HP}(G) \geq |V(G)| \cdot C^{HC}(G)$, $C^{ST}(G) \geq C^{HC}(G)$ and $C^{ST_{1,3}}(G) \leq C^{ST}(G)$.

The solutions can be found here: `http://home.wxs.nl/~faase009/Cresults.html`

7.3 Spanning trees with degree 1 and 3

The property that a graph does have a Hamilton cycle is studied extensively in literature, because it is a very strong property. This is also reflected by the fact that the values for $C_G^{HC}(n)$ are usually low compared to that of other properties. However it turns out that the values for $C_G^{ST_{1,3}}(n)$ are also comparatively low. When comparing $C_G^{HC}(n)$ and $C_G^{ST_{1,3}}(n)$ for the graphs in the above sections, it appears that $C_G^{ST_{1,3}}(n) < C_G^{HC}(n)$ for graphs G with few edges and $C_G^{HC}(n) < C_G^{ST_{1,3}}(n)$ for graphs G with many edges.

At least some conditions have been found for which a graph G cannot have a spanning tree with only degrees 1 and 3. When G is bipartite, let x and y be the number of vertices in the two vertex classes, then G can only have a spanning subtree with degrees 1 and 3, if x and y are odd, and when $x > (y-1)/2$ and $y > (x-1)/2$. This explains almost all of the (non-trivial) cases for which $C_G^{ST_{1,3}}(n) = 0$ as mentioned in the above results.

8 Further research

Research could be done for how an M_G^P can be constructed for other properties. It seems to be possible to give a construction method for M_G^P where P is one of the following properties: restrictions on the number of components, restrictions on the number of vertices in a component, restrictions on the number of edges in a component, and restrictions on the degree per separate vertex of G. It seems also be possible to construct M_G^P for counting the number of different vertex or edge colourings for both $G \times P_n$, as for all spanning subgraphs of $G \times P_n$.

Another research question is: for which of the above properties an M_G^P can be constructed for $G \times C_n$, where C_n is the cycle graph on n points. It seems that only the properties connected and acyclic are excluded.

Of course research could be done for necessary and sufficient conditions for a graph to have a $ST_{1,3}$ spanning subgraph.

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Soli Gloria Deo.

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