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Pellian sequence relationships among π, e, $\sqrt{2}$

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Abstract: The numerators and denominators of the convergents of the continued fractions of π, e and $\sqrt{2}$ are shown to be elements of second order recurrence sequences of the Pellian or Fibonacci variety which are related to Pythagorean triples ($c^2 = b^2 + a^2$, $b > a$). π and $\sqrt{2}$ have surprisingly similar structures except that $\sqrt{2}$ has primitive Pythagorean triples with $c - b = 1$ or $b - a = 1$, whereas π has $c - b$ even and not constant and $b - a$ not constant, although the right-end-digits are constant.

Keywords: Integer structure analysis, Modular rings, Prime numbers, Fibonacci numbers, Infinite series, Pell sequence, Continued fractions, Primitive Pythagorean triples, Right-end-digits.

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1 Introduction

We have recently shown that

$$\pi = 2\sqrt{2}\,Q \tag{1.1}$$

where Q is the ratio of the quarter circumference of a circle to the side of the inscribed square [13]. Here we extend the study to the structure of the irrationals e and $\sqrt{2}$ and compare with π. The first six convergents of their continued fractions are set out in Table 1.

Number	Convergents					
$\sqrt{2}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{17}{12}$	$\frac{41}{29}$	$\frac{99}{70}$
π	$\frac{3}{1}$	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103993}{33102}$	$\frac{104348}{33215}$
e	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{8}{3}$	$\frac{11}{4}$	$\frac{19}{7}$	$\frac{87}{32}$

Table 1. First six convergents for $\sqrt{2}$, π and e

2 The convergents for $\sqrt{2}$

The convergents $\frac{N}{D}$ from the first row of Table 1 satisfy the second order linear recurrence relations [8]:

$$N_{n+1} = 2N_n + N_{n-1},\ D_{n+1} = 2D_n + D_{n-1}. \quad (2.1)$$

with initial terms 1 and 3 in the numerators $\{N_n\}$, and 1 and 2 in the denominators $\{D_n\}$ (the standard Pell sequence [8]).

From the relationship between the Pell and Pell-Lucas sequences [8], it has been shown [12] that Pellian sequences can be generated from the *z*–*j* grid [11] set up to characterise Pythagorean triples (Table 2)

$$c^2 = b^2 + a^2,\ b > a$$

Two questions arise:

- Are the sequences, $\{N_n\}$ and $\{D_n\}$ related to primitive Pythagorean triples (pPts)?, and
- Are there similar structures for π and e?

	$z = c - b,\ b > a$	c	b	a	$y = b - a$
odd	$(2K-1)^2$	$j^2 + (j + z^{½})^2$ $d^2 + f^2$	$2j(j + z^{½})$ $2df$	$z^{½}(2j + z^{½})$ $f^2 - d^2$	$2j^2 - z$
even	$2K^2$	$[(\frac{1}{2}z)^{½} + 2j - 1]^2 + \frac{1}{2}z$ $d^2 + f^2$	$[(\frac{1}{2}z)^{½} + 2j - 1]^2 - \frac{1}{2}z$ $f^2 - d^2$	$[(\frac{1}{2}z)^{½} + 2j - 1] \times 2(\frac{1}{2}z)^{½}$ $2df$	$(2j-1)^2 - z$

Table 2. *z*–*j* grid for Pythagorean triples: *j* is the integer counter;
criterion for generating pPts is $(j, z^{½}) = 1$ when $z > 1$; if $z = 1$ only pPts are obtained.

The elements of the numerator sequence, $\{N_n\}$, are all odd and it is found that they equal d (= j) for pPts with $z = 1$ (Table 3).

n	$d = j$	$f = j + z^{½}$	pPts	$y = 2j^2 - z$	$z = 2j^2 - y$
1	1	2	5, 4, 3	1	1
2	3	4	25, 24, 7	17	1
3	7	8	113, 112, 15	97	1
4	17	18	613, 612, 35	577	1
5	41	42	3445, 3444, 83	3361	1
6	99	100	19801, 19800, 199	19601	1
7	239	240	114721, 114720, 479	114241	1
8	577	578	667013, 667012, 1155	665857	1

Table 3. Numerators and pPts

The sequences $\{f\}$ and $\{y\}$ are not present in [18]. However, $\{f\} \equiv \{2u_n\}$ where

$$u_{n+1} = 2u_n + u_{n-1} - 1.$$

a Pellian non-homogeneous second order recurrence relation with initial terms, 1 and 2 [7]. That is, $\{f\}$ satisifies

$$f_{n+1} = 2f_n + f_{n-1} - 2.$$

The other internal parameters are z = 1, and y ($c - b$) which also satisfies a Pellian non-homogeneous recurrence relation:

$$y_{n+1} = 6y_n - y_{n-1} + a \tag{2.2}$$

in which

$$a = \begin{cases} -4 & n \text{ even}, \\ 12 & n \text{ odd}. \end{cases}$$

The elements of the sequence of denominators, $\{D_n\}$, equal d, f pairs for pPts with y ($b - a$) = 1 (Table 4).

n	$d = j$	$f = j + z^{½}$	pPts	z	$\sqrt{z}$	y
1, 2	1	2	5, 4, 3	1	1	1
3, 4	5	12	169, 120, 119	49	7	1
5, 6	29	70	5741, 4060, 4059	1681	41	1
7, 8	169	408	195025, 137904, 137903	57121	239	1

Table 4. Denominators and pPts

Again we can find Pellian-type recurrence relations; for instance, $\{\sqrt{z}\}$ satisfies (2.2) with a = 0.

3 Convergence of *e* and *π*

The convergents of e in the third row of Table 1 oscillate between Pellian and Fibonacci sequences (Table 5).

n	**1,2,3**	**2,3,4,5**	**4,5,6**
Recurrence relation	$N_n = 2N_{n-1} + N_{n-2}$	$N_n = N_{n-1} + N_{n-2}$	$N_n = 4N_{n-1} + N_{n-2}$
Type	Pellian	Fibonacci	Pellian

Table 5. Recurrence relations for convergents of e

The convergents of π in the second row of Table 1 also oscillate between Pellian and Fibonacci sequences (Table 6).

n	**1,2,3**	**2,3,4**	**3,4,5**	**4,5,6**
Recurrence relation	$N_n = 15N_{n-1} + N_{n-2}$	$N_n = N_{n-1} + N_{n-2}$	$N_n = 292N_{n-1} + N_{n-2}$	$N_n = N_{n-1} + N_{n-2}$
Type	Pellian	Fibonacci	Pellian	Fibonacci

Table 6. Recurrence relations (Rr) for convergents of π

The Pellian-type sequences are again associated with pPts. For example, for π the first two N_n are the d and f of the triple: {493, 475, 132} with z = 18 and y = 343. If N_3 (333) is taken as d, then the triple is {757305, 535527, 535464} with z = 221778 (18 × 12321) and y = 63. When D_n = 7, 106 for d and f, this yields the triple {11285, 11187, 1484} with z = 98 and y = 9703. The value of z is even and has a right-end-digit (RED) of 8 while y has a RED of 3. The REDs remain constant while the values of z and y vary. This is in contrast to the $\sqrt{2}$ system where z = 1 for $\{N_n\}$ and y = 1 for $\{D_n\}$.

The continued fraction for π is:

$$\pi = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\ldots \tag{3.1}$$

The occurrence of 15, 1 and 292, 1 and 2, 1 in the partial quotients and the coefficients in the Pellian-type recurrence relations invites further investigation. These partial quotients, unlike those for e and $\sqrt{2}$, have not been found to obey any simple laws [2]. It is somewhat surprising then that the Pellian relations for e and π have similar patterns which are a mix of the second order recurrence relations [14]. The recurrence relations associated with $\sqrt{2}$ follow different patterns in their links with pPts, in contrast to those of e and π.

4 Concluding comments

Topics for further research readily emerge. For instance, if we take the recurrence relation (2.2) and generalize it to the homogeneous form

$$w_{m,n} = 6w_{m,n-1} - w_{m,n-2} \tag{4.1}$$

with initial conditions $w_{m,1} = 1, w_{m,2} = m$, m = 1,2,…,7, we get the tableau in Table 7:

m \ n	1	2	3	4	5	6	Reference
1	1	1	5	29	169	985	[5]
2	1	2	11	64	373	2174	[4]
3	1	3	17	99	577	3363	[9]
4	1	4	23	134	781	4552	[1]
5	1	5	29	169	985	5741	[5]
6	1	6	35	204	1189	6930	[6]
7	1	7	41	239	1393	8119	[15]
Δ_2	0	1	6	35	204	1189	[6]

Table 7. Examples of recursive sequence defined by (4.1);
$\Delta_2 w_{m,n} = w_{m+1,n} - w_{m,n}$

We notice that

$$w_{m,n} = w_{m-1,n} + w_{6,n-1}, \tag{4.2}$$

and if $\Delta_1 w_{m,n} = w_{m,n+1} - w_{m,n}$, then $\Delta_1 w_{m,6} = w_{6,5m}$ and $\Delta_2 w_{6,n} = w_{5,n+1}$.

To what extent can these results be generalized?

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