ON THE SEQUENCE REPRESENTING NARROWEST INTERVALS CONTAINING TWO PAIRS OF INTEGERS WITH EQUAL CUBE SUMS

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Abstract. For every positive integer n, define $f(n) > n$ to be the smallest integer for which there exist two integers $u, v > n$ satisfying $n^3 + f^3(n) = u^3 + v^3$. This sequence is now denoted A360619 in the On-Line Encyclopedia of Integer Sequences. In this note, we explore several properties of it. In particular, the Mth local minima of the ratio $f(n)/n$. This is another new sequence A360427.

1. Calculations for Euler's equation

Euler's cubic equation is an equation $x^3 + y^3 = u^3 + v^3 = N$ to be solved in integers.

Definition 1 (Sequence $f(n)$). *For any positive integer n, let $f(n) \in \mathbb{N}$, $f(n) > n$, be the smallest positive integer such that there exists $u, v \in \mathbb{N}$, $u, v > n$, so that*

$$n^3 + f^3(n) = u^3 + v^3.$$

For example, identities $1209 = 1^3 + 12^3 = 9^3 + 10^3$ and $4104 = 2^3 + 16^3 = 9^3 + 15^3$ represent two values: $f(1) = 12$ and $f(2) = 16$. Since the identities $x^3 + (12x)^3 = (9x)^3 + (10x)^3$ and $(2x)^3 + (16x)^3 = (9x)^3 + (15x)^3$ also hold, it follows that $f(n) \leq 12n$, and $f(n) \leq 8n$ if n is even. Here is the first 26 terms[1] with an offset 1:

$$12, 16, 36, 32, 60, 48, 84, 53, 34, 27, 93, 40, 156, 112, 80, 106, 39, 68, 228, 54, 238, 176, 94, 80, 67, 156.$$

Let $R(n) = \frac{f(n)}{n}$. As we have seen, $R(n) \leq 12$. Figure 1 shows the pointplot of this function for integral x, $1 \leq x \leq 1200$.

Definition 2 (Sequence $\mathcal{U}(m)$). *Let $\mathcal{U}(m)$ be a positive integer for which the function $\frac{R(n)}{n}$ attains its mth local minima.*

This sequence is labelled A360427 in [**?**]. In practice, when a program finds a minima record M, the next iteration for computations can by narrowed to search the existence of solutions to $a^3 + b^3 = c^3 + d^3$ with $\frac{b}{a} < M$. Table 1 provides all 24 terms under 10^5.

Problem. *We ask whether there exist the limit*

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^{x} R(n).$$

Definition 3 (Primitive quadruples). *Let us consider only those $n \in \mathbb{N}$ for which the corresponding quadruple $(n, f(n); u, v)$ satisfies an additional condition* $\mathrm{GCD}(n, f(n), u, v) = 1$. *The ordered sequence of these n's is the sequence L.*

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[1]1200 terms are contained in this file: `https://web.vu.lt/mif/g.alkauskas/math/cubic-sums.txt`

Sequence $\mathcal{M}$		Supplementary sequences				
m	$n(m)$	$f(n)$	$u(n)$	$v(n)$	$n^3+f^3(n)=u^3+v^3$	$R(n)$
1	1	12	9	10	1729	12
2	2	16	9	15	4104	8
3	8	53	29	50	149389	6.625
4	9	34	16	33	40033	3.7777
5	10	27	19	24	20683	2.7
6	17	39	26	36	64232	2.2941
7	30	67	51	58	327763	2.2333
8	42	69	56	61	402597	1.6428
9	51	82	64	75	684019	1.6078
10	135	202	166	183	10702783	1.4962
11	156	225	169	218	15187041	1.4423
12	285	352	309	334	66763333	1.2350
13	792	901	829	870	1228225789	1.1376
14	1634	1858	1707	1797	10776828816	1.1370
15	3751	4172	3878	4063	125392670199	1.1122
16	4026	4267	4104	4195	142946631739	1.0598
17	6192	6555	6259	6494	519062033763	1.0586
18	14934	15697	15090	15553	7198320746377	1.0510
19	15768	16533	16130	16189	8439526756269	1.0485
20	16147	16626	16299	16480	8805759770899	1.0296
21	45121	46440	45238	46329	192017975290561	1.0292
22	58230	59467	58683	59026	407736961367563	1.0212
23	61389	62274	61733	61936	472852910313693	1.0144
24	79876	80937	80209	80610	1039824823550329	1.0132

TABLE 1. Computations for consecutive local minima of $R(n)$

Here are the first 30 terms:[2]

$1, 2, 8, 9, 10, 11, 12, 15, 17, 21, 23, 25, 26, 29, 30, 31, 32, 35, 37, 41, 42, 47, 49, 50, 51, 53, 55, 57, 61, 62.$

Definition 4 (GCD of a quadruple)**.** *Let* $g(n) = \mathrm{GCD}(n, f(n), u, v)$.

By definition, $g(n)$ is a divisor of n. Here is the beginning of this sequence:[3]

$$1, 1, 3, 2, 5, 3, 7, 1, 1, 1, 1, 1, 13, 7, 1, 2, 1, 2, 19, 2, 1, 11, 1, 2, 1, 1, 3, 14.$$

Now we will define yet another object.

Definition 5. *For each* $x \in L(\mathbb{N})$ *(this is the set defined by the sequence* $f(n)$*), define a set* $\mathfrak{S}(x)$ *by the condition*

$$N \in \mathfrak{S}(x), \text{ if } (Nx) = N.$$

Consider for example $x = 1$. This condition requires $g(N) = N$. That is, in the quadruple $(N, f(N), v, y)$, all terms are divisible by N. Consequently, this quadruple is exactly

[2]All 592 terms ≤ 1200 are here: `https://web.vu.lt/mif/g.alkauskas/math/cubic-sums2.txt`

[3]1200 first terms are here: `https://web.vu.lt/mif/g.alkauskas/math/cubic-sums3.txt`

Sets $\mathfrak{S}$		
n	$L(n)$	$\mathfrak{S}(x)$
1	1	$1\{1,3,5,7,13,19,39,43,59,65,91,97,169,211,293,313,431,845\}$
2	2	$\{1,2,3,7,11,14,19,22\}$
3	8	$\{1,2,7\}$
4	9	$\{1,2,3,5,6,7,13,21,23,26,65,67,91\}$
5	10	$\{1,2,4,7,8,13,14,16,19,23,26,28,29,32,41,43,$ $46,49,52,53,56,64,82,86,91,98,104,109,112\}$
6	11	$\{1,3,7,53\}$
7	12	$\{1,2,3,4,8,9,16,32,59,64\}$
8	15	$\{1,5,13\}$
9	17	$\{1,2,4,5,8,10,11,13,16,17,19,20,20,23,26,31,37,38,40,41,43,53,67\}$
10	21	$\{1\}$
11	23	$\{1,2,4,5,7,11,13,14,22,25,26,37,43,44,49\}$
12	25	$\{1\}$
13	26	$\{1,2,3,4,7,8,11,13,14,26,28\}$
14	29	$\{1,2,7,11,13,14,19,22,23,26,29,31\}$
15	30	$\{1,2,3,4,6,8,13,16,23,29,39\}$
16	31	$\{1,7,13\}$
17	32	$\{1,13\}$

TABLE 2. Computations

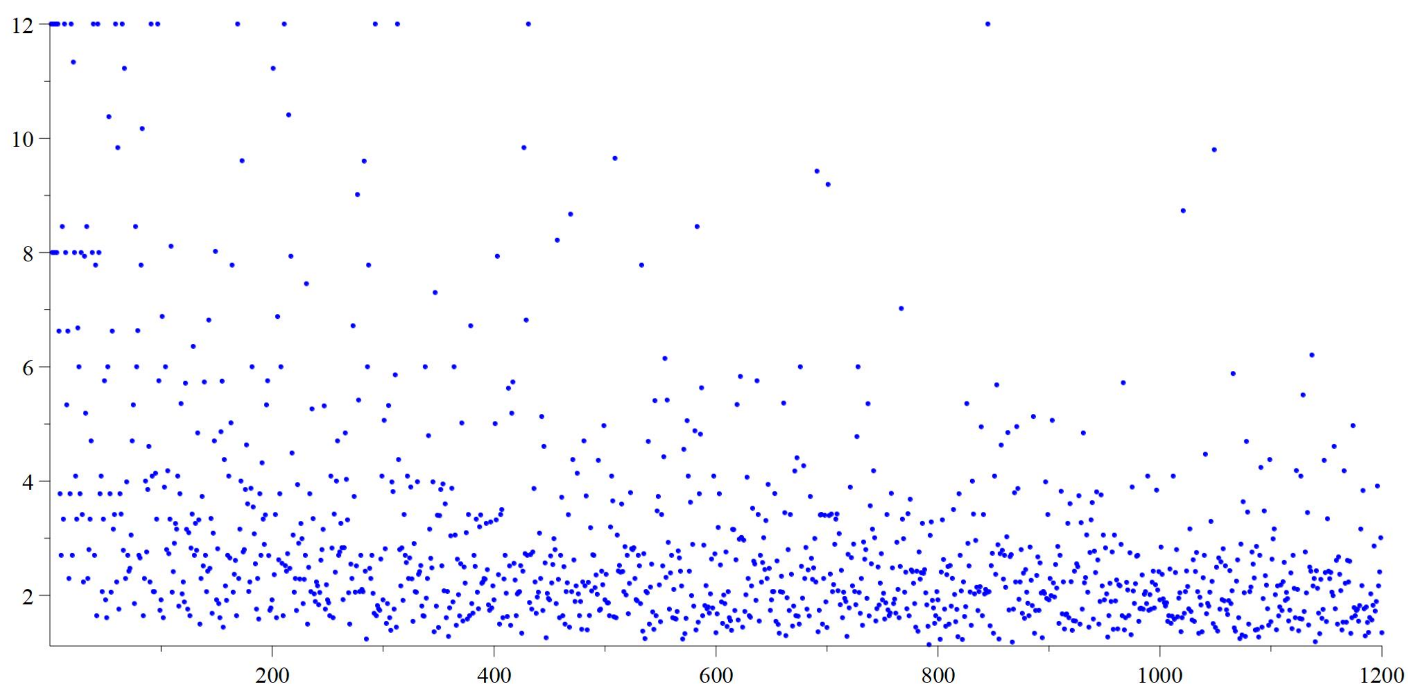

FIGURE 1. Pointplot for the function $R(n)$, $1 \leq n \leq 1200$.

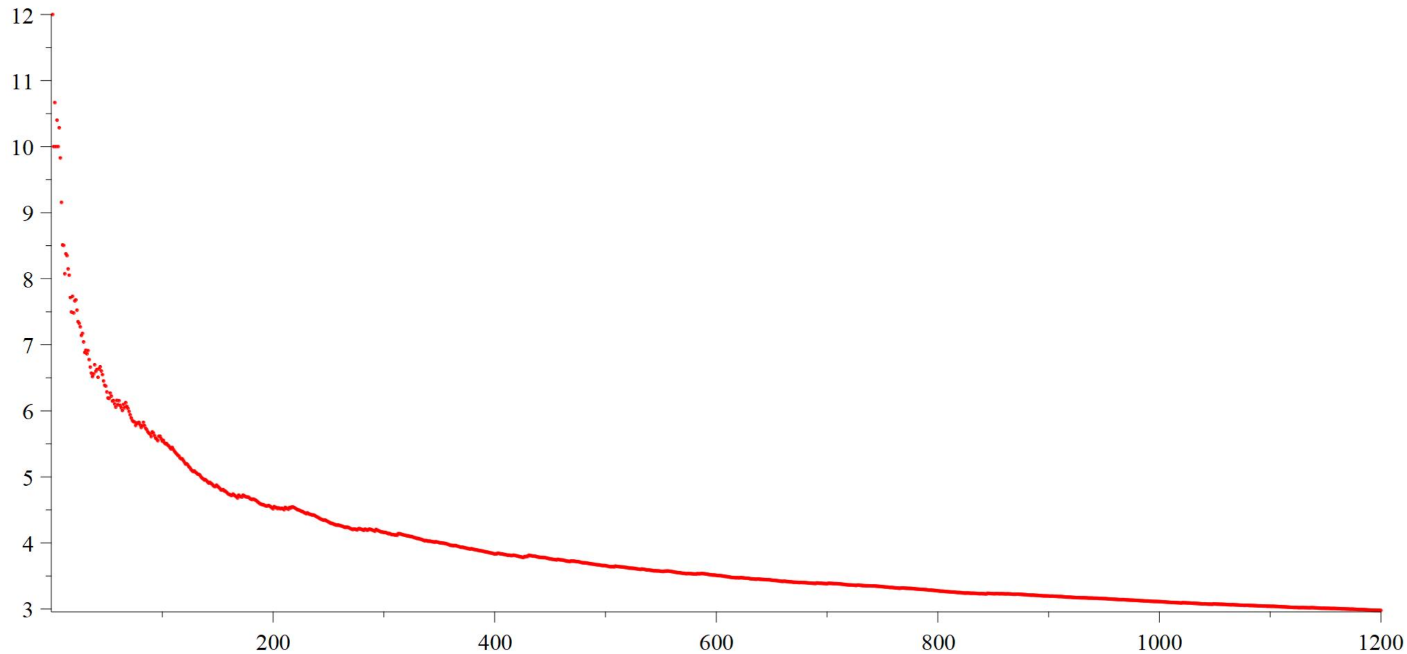

FIGURE 2. Pointplot for the function $(\sum_{n=1}^{x} R(n))/x$, $1 \leq x \leq 1200$.

$(N, 12N, 9N, 10N)$. For N in the range $[1, 1200]$, the following terms belong to $\mathfrak{S}(1)$:

$$1, 3, 5, 7, 13, 19, 39, 43, 59, 65, 91, 97, 169, 211, 293, 313, 431, 845.$$

As an corollary, these calculations yield the following result.

Proposition 1. *Let $x = 845$. Then the smallest integral $y > x$ such that there exists integers $x < u < v < y$ with $x^3 + y^3 = u^3 + v^3$ is equal to $12 \cdot 845$.*

Problem. *Is the set $\mathfrak{S}(1)$ infinite?*

2. Explicit formulas

Complete list of integer solutions to $x^3 + y^3 = u^3 + v^3$ was given by Choudhry [?], and they read as

$$\begin{cases} x = c(b-a)(a^2+ab+b^2) + c^4, \\ y = (a^2+ab+b^2)^2 + (2a+b)c^3, \\ u = (a^2+ab+b^2)^2 + (b-a)c^3 \\ v = (a^2+ab+b^2)(2a+b)c + c^4, \end{cases}$$

with additional allowance to factor out GCD of the quadruple $x, y; u, v$. Consider a carefully chosen special example $a = N - 1$, $b = N^2 - N + 2$, $c = N^2 - N + 1$. After factoring out c^2, we obtain

$$\begin{cases} x = 2N^4 - 4N^3 + 9N^2 - 8N + 10, \\ y = 2N^4 + 6N^2 + N + 9, \\ u = 2N^4 - 3N^3 + 12N^2 - 5N + 12, \\ v = 2N^4 - N^3 + 6N^2 + N + 1. \end{cases}$$

Since $\frac{y}{x} \approx 1 + \frac{2^{5/4}}{x^{1/4}}$, this shows that $\liminf_{x\to} R(n) = 1$.

Proposition 2. *The sequence $\mathcal{M}$ is infinite.*

Avery primitive quadruple can be represented in two different ways, a

$$L_1 = \Big(X(a,b,c_1),\, Y(a,b,c_1),\, Z(a,b,c_1),\, W(a,b,c_1)\Big)$$
$$L_2 = \Big(Z(a,b,c_2),\, W(a,b,c_1),\, X(a,b,c_2),\, Y(a,b,c_2)\Big)$$

and the following holds

$$L_1 c_2^2 = L_2 c_1^2.$$

$$\Big(c_1(b-a)(a^2+ab+b^2)+c_1^4\Big)c_2^2 = \Big((a^2+ab+b^2)^2+(b-a)c_2^3\Big)c_1^2$$

This rewrites as

$$(a^2+ab+b^2) = c_1c_2.$$

Observation. *The minimum of the function $R(X)$ is achieved when*

- ($\star$) *a is sufficiently small compared to b,*
- ($\star$) *$a^2+ab+b^2 = c_1c_2$, where $c_1 \approx c_2$.*

Consider the identity

$$N^2 + N(N^2+N+2) + (N^2+N+2)^2 = (N^2+N+1)(N^2+2N+4).$$

Let $a = N$, $b = N^2+N+2$, $c_1 = c = N^2+N+1$.

$$\begin{cases} X = (b-a)c_1^2c_2 + c_1^4, \\ Y = c_1^2c_2^2 + (2a+b)c_1^3, \\ Z = c_1^2c_2^2 + (b-a)c_1^3 \\ W = c_1^3c_2(2a+b) + c_1^4. \end{cases}$$

Suppose $c_1 < c_2$. Then

$$Y - X = c_1^3(2a+b-c_1) + c_1^2c_2(a+c_2-b),$$

Fore example. Here is MAPLE code which does all computations up to $x \leq 1200$. The array `Es1[W]` (`W` is a permutation of $\{1,\dots 1200\}$) is the sequence x's: $1,2,3,\dots.$ The array `Es2[W]` is the sequence of y's: $12,16,36,32,60,48,\dots$ (that is, A_1). `Es3[W]` is u's, `Es4[W]` is v's, and `Es5[W]` is the sequence A_3.

```
restart:
N:=0:
Por:=90660270:
Sk:=1200: #upper bound for a(n)
f:=Array(1..Por,  datatype=integer):
Ar:=Array(1..Por, datatype=integer):
Br:=Array(1..Por, datatype=integer):
fc1:=Array(1..Por,datatype=integer[2]):
fc2:=Array(1..Por,datatype=integer[2]):
Es1:=Array(1..Sk, datatype=integer[2]):
Es2:=Array(1..Sk, datatype=integer[2]):
Es3:=Array(1..Sk, datatype=integer[2]):
Es4:=Array(1..Sk, datatype=integer[2]):
Es5:=Array(1..Sk, datatype=integer[2]):
```

```
Rado:=Array(1..Sk,datatype=boolean):
for t from 1 to Sk do Rado[t]:=false end do:
for X from 1 to Sk do
for Y from X to 12*X do
N:=N+1:  f[N]:=Y^3+X^3: fc1[N]:=Y: fc2[N]:=X:
end do:
end do:
for X from Sk+1 to floor(9.53*Sk) do
for Y from X to 12*Sk do
N:=N+1: f[N]:=Y^3+X^3: fc1[N]:=Y: fc2[N]:=X:
end do:
end do:
Ar,Br:=sort(f,'output=[sorted,permutation]'):
K:=0:
for i from 1 to N-1 do if (Ar[i]=Ar[i+1]) and (fc2[Br[i]]<=Sk) then
if (not Rado[fc2[Br[i]]]) then
Rado[fc2[Br[i]]]:=true:
K:=K+1:
Es1[K]:=fc2[Br[i]]:
Es2[K]:=fc1[Br[i]]:
Es3[K]:=fc2[Br[i+1]]:
Es4[K]:=fc1[Br[i+1]]:
Es5[K]:=igcd(fc1[Br[i]], fc2[Br[i]], fc1[Br[i+1]], fc2[Br[i+1]]):
end if:
end if:
end do:
Ess:=Array(1..K):
for k from 1 to K do Ess[k]:=Es1[k] end do:
W:=sort(Ess,'output=permutation'):
```

References

1. Ono, Ken, Trebat-Leder, Sarah The 1729 K3 surface, *Research in Number Theory*, **2** (26), 2016, https://link.springer.com/article/10.1007/s40993-016-0058-2.

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