Ercole Suppa

Eötvös-Kürschák Competitions

Eötvös-Kürschák Competitions

Mathematical and Physical Society

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Sommario

There are several separate competitions in Hungary. The oldest modern mathematical competition, not only in Hungary but also in the world, is the Kürschák Mathematical Competition, founded in 1894, but known as Eötvös Mathematical Competition until 1938. This competition is for students up to the first year of university and consists of 3 problems. This competition changed its name from Eötvös to Kürschák after the second world war. The Eötvös was not held in the years 1919,1920,1921,1944,1945,1946. The Kürschák was not held in the year 1956.

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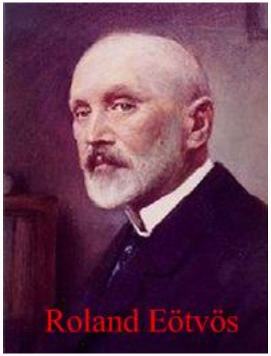

In 1894 baron Roland Eötvös was asked to serve as minister of education in the Hungarian government, to help the acceptance of civil rights and religious freedom in the Hungarian Parliament. The Hungarian Mathematical and Physical Society decided, that, to commemorate this period, launches yearly competitions to secondary school graduates. This has got the name Eötvös Competition. It was organized first in the fall of 1884, and it runs in every year (with the exception of a few war years).

The problems given at this competitions are intended to assess the creativity (and not the memorized knowledge) of the students. The problems have to be solved in a closed room, supervised by impartial observers, within an afternoon. The respect of this competition is created, among others, by the fact that the 10 bests in mathematics and the 10 bests in physics have free admission to the university. (It has to be noted that students are accepted at scientific, engineering and medical faculties only after passing very competitive entrance examinations. The Eötvös Competition offers a route for wild talents to the university.) The actual organization of the Eötvös Competition happens mostly at the Eötvös University under the patronage of the Eötvös Society. Local physics competitions are organized in several districts of Hungary.

Among the prize winners one finds the names of Theodore von Kármán in mathematics (1897), Leo Szilard in physics (1916), Edward Teller, in both, mathematics and physics (1925), in physics together with Laszlo Tisza, John Harsanyi in mathematics (1937), Ferenc Mezei in physics (1960), and so on. These names indicate the century-long tradition.

The problem-solving student journal, Középiskolai Matematikai és Fizikai Lapok was launched in the same year. This journal is published by the Eötvös Society in thousands of copies monthly.

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The respect of the Eötvös Competition is expressed by the fact that John von Neumann tried to introduce a similar competition in Germany in the 1920s, furthermore Gabriel Szegõ (the Hungarian mathematics professor at Stanford University) organized competitions in California, following the pattern of the Eötvös Competitions, after World War 2. At the centenary celebration Kai-hua Zhao stressed that the Eötvös Competition might be considered as the forerunner of the International Physics Student Olympiads. These Olympiads were created at joint Czech-Hungarian-Polish initiative in 1964, now students from about 50 countries participate on them. In 1991 the International Union for Pure and Applied Physics gave its educational medal to the Physics Olympiad and to its three initiators. Inspired by the Student Olympiads, nowadays national student competitions are organized in many countries.

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Eötvös-Kürschák Competitions

1st Eötvös Competition 1894

Organized by Mathematical and Physical Society

1. Prove that the expressions
$$2x + 3y \quad \text{and} \quad 9x + 5y$$
are divisible by 17 for the same set of integral values of x and y.

2. Given a circle and two points, P and Q: construct an inscribed right triangle such that one of its legs goes through the given point P and the other through the given point Q. For what position of the points P and Q is this construction impossible?

3. The lengths of the sides of a triangle form an arithmetic progression with difference d. The area of the triangle is t. Find the sides and angles of this triangle. Solve this problem for the case $d = 1$ and $t = 6$.

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2nd Eötvös Competition 1895

Organized by Mathematical and Physical Society

1. Prove that there are $2\left(2^{n-1}-1\right)$ ways of dealing n cards to two person. (The players may receive unequal numbers of cards.)

2. Give a right triangle ABC, construct a point N inside the triangle such that the angles $\angle NBC$, $\angle NCA$ and $\angle NAB$ are equal.

3. Given the following information about a triangle: the radius R of its circumscribed circle, the length c of one of its sides, and the ratio a/b of the lengths of the other two sides; determine all three sides and angles of this triangle.

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3rd Eötvös Competition 1896

Organized by Mathematical and Physical Society

1. Prove that
$$\log n \geq k \cdot \log 2$$
where n is a natural number and k the number of distinct primes that divide n.

2. Prove that the equations
$$x^2 - 3xy + 2y^2 + x - y = 0$$
and
$$x^2 - 2xy + y^2 - 5x + 7y = 0$$
imply the equation
$$xy - 12x + 15y = 0.$$

3. Construct a triangle, given the feet of its altitudes. Express the lengths of the sides of the solution triangle Y in terms of the lengths of the sides of the solution triangle X whose vertces are the feet of the altitudes of triangle Y.

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4th Eötvös Competition 1897

Organized by Mathematical and Physical Society

1. Prove, for angles α, β and γ of a right triangle, the following relation:

$$\sin\alpha\sin\beta\sin(\alpha-\beta)+\sin\beta\sin\gamma\sin(\beta-\gamma)+\sin\gamma\sin\alpha\sin(\gamma-\alpha)+$$
$$+\sin(\alpha-\beta)\sin(\beta-\gamma)+\sin(\gamma-\alpha)=0$$

2. Show that, if α, β and γ are angles of an arbitrary triangle,

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}<\frac{1}{4}$$

3. Let $ABCD$ be a rectangle and let M, N and P, Q be the points of intersections of some line e with the sides AB, CD and AD, BC, respectively (or their extensions). Given the points M, N, P, Q and the length p of side AB, construct the rectangle. Under what conditions can this problem be solved, and how many solutions does it have ?

5th Eötvös Competition 1898

Organized by Mathematical and Physical Society

1. Determine all positive integers n for which $2^n + 1$ is divisible by 3.

2. Prove the following theorem: If two triangles have a common angle, then the sum of the sines of the angles will be larger in that triangle where the difference of the remaining two angles is smaller.
   On the basis of this theorem, determine the shape of that triangle for which the sum of the sines of its angles is a maximum.

3. Let A, B, C, D be four given points on a straight line e. Construct a square such that two of its parallel sides (or their extensions) go trough A and B respectively, and the other two sides (and their extensions) go trough C and D respectively

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6th Eötvös Competition 1899

Organized by Mathematical and Physical Society

1. The points $A_0, A_1, A_2, A_{,3}, A_4$ divide a unit circle (circle of radius 1) into five equal parts. Prove that the chords A_0A_1, A_0A_2 satisfy

$$(A_0A_1 \cdot A_0A_2)^2 = 5$$

2. Let x_1 and x_2 be the roots of the equation

$$x^2 - (a+d)x + ad - bc = 0.$$

   Show that x_1^3 and x_2^3 are the roots of

$$y^3 - \left(a^3 + d^3 + 3abc + 3bcd\right) y + (ad - bc)^3 = 0$$

3. Prove that, for any natural number n, the expression

$$A = 2903^n - 803^n - 464^n + 261^n$$

   is divisible by 1897.

7th Eötvös Competition 1900

Organized by Mathematical and Physical Society

1. Let a, b, c, d be fixed integerswith d not divisible by 5. Assume thet m is an integer for which
$$am^3 + bm^2 + cm + d$$
is divisible by 5. Prove that there exists an integer n for which
$$dn^3 + cn^2 + bn + a$$
is also divisible by 5.

2. Construct a triangle ABC, given the length c of its side AB, the radius r of its inscribed circle, and the radius r_c of its ex-circle tangent to the side AB and the extensions of BC and CA.

3. A cliff is 300 meters high. Consider two free-falling raindropssuch that the second one leaves the top of the cliff when the first one has already fallen 0.001 millimeters. What is the distance between the drops at the moment the first hits the ground? (Compute the answer to within 0.1 mm Neclect air resistance, ect.)

8th Eötvös Competition 1901

Organized by Mathematical and Physical Society

1. Prove that, for any positive integer n,

$$1^n + 2^n + 3^n + 4^n$$

is divisible by 5 if and only if n is not divisible by 4.

2. If

$$u = \cot 22^\circ 30' \quad , \quad v = \frac{1}{\sin 22^\circ 30'}$$

prove that u satisfies a quadratic and v a quartic (4th degree) equation with integral coefficients and with leading coefficients 1.

3. Let a and b two natural numbers whose greatest common divisor is d. Prove that exactly d of the numbers

$$a,\ 2a,\ 3a,\ \ldots,\ (b-1)a,\ ba$$

is divisible by b.

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9th Eötvös Competition 1902

Organized by Mathematical and Physical Society

1. Prove that any quadratic expression

$$Q(x) = Ax^2 + Bx + C$$

   (a) can be put into the form

$$Q(x) = k\frac{x(x-1)}{1 \cdot 2} + lx + m$$

   where k, l, m depend on the coefficients A, B, C and

   (b) $Q(x)$ takes on integral values for every integer x if and only if k, l, m are integers.

2. Let S be a given sphere with center O and radius r. Let P be any point outside the sphere S, and let S' be the sphere with center $P$and radius PO. Denote by F the area of the surface of the part of S' that lies inside S. Prove that F is independent of the particular point P chosen.

3. The area T and an angle γ of a triangle are given. Determine the lengths of the sides a and b so that the side c, opposite the angle γ, is as short as possible.

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10th Eötvös Competition 1903

Organized by Mathematical and Physical Society

1. Let $n = 2^{p-1}(2^p - 1)$, and let $2^p - 1$ be a prime number. Prove that the sum of all (positive) divisors of n (not including n itself) is exactly n.

2. For a given pair of values x and y satisfying $x = \sin\alpha$, $y = \sin\beta$, there can be four different values of $z = \sin(\alpha + \beta)$.

   (a) Set up a relation between x, y and z not involving trigonometric functions or radicals.

   (b) Find those pairs of values (x, y) for which $z = \sin(\alpha + \beta)$ takes on fewer than four distinct values.

3. Let A, B, C, D be the vertices of a rhombus; let k_1 be the circle through B, C and D; let k_2 be the circle through A, C and D; let k_3 be the circle through A, B and D; let k_4 be the circle through A, B and C. Prove that the tangents to k_1 and k_3 at B form the same angle as the tangents to k_2 and k_4 at A.

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11th Eötvös Competition 1904

Organized by Mathematical and Physical Society

1. Prove that, if a pentagon (five-sided polygon) inscribed in a circle has equal angles, then its sides are equal.

2. If a is a natural number, show that the number of positive integral solutions of the indeterminate equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = a \tag{1}$$

is equal to the number of non-negative integral solutions of

$$y_1 + 2y_2 + 3y_3 + \cdots + ny_n = a - \frac{n(n+1)}{2} \tag{2}$$

[By a *solution* of equation (1), we mean a set of numbers $\{x_1, x_2, \ldots, x_n\}$ which satisfies equation (1)].

3. Let A_1A_2 and B_1, B_2 be the diagonals of a rectangle, and let O be its center. Find and construct the set of all points P that satisfy simultaneously the four inequalides

$$A_1P > OP \quad , \quad A_2P > OP \quad , \quad B_1P > OP \quad , \quad B_2P > OP.$$

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12th Eötvös Competition 1905

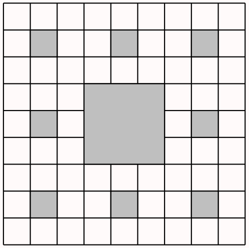
Organized by Mathematical and Physical Society

1. For given positive integers n and p, find neaessary and suffcient conditions for the system of equations
$$x + py = n \quad , \quad x + y = p^2$$
to have a solution (x, y, z) of positive integers. Prove also that there is at most one such solution.

2. Divide the unit square into 9 equal squares by means of two pairs of lines parallel to the sides (see figure). Now remove the central square. Treat the remaining 8 squares the same way, and repeat the process n times.

   (a) How many squares of side length $1/3^n$ remain?

   (b) What is the sum of the areas of the removed squares as n becomes infinite?



3. Let C_1 be any point on side AB of a triangle ABC, and draw C_1C. Let A_1 be the intersection of BC extended and the line through A parallel to CC_1; similarly let B_1, be the intersection of AC extended and the line through B parallel to CC_1. Prove that
$$\frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$$

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13th Eötvös Competition 1906

Organized by Mathematical and Physical Society

1. Prove that, if $\tan(\alpha/2)$ is rational (or else, if α is an odd multiple of π so that $\tan(\alpha/2)$ is not defined), then $\cos\alpha$ and $\sin\alpha$ are rational; and, conversely, if $\cos\alpha$ and $\sin\alpha$ are rational, then $\tan(\alpha/2)$ is rational unless α is an odd multiple of Π so that $\tan(\alpha/2)$ is not defined.

2. Let K, L, M, N designate the centers of the squaxes erected on the four sides (outside) of a rhombus. Prove that the polygon $KLMN$ is a square.

3. Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, \ldots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2)\cdots(a_n - n)$$

is an even number.

14th Eötvös Competition 1907

Organized by Mathematical and Physical Society

1. If p and q are odd integers, prove that the equation
$$x^2 + 2px + 2q = 0 \tag{1}$$
has no rational roots.

2. Let P be any point inside the parallelogram $ABCD$ and let R be the radius of the circle through A, B, and C. Show that the distance from P to the nearest vertex is not greater than R.

3. Let
$$\frac{r}{s} = 0.k_l k_2 k_3 \cdots$$
be the decimal expansion of a rational number (If this is a terminating decimal, all k_i from a certain one on are 0). Prove that at least two of the numbers
$$\sigma_1 = 10\frac{r}{s} - k_i \quad , \quad \sigma_2 = 10^2 - (10k_1 + k_2),$$
$$\sigma_3 = 10^2 - (10^2 k_1 + 10k_2 + k_3) \quad , \quad \ldots$$
are equal.

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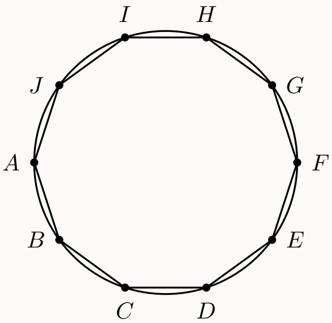
15th Eötvös Competition 1908

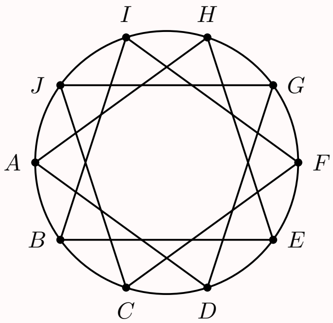
Organized by Mathematical and Physical Society

1. Given two odd integers a and b; prove that $a^3 - b^3$ is divisible by 2^n if and only if $a - b$ is divisible by 2^n.

2. Let n be an integer greater than 2. Prove that the nth power of the length of the hypotenuse of a right triangle is greater than the sum of the nth powers of the lengths of the legs.

3. A regular polygon of 10 sides (a regular decagon) may be inscribed in a circle in the following two distinct ways: Divide the circumference into 10 equal arcs and (1) join each division point to the next by straight line segments, (2) join each division point to the next but two by straight line segments. (See figures). Prove that the difference in the side lengths of these two decagons is equal to the radius of their circumscribed circle.





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16th Eötvös Competition 1909

Organized by Mathematical and Physical Society

1. Consider any three consecutive natural numbers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

2. Show that the radian measure of an acute angle is less than the arithmetic mean of its sine and its tangent.

3. Let A_1, B_1, C_1, be the feet of the altitudes of $\triangle ABC$ drawn from the vertices A, B, C respectively, and let M be the orthocenter (point of intersection of altitudes) of $\triangle ABC$. Assume that the orthic triangle (i.e. the triangle whose vertices are the feet of the altitudes of the original triangle) A_1, B_1, C_1 exists. Prove that each of the points M, A, B, and C is the center of a circle tangent to all three sides (extended if necessary) of $\triangle A_1B_1C_1$. What is the difference in the behavior of acute and obtuse triangles ABC?

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17th Eötvös Competition 1910

Organized by Mathematical and Physical Society

1. If a, b, c are real numbers such that
$$a^2 + b^2 + c^2 = 1$$
prove the inequalities
$$-\frac{1}{2} \leq ab + bc + ca \leq 1.$$

2. Let a, b, c, d and u be integers such that each of the numbers
$$ac \quad , \quad bc + ad \quad , \quad bd$$
is a multiple of u. Show that bc and ad are multiples of u.

3. The lengths of sides CB and CA of $\triangle ABC$ are a and b, and the angle between them is $\gamma = 120°$. Express the length of the bisector of γ in terms of a and b.

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18th Eötvös Competition 1911

Organized by Mathematical and Physical Society

1. Show that, if the real numbers a, b, c, A, B, C satisfy

$$aC - 2bB + cA = 0 \quad \text{and} \quad ac - bz > 0,$$

then

$$AC - B^2 < 0.$$

2. Let Q be any point on a circle and let $P_1P_2P_3 \cdots P_8$ be a regular inscribed octagon. Prove that the sum of the fourth powers of the distances from Q to the diameters P_1P_5, P_2P_6, P_3P_7, P_4P_8 is independent of the position of Q.

3. Prove that $3^n + 1$ is not divisible by 2^n for any integer $n > 1$.

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19th Eötvös Competition 1912

Organized by Mathematical and Physical Society

1. How many positive integers of n digits exist such that each digit is 1, 2, or 3? How many of these contain all three of the digits 1, 2, and 3 at least once?

2. Prove that for every positive integer n, the number

$$A_n = 5^n + 2 \cdot 3^{n-1} + 1$$

is a multiple of 8.

3. Prove that the diagonals of a quadrilateral are perpendicular if and only if the sum of the squares of one pair of opposite sides equals that of the other.

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20th Eötvös Competition 1913

Organized by Mathematical and Physical Society

1. Prove thatfor every integer $n > 2$

$$(1 \cdot 2 \cdot 3 \cdots n)^2 > n^n.$$

2. Let O and O' designate two dìagonally opposite vertices of a cube. Bisect those edges of the cube that contain neither of the points O and O'. Prove that these midpoints of edges lie in a plane and form the vertices of a regular hexagon.

3. Let d denote the greatest common divisor of the natural numbers a and b, and let d' denote the greatest common divisor of the natural numbers a' and b'. Prove that dd' is the greatest common divisor of the four numbers

$$aa' \quad , \quad ab' \quad , \quad ba' \quad , \quad bb'.$$

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21st Eötvös Competition 1914

Organized by Mathematical and Physical Society

1. Let A and B be points on a circle k. Suppose that an arc k' of another circle, ℓ, connects A with B and divides the area inside the circle k into two equal parts. Prove that arc k' is longer than the diameter of k.

2. Suppose that
$$-1 \le ax^2 + bx + c \quad \le 1\text{for} \quad -1 \le x \le 1,$$
where a, b, c are real numbers. Prove that
$$-4 \le 2ax + b \le 4 \quad \le 1\text{for} \quad -1 \le x \le 1.$$

3. The circle k intersects the sides BC, CA, AB of triangle ABC in points A_1, A_2; B_1, B_2; C_1, C_2. The perpendiculars to BC, CA, AB through A_1, B_1, C_1, respectively, meet at a point M. Prove that the three perpendiculars to BC, CA, AB through A_2, B_2, and C_2, respectively, also meet in one point.

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22nd Eötvös Competition 1915

Organized by Mathematical and Physical Society

1. Let A, B, C be any three real numbers. Prove that there exists a number ν such that

$$An^2 + Bn + < n!$$

for every natural number $n > \nu$.

2. Triangle ABC lies entirely inside a polygon. Prove that the perimeter of triangle ABC is not greater than that of the polygon.

3. Prove that a triangle inscribed in a parallelogram has at most half the area of the parallelogram.

23rd Eötvös Competition 1916

Organized by Mathematical and Physical Society

1. If a and b are positive numbers, prove that the equation

$$\frac{1}{x}+\frac{1}{x-z}+\frac{1}{x-b}=0$$

   has two real roots, one between $a/3$ and $2a/3$, and one between $-2b/3$ and $-b/3$.

2. Let the bisector of the angle at C of triangle ABC intersect side AB in point D. Show that the segment CD is shorter than the *geometric mean* of the sides CA and CB. (The geometric mean of two positive numbers is the square root of their product; the geometric mean of n numbers is the nth root of their product.

3. Divide the numbers

$$1, 2, 3, 4, 5$$

   into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.

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24th Eötvös Competition 1917

Organized by Mathematical and Physical Society

1. If a and b are integers and if the solutions of the system of equations

$$\begin{cases} y - 2x - a = 0 \\ y^2 - xy + x^2 - b = 0 \end{cases}$$

are rational, prove that the solutions are integers.

2. In the square of an integer a, the tens' digit is 7. What is the units' digit of a^2?

3. Let A and B be two points inside a given circle k. Prove that there exist (infinitely many) circles through A and B which lie entirely in k.

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25th Eötvös Competition 1918

Organized by Mathematical and Physical Society

1. Let AC be the longer of the two diagonals of the parallelogram $ABCD$. Drop perpendiculars from C to AB and AD extended. If E and F are the feet of these perpendiculars, prove that
$$AB \cdot AE + AD \cdot AF = (AC)^2.$$

2. Find three distinct natural numbers such that the sum of their reciprocals is an integer.

3. If $a, b, c; p, q, r$ are real numbers such that, for every real number x,
$$ax^2 - 2bx + c \geq 0 \quad \text{and} \quad px^2 + 2qx + r \geq 0,$$
prove that then
$$apx^2 + bqx + cr \geq 0$$
for all real x.

Remark. No contests were held in the years 1919-1921.

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26th Eötvös Competition 1922

Organized by Mathematical and Physical Society

1. Given four points A, B, C, D in space, find a plane, S, equidistant from all four points and having A and C on one side, B and D on the other.

2. Prove that
$$x^4 + 2x^2 + 2x + 2$$
is not the product of two polynomials
$$x^2 + ax + b \quad \text{and} \quad x^2 + cx + d$$
in which a, b, c, d are integers.

3. Show that, if $a, b, \ldots, n$ are distinct natural numbers, none divisible by any primes greater than 3, then
$$\frac{1}{a} + \frac{1}{b} + \cdots \frac{1}{n} < 3$$

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27th Eötvös Competition 1923

Organized by Mathematical and Physical Society

1. Three circles through the point o and of radius r intersect pairwise in the additional points A, B, C. Prove that the circle through the points A, B, and C also has radius r.

2. If
$$s_n = 1 + q + q^2 + \cdots + q^n$$
and
$$S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n,$$
prove that
$$\binom{n+1}{1} + \binom{n+1}{2} s_1 + \binom{n+1}{3} s_2 + \cdots + \binom{n+1}{n+1} s_n = 2^n S_n$$

3. Prove that, if the terms of an infinite arithmetic progression of natural numbers are not all equal, they cannot all be primes.

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28th Eötvös Competition 1924

Organized by Mathematical and Physical Society

1. Let a, b, c be fixed natural numbers. Suppose that, for every positive integer n, there is a triangle whose sides have lengths a^n, b^n, and c^n respectively. Prove that these triangles are isosceles.

2. If O is a given point, ℓ a given line, and a a given positive number, find the locus of points P for which the sum of the distances from P to O and from P to ℓ is a.

3. Let A, B, and C be three given points in the plane; construct three cirdes, k_1, k_2, and k_3, such that k_2 and k_3 have a common tangent at A, k_3 and k_1 at B, and k_1 and k_2 at C.

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29th Eötvös Competition 1925

Organized by Mathematical and Physical Society

1. Let a, b, c, d be four integers. Prove that the product of the six differences
$$b-a,\ c-a,\ d-a,\ d-c,\ d-b,\ c-b$$
is divisible by 12.

2. How maay zeros are there at the end of the number
$$1000! = 1 \cdot 2 \cdot 3 \cdots 999 \cdot 1000?$$

3. Let r be the radius of the inscribed circle of a right triangle ABC. Show that r is less than half of either leg and less than one fourth of the hypotenuse.

30th Eötvös Competition 1926

Organized by Mathematical and Physical Society

1. Prove that, if a and b are given integers, the system of equatìons

$$\begin{cases} x + y + 2z + 2t = a \\ 2x - 2y + z - t = b \end{cases}$$

   has a solution in integers x, y, z, t.

2. Prove that the product of four consecutive natural numbers cannot be the square of an integer.

3. The circle k rolls along the inside of circle k; the radius of k is twice the radius of k'. Describe the path of a point on k.

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31st Eötvös Competition 1927

Organized by Mathematical and Physical Society

1. Let the integers a, b, c, d be relatively prime to
$$m = ad - bc.$$
Prove that the pairs of integers (x, y) for which $ax + by$ is a multiple of m are identical with those for which $cx + dy$ is a multiple of m.

2. Find the sum of all distinct four-digit numbers that contain only the digits 1, 2, 3, 4, 5, each at most once.

3. Consider the four circles tangent to all three lines containing the sides of a triangle ABC; let k and k_c be those tangent to side AB between A and B. Prove that the geometric mean of the radii of k and k_c, does not exceed half the length of AB.

32nd Eötvös Competition 1928

Organized by Mathematical and Physical Society

1. Prove that, among the positive numbers
$$a, 2a, \ldots, (n-1)a,$$
there is one that differs from an integer by at most $1/n$.

2. Put the numbers $1, 2, 3, \ldots, n$ on the circumference of a circle in such a way that the difference between neighboring numbers is at most 2. Prove that there is just one solution (if regard is paid only to the order in which the numbers are arranged).

3. Let ℓ be a given line, A and B given points of the plane. Choose a point P on ℓ so that the longer of the segments AP, BP is as short as possible. (If $AP = BP$, either segment may be taken as the *longer* one.)

Ercole Suppa

33rd Eötvös Competition 1929

Organized by Mathematical and Physical Society

1. In how many ways can the sum of 100 fillér be made up with coins of denominations 1, 2, 10, 20 and 50 fillér?

2. Let $k \leq n$ be positive integers and x be a real number with $0 \leq x < 1/n$. Prove that

$$\binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots + (-1)^k \binom{n}{k} x^k > 0$$

3. Let p, q and r be three concurrent lines in the plane such that the angle between any two of them is $60°$. Let a, b and c be real numbers such that $0 < a \leq b \leq c$.

   (a) Prove that the set of points whose distances from p, q and r are respectively less than a, b and c consists of the interior of a hexagon if and only if $a + b > c$.

   (b) Determine the length of the perimeter of this hexagon when $a + b > c$.

Ercole Suppa

Eötvös-Kürschák Competitions

34th Eötvös Competition 1930

Organized by Mathematical and Physical Society

1. How many five-digit multiples of 3 end with the digit 6 ?

2. A straight line is drawn across an 8×8 chessboard. It is said to pierce a square if it passes through an interior point of the square. At most how many of the 64 squares can this line pierce?

3. Inside an acute triangle ABC is a point P that is not the circumcenter. Prove that among the segments AP, BP and CP, at least one is longer and at least one is shorter than the circumradius of ABC.

Ercole Suppa

35th Eötvös Competition 1931

Organized by Mathematical and Physical Society

1. Let p be a prime greater than 2. Prove that $\frac{2}{p}$ can be expressed in exactly one way in the form
$$\frac{1}{x}+\frac{1}{y}$$
where x and y are positive integers with $x > y$.

2. Let $a_1^2+a_2^2+a_3^2+a_4^2+a_5^2=b^2$, where a_l, a_2, a_3, a_4, a_5, and b are integers. Prove that not all of these numbers can be odd.

3. Let A and B be two given points, distance 1 apart. Determine a point P on the line AB such that
$$\frac{1}{1+AP}+\frac{1}{1+BP}$$
is a maximum.

Ercole Suppa

Eötvös-Kürschák Competitions

36th Eötvös Competition 1932

Organized by Mathematical and Physical Society

1. Let a, b and n be positive integers such that b is divisible by a^n. Prove that $(a+1)^b - 1$ is divisible by a^{n+1}.

2. In triangle ABC, $AB \neq AC$. Let AF, AP and AT be the median, angle bisector and altitude from vertex A, with F, P and T on BG or its extension.

   (a) Prove that P always lies between F and T.

   (b) Prove that $\angle FAP < \angle PAT$ if ABC is an acute triangle.

3. Let α, β and γ be the interior angles of an acute triangle. Prove that if $\alpha < \beta < \gamma$, then $\sin 2\alpha > \sin 2\beta > \sin 2\gamma$.

Ercole Suppa

37th Eötvös Competition 1933

Organized by Mathematical and Physical Society

1. Let a, b, c and d be rea] numbers such that $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. Determine the value of $ab + cd$.

2. Sixteen squares of an 8×8 chessboard are chosen so that there are exactly lwo in each row and two in each column. Prove that eight white pawns and eight black pawns can be placed on these sixteen squares so that there is one white pawn and one black pawn in each row and in cach colunm.

3. The circles k_1 and k_2 are tangent at the point P. A line is drawn through P, cutting k_1 at A_1 and k_2 at A_2. A second line is drawn through P, cutting k_1 at B_l and k_2 at B_2. Prove that the triangles PA_1B_1 and PA_2B_2 are similar.

Ercole Suppa

Eötvös-Kürschák Competitions

38th Eötvös Competition 1934

Organized by Mathematical and Physical Society

1. Let n be a given positive integer and
$$A = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$
Prove that at least one term of the sequence A, $2A, 4A, 8A, \ldots, 2^k A, \ldots$ is an integer.

2. Which polygon inscribed in a given circle has the property that the sum of the squares of the lengths of its sides is maximum?

3. We are given an infinite set of rectangles in the plane, each with vertices of the form $(0,0)$, $(0,m)$, $(n,0)$ and (n,m), where m and n are positive integers. Prove that there exist two rectangles in the set such that one contains the other.

Ercole Suppa

39th Eötvös Competition 1935

Organized by Mathematical and Physical Society

1. Let n be a positive integer. Prove that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots \frac{a_n}{b_n} \geq n,$$

   where $(b_l, b_2, \ldots, b_n)$ is any permutation of the positive real numbers $a_1, a_2, \ldots, a_n$.

2. Prove that a finite point set cannot have more than one center of symmetry.

3. A real number is assigned to each vertex of a triangular prism so that the number on any vertex is the arithmetic mean of the numbers on the three adjacent vertices. Prove that all six numbers are equal.

Ercole Suppa

40th Eötvös Competition 1936

Organized by Mathematical and Physical Society

1. Prove that for all positive integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots\frac{1}{(2n-1)2n}=\frac{1}{n+1}\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

2. S is a point inside triangle ABC such that the areas of the triangles ABS, BCS and CAS are all equal. Prove that S is the centroid of ABC.

3. Let a be any positive integer. Prove that there exists a unique pair of positive integers x and y such that

$$x+\frac{1}{2}(x+y-1)(x+y-2)=a.$$

41st Eötvös Competition 1937

Organized by Mathematical and Physical Society

1. Let n be a positive integer. Prove that $a_1!a_2!\cdots a_n! < k!$, where k is an integer which is greater than the sum of the positive integers $a_1, a_2, \ldots, a_n$.

2. Two circles in space are said to be tangent to each other if they have a corni-non tangent at the same point of tangency. Assume that there are three circles in space which are mutually tangent at three distinct points. Prove that they either alI lie in a plane or all lie on a sphere.

3. Let n be a positive integer. Let $P, Q, A_1, A_2, \ldots, A_n$ be distinct points such that $A_1, A_2, \ldots, A_n$ are not collinear. Suppose that $PA_1 + PA_2 + \cdots + +PA_n$, and $QA_1 + QA_2 + \cdots + QA_n$, have a common value s for some reat number s. Prove that there exists a point R such that

$$RA_1 + RA_2 + \cdots + RA_n < s.$$

Ercole Suppa

Eötvös-Kürschák Competitions

42nd Eötvös Competition 1938

Organized by Mathematical and Physical Society

1. Prove that an integer n can be expressed as the sum of two squares if and only if $2n$ can be expressed as the sum of two squares.

2. Prove that for all integers $n > 1$,

$$\frac{1}{n} + \frac{1}{n+1} + \cdots \frac{1}{n^2-1} + \frac{1}{n^2} > 1$$

3. Prove that for any acute triangle, there is a point in space such that every line segment from a vertex of the triangle to a point on the line joining the other two vertices subtends a right angle at this point.

Ercole Suppa

Eötvös-Kürschák Competitions

43rd Eötvös Competition 1939

Organized by Mathematical and Physical Society

1. Let a_1, a_2, b_1, b_2, c_1 and c_2 be real numbers for which $a_1a_2 > 0$, $a_1c_1 \geq b_1^2$ and $a_2c_2 > b_2^2$. Prove that
$$(a_1 + a_2)(c_1 + c_2) \geq (b_1 + b_2)^2$$

2. Determine the highest power of 2 that divides $2^n!$.

3. ABC is an acute triangle. Three semicircles are constructed outwardly on the sides BC, CA and AB respectively. Construct points A', B' and C' on these semicìrcles respectively so that $AB' = AC'$, $BC' = BA'$ and $CA' = CB'$.

Ercole Suppa

44th Eötvös Competition 1940

Organized by Mathematical and Physical Society

1. In a set of objects, each has one of two colors and one of two shapes. There is at least one object of each color and at least one object of each shape. Prove that there exist two objects in the set that are different both in color and in shape.

2. Let m and n be distinct positive integers. Prove that $2^{2^m}+1$ and $2^{2^n}+1$ have no common divisor greater than 1.

3. (a) Prove that for any triangle H_1, there exists a triangle H_2 whose side lengths are equal to the lengths of the medians of H_1. (b) If H_3 is the triangle whose side lengths are equal to the lengths of the medians of H_2, prove that H_1 and H_3 are similar.

Ercole Suppa

45th Eötvös Competition 1941

Organized by Mathematical and Physical Society

1. Prove that

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^{k-1}}) = 1+x+x^2+x^3+\cdots+x^{2^k-1}.$$

2. Prove that if all four vertices of a parallelogram are lattice points and there are some other lattice points in or on the parallelogram, then its area exceeds 1.

3. The hexagon $ABCDEF$ is inscribed in a circle. The sides AB, CD and EF are all equal in length to the radius. Prove that the midpoints of the other three sides determine an equilateral triangle.

Ercole Suppa

46th Eötvös Competition 1942

Organized by Mathematical and Physical Society

1. Prove that in any triangle, at most one side can be shorter than the altitude from the opposite vertex.

2. Let a, b, c and d be integers such that for all integers m and n, there exist integers x and y such that $ax + by = m$, and $cx + dy = n$. Prove that $ad - bc = \pm 1$.

3. Let A', B' and C' be points on the sides BC, CA and AB, respectively, of an equilateral triangle ABC. If $AC' = 2C'B$, $BA' = 2A'C$ and $CB' = 2B'A$, prove that the lines AA', BB' and CC' enclose a triangle whose area is $1/7$ that of ABC.

47th Eötvös Competition 1943

Organized by Mathematical and Physical Society

1. Prove that in any group of people, the number of those who know an odd number of the others in the group is even. Assume that *knowing* is a symmetric relation.

2. Let P be any point inside an acute triangle. Let D and d be respectively the maximum and minimum distances from P to any point on the perimeter of the triangle.

   (a) Prove that $D \geq 2d$.

   (b) Determine when equality holds.

3. Let $a < b < c < d$ be real numbers and (x, y, z, t) be any permutation of a, b, c and d. What are the maximum and minimum values of the expression

$$(x-y)^2 + (y-z)^2 + (z-t)^2 + (t-x)^2?$$

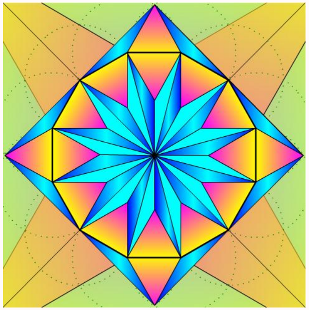
Remark. No contests were held in the years 1944-1945-1946.

2. Kürschák Competitions

József Kürschák's mother was Jozefa Teller and his father was András Kürschák, a manual worker who died when József was six years old. After the death of his father, József was brought up in Budapest by his mother. It is correct to say that József, although born in Buda, was brought up in Budapest after the death of his father since Budapest was created by the unification of Buda, Obuda and Pest in 1872. Kürschák attended secondary school in the flourishing city that became not only the capital of Hungary but also a major centre for industry, trade, communications, and architecture. Most importantly for the young boy, he was growing up in a city which was a centre for education, and for intellectual and artistic life.

Kürschák entered the Technical University of Budapest in 1881 and graduated in 1886 with qualifications to teach mathematics and physics in secondary schools. He then taught at a school in Roznyo, Slovakia, for two years before returning to the Technical University of Budapest to undertake research. He received his doctorate in 1890 and then taught in Budapest, at the Technical University, for the whole of his career. He was appointed in 1891 and successively promoted, achieving the rank of professor in 1900.

In [4] the authors describe a paper by Kürschák written in 1898 in which a regular dodecagon inscribed in a unit circle is investigated. A trigonometric argument can be used to show that its area of the dodecagon is 3 but Kürschák gives a purely geometric proof. He proves that the dodecagon can be dissected into a set of triangles which can be rearranged so as to fill three squares with sides having length 1. Kürschák's tile, which occurs in the title of [4], is constructed as follows. Start with a square and construct an equilateral triangle drawn inwards on each side. A regular dodecagon has the following 12 points as vertices: the 8 intersections of the corresponding sides of adjacent equilateral triangles and the 4 midpoints of the sides of the new square formed by the vertices of the four triangles. In fact the first paper which Kürschák wrote was concerned with polygons. This was Über dem Kreise ein-und umgeschriebene Vielecke (1887) which investigated extremal properties of polygons inscribed in, and circumscribed about, a circle.



Another topic which Kürschák investigated was the differential equations of the calculus of variations. Papers such as Über partielle Differentialgleichungen zweiter Ordnung mit gleichen Charakteristiken (1890), Über die partielle Differentialgleichung des Problems (1894), Über die Transformation der partiellen Differentialgleichungen der Variationsrechnung (1902), and Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung (1905) are examples of his work in this area. He proved invariance of the differential equations he was considering under contact transformations. Another problem he solved [3] in this area was to find, in his 1905 paper, necessary and sufficient conditions for:

. . . a second-order differential expressions to provide the equation belonging to the variation of a multiple integral.

Led to consider linear algebra in the context of the above work, he wrote a number of papers on matrices and determinants such as Über symmetrische Matrices (1904) and Ein irreduzibilitätssatz in der Theorie der symmetrischen matrizen (1921).

Kürschák's most important work, however, was in 1912 when he founded the theory of valuations. His idea was to define $|x|_p$ for a rational number x as follows. Express x in lowest terms as $p^n \frac{a}{b}$ and then define

$$|x|_p = p^{-n}$$

For example

$$\left|\frac{20}{11}\right|_2 = \frac{1}{4} \quad , \quad \left|\frac{20}{11}\right|_3 = 1 \quad , \quad \left|\frac{20}{11}\right|_5 = \frac{1}{5} \quad , \quad \left|\frac{20}{11}\right|_{11} = 11$$

The rationals are not complete with this metric and their completion is the field of p-adic numbers. Kürschák's work was inspired by earlier work of Julius König, Steinitz and Hensel. The importance of Kürschák's valuations is that they allow notions of convergence and limits be used in the theory of abstract fields and greatly enrich the topic.

Dénes Kónig and Von Neumann were both students of Kürschák and many other famous mathematicians benefited from his teaching. Dénes Kónig received his doctorate in 1907 for a thesis written under Kürschák's supervision. Peter writes in [5] that:

. . . outstanding mathematicians such as Hunyadi, Julius König, Kürschák and Rados have contributed to the high standard of mathematical education at the Technical University [of Budapest]. Their scientific and teaching activity affected mathematical life in the whole country and laid the foundation of the internationally recognized mathematical school in Hungary.

Indeed Kürschák achieved this through his excellent teaching as well as bringing the very best out of his students. He was one of the main organisers of mathematical competitions and to honour his outstanding contributions in this area the Loránd Eötvös Mathematics Competition, started in 1925, was renamed the József Kürschák Mathematics Competition in 1949.

Kürschák was elected to the Hungarian Academy of Sciences in 1897.

Ercole Suppa

48th Eötvös-Kürschák Competition 1947

Organized by János Bolyai Mathematical Society

1. Prove that $46^{2n+1} + 296 \cdot 13^{2n+1}$ is divisible by 1947.
2. Show that any graph with 6 points has a triangle or three points which are not joined to each other.
3. What is the smallest number of disks radius $\frac{1}{2}$ that can cover a disk radius 1?

49th Eötvös-Kürschák Competition 1948

Organized by János Bolyai Mathematical Society

1. Knowing that 23 October 1948 was a Saturday, which is more frequent for New Year's Day, Sunday or Monday?

2. A convex polyhedron has no diagonals (every pair of vertices are connected by an edge). Prove that it is a tetrahedron.

3. Prove that among any n positive integers one can always find some (at least one) whose sum is divisible by n.

Ercole Suppa

50th Eötvös-Kürschák Competition 1949

Organized by János Bolyai Mathematical Society

1. Prove that $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x > 0$ for $0 < x < 180°$.
2. P is a point on the base of an isosceles triangle. Lines parallel to the sides through P meet the sides at Q and R. Show that the reflection of P in the line QR lies on the circumcircle of the triangle.
3. Which positive integers cannot be represented as a sum of (two or more) consecutive integers?

51st Eötvös-Kürschák Competition 1950

Organized by János Bolyai Mathematical Society

1. Several people visited a library yesterday. Each one visited the library just once (in the course of yesterday). Amongst any three of them, there were two who met in the library. Prove that there were two moments T and T' yesterday such that everyone who visited the library yesterday was in the library at T or T' (or both).

2. Three circles C_1, C_2, C_3 in the plane touch each other (in three different points). Connect the common point of C_1 and C_2 with the other two common points by straight lines. Show that these lines meet C_3 in diametrically opposite points.

3. (x_1, y_1, z_1) and (x_2, y_2, z_2) are triples of real numbers such that for every pair of integers (m, n) at least one of $x_1m + y_1n + z_1$, $x_2m + y_2n + z_2$ is an even integer. Prove that one of the triples consists of three integers.

Ercole Suppa

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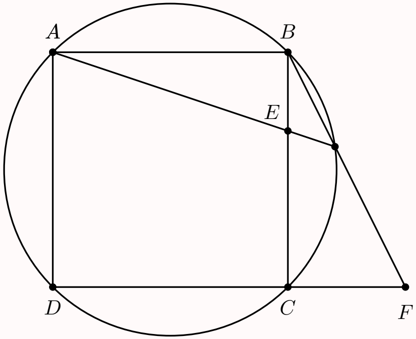
52nd Eötvös-Kürschák Competition 1951

Organized by János Bolyai Mathematical Society

1. $ABCD$ is a square. E is a point on the side BC such that $BE = \frac{1}{3}BC$, and F is a point on the ray DC such that $CF = \frac{1}{2}DC$. Prove that the lines AE and BF intersect on the circumcircle of the square.



2. For which $m > 1$ is $(m-1)!$ divisible by m?

3. An open half-plane is the set of all points lying to one side of a line, but excluding the points on the line itself. If four open half-planes cover the plane, show that one can select three of them which still cover the plane.

Ercole Suppa

Eötvös-Kürschák Competitions

53rd Eötvös-Kürschák Competition 1952

Organized by János Bolyai Mathematical Society

1. A circle C touches three pairwise disjoint circles whose centers are collinear and none of which contains any of the others. Show that its radius must be larger than the radius of the middle of the three circles.

2. Show that if we choose any $n+2$ distinct numbers from the set $\{1, 2, 3, \ldots, 3n\}$ there will be two whose difference is greater than n and smaller than $2n$.

3. ABC is a triangle. The point A' lies on the side opposite to A and $BA'/BC = k$, where $\frac{1}{2} < k < 1$. Similarly, B' lies on the side opposite to B with $CB'/CA = k$, and C' lies on the side opposite to C with $AC'/AB = k$. Show that the perimeter of $A'B'C'$ is less than k times the perimeter of ABC.

Ercole Suppa

Eötvös-Kürschák Competitions

54th Eötvös-Kürschák Competition 1953

Organized by János Bolyai Mathematical Society

1. A and B are any two subsets of $\{1, 2, \dots, n-1\}$ such that $|A| + |B| > n - 1$. Prove that one can find a in A and b in B such that $a + b = n$.

2. n and d are positive integers such that d divides $2n^2$. Prove that $n^2 + d$ cannot be a square.

3. $ABCDEF$ is a convex hexagon with all its sides equal. Also $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$. Show that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

Ercole Suppa

55th Eötvös-Kürschák Competition 1954

Organized by János Bolyai Mathematical Society

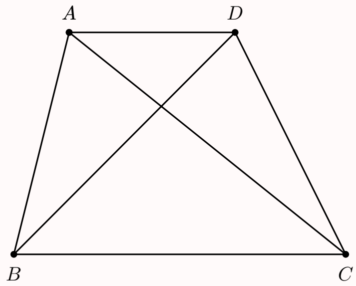
1. $ABCD$ is a convex quadrilateral with $AB + BD = AC + CD$. Prove that $AB < AC$.
2. Every planar section of a three-dimensional body B is a disk. Show that B must be a ball.
3. A tournament is arranged amongst a finite number of people. Every person plays every other person just once and each game results in a win to one of the players (there are no draws). Show that there must a person X such that, given any other person Y in the tournament, either X beat Y, or X beat Z and Z beat Y for some Z.

Ercole Suppa

56th Eötvös-Kürschák Competition 1955

Organized by János Bolyai Mathematical Society

1. Prove that if the two angles on the base of a trapezoid are different, then the diagonal starting from the smaller angle is longer than the other diagonal.



2. How many five digit numbers are divisible by 3 and contain the digit 6?

3. The vertices of a triangle are lattice points (they have integer coordinates). There are no other lattice points on the boundary of the triangle, but there is exactly one lattice point inside the triangle. Show that it must be the centroid.

Ercole Suppa

Eötvös-Kürschák Competitions

57th Eötvös-Kürschák Competition 1957

Organized by János Bolyai Mathematical Society

1. ABC is an acute-angled triangle. D is a variable point in space such that all faces of the tetrahedron $ABCD$ are acute-angled. P is the foot of the perpendicular from D to the plane ABC. Find the locus of P as D varies.

2. A factory produces several types of mug, each with two colors, chosen from a set of six. Every color occurs in at least three different types of mug. Show that we can find three mugs which together contain all six colors.

3. What is the largest possible value of

$$|a_1 - 1| + |a_2 - 2| + \cdots + |a_n - n|$$

where $a_1, a2, \ldots, a_n$ is a permutation of $1, 2, \ldots, n$?

58th Eötvös-Kürschák Competition 1958

Organized by János Bolyai Mathematical Society

1. Given any six points in the plane, no three collinear, show that we can always find three which form an obtuse-angled triangle with one angle at least 120°.

2. Show that if m and n are integers such that $m^2 + mn + n^2$ is divisible by 9, then they must both be divisible by 3.

3. The hexagon $ABCDEF$ is convex and opposite sides are parallel. Show that the triangles ACE and BDF have equal area.

Ercole Suppa

Ercole Suppa

59th Eötvös-Kürschák Competition 1959

Organized by János Bolyai Mathematical Society

1. a, b, c are three distinct integers and n is a positive integer. Show that

$$\frac{a^n}{(a-b)(a-c)} + \frac{b^n}{(b-a)(b-c)} + \frac{c^n}{(c-a)(c-b)}$$

   is an integer.

2. The angles subtended by a tower at distances 100, 200 and 300 from its foot sum to 90°. What is its height?

3. Three brothers and their wives visited a friend in hospital. Each person made just one visit, so that there were six visits in all. Some of the visits overlapped, so that each of the three brothers met the two other brothers' wives during a visit. Show that one brother must have met his own wife during a visit.

Ercole Suppa

60th Eötvös-Kürschák Competition 1960

Organized by János Bolyai Mathematical Society

1. Among any four people at a party there is one who has met the three others before the party. Show that among any four people at the party there must be one who has met everyone at the party before the party.

2. Let $a_1 = 1, a_2, a_3, \dots$ be a sequence of positive integers such that

$$a_k < 1 + a_1 + a_2 + \cdots + a_{k-1}$$

for all $k > 1$. Prove that every positive integer can be expressed as a sum of ais.

3. E is the midpoint of the side AB of the square $ABCD$, and F, G are any points on the sides BC, CD such that EF is parallel to AG. Show that FG touches the inscribed circle of the square.

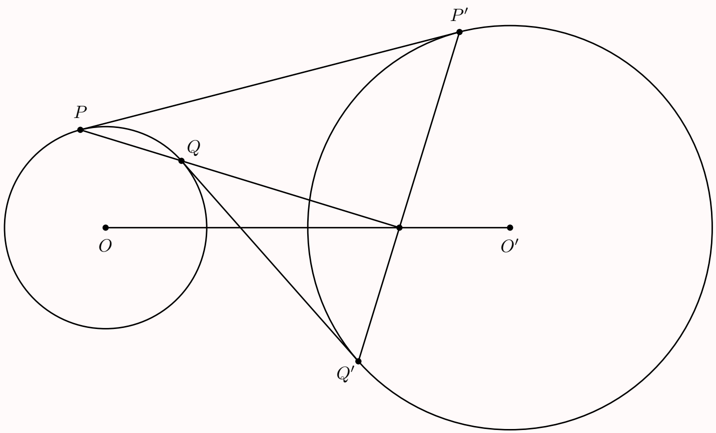
61st Eötvös-Kürschák Competition 1961

Organized by János Bolyai Mathematical Society

1. Given any four distinct points in the plane, show that the ratio of the largest to the smallest distance between two of them is at least $\sqrt{2}$.

2. x, y, z are positive reals less than 1. Show that at least one of $(1-x)y$, $(1-y)z$ and $(1-z)x$ does not exceed $\frac{1}{4}$.

3. Two circles centers O and O' are disjoint. PP' is an outer tangent (with P on the circle center O, and P' on the circle center O'). Similarly, QQ' is an inner tangent (with Q on the circle center O, and Q' on the circle center O'). Show that the lines PQ and $P'Q'$ meet on the line OO'.



Ercole Suppa

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62nd Eötvös-Kürschák Competition 1962

Organized by János Bolyai Mathematical Society

1. Show that the number of ordered pairs (a, b) of positive integers with lowest common multiple n is the same as the number of positive divisors of n^2.
2. Show that given any $n+1$ diagonals of a convex n-gon, one can always find two which have no common point.
3. P is any point of the tetrahedron $ABCD$ except D. Show that at least one of the three distances DA, DB, DC exceeds at least one of the distances PA, PB and PC.

Ercole Suppa

Ercole Suppa

63rd Eötvös-Kürschák Competition 1963

Organized by János Bolyai Mathematical Society

1. mn students all have different heights. They are arranged in $m > 1$ rows of $n > 1$. In each row select the shortest student and let A be the height of the tallest such. In each column select the tallest student and let B be the height of the shortest such. Which of the following are possible: $A < B$, $A = B$, $A > B$? If a relation is possible, can it always be realized by a suitable arrangement of the students?

2. A is an acute angle. Show that

$$(1 + \frac{1}{\operatorname{sen} A})(1 + \frac{1}{\cos A}) > 5$$

3. A triangle has no angle greater than $90°$. Show that the sum of the medians is greater than four times the circumradius.

Ercole Suppa

Eötvös-Kürschák Competitions

64th Eötvös-Kürschák Competition 1964

Organized by János Bolyai Mathematical Society

1. ABC is an equilateral triangle. D and D' are points on opposite sides of the plane ABC such that the two tetrahedra $ABCD$ and $ABCD'$ are congruent (but not necessarily with the vertices in that order). If the polyhedron with the five vertices A, B, C, D, D' is such that the angle between any two adjacent faces is the same, find $\frac{DD'}{AB}$.

2. At a party every girl danced with at least one boy, but not with all of them. Similarly, every boy danced with at least one girl, but not with all of them. Show that there were two girls G and G' and two boys B and B', such that each of B and G danced, B' and G' danced, but B and G' did not dance, and B' and G did not dance.

3. Show that for any positive reals w, x, y, z we have

$$\sqrt{\frac{w^2+x^2+y^2+z^2}{4}} \geq \sqrt[3]{\frac{wxy+wxz+wyz+xyz}{4}}$$

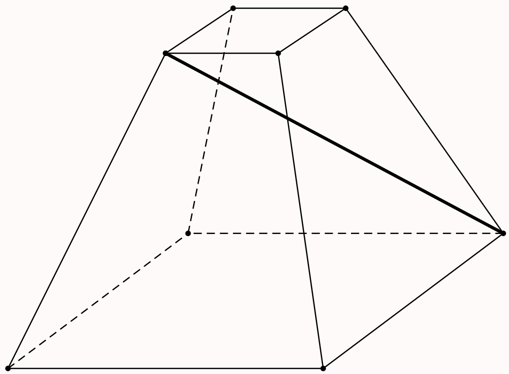
65th Eötvös-Kürschák Competition 1965

Organized by János Bolyai Mathematical Society

1. What integers a, b, c satisfy $a^2 + b^2 + c^2 + 3 < ab + 3b + 2c$?

2. D is a closed disk radius R. Show that among any 8 points of D one can always find two whose distance apart is less than R.

3. A pyramid has square base and equal sides. It is cut into two parts by a plane parallel to the base. The lower part (which has square top and square base) is such that the circumcircle of the base is smaller than the circumcircles of the lateral faces. Show that the shortest path on the surface joining the two endpoints of a spatial diagonal lies entirely on the lateral faces.



Ercole Suppa

Ercole Suppa

66th Eötvös-Kürschák Competition 1966

Organized by János Bolyai Mathematical Society

1. Can we arrange 5 points in space to form a pentagon with equal sides such that the angle between each pair of adjacent edges is $90°$?
2. Show that the n digits after the decimal point in $\left(5+\sqrt{26}\right)^n$ are all equal.
3. Do there exist two infinite sets of non-negative integers such that every non-negative integer can be uniquely represented in the form $a+b$ with a in A and b in B?

Ercole Suppa

Eötvös-Kürschák Competitions

67th Eötvös-Kürschák Competition 1967

Organized by János Bolyai Mathematical Society

1. A is a set of integers which is closed under addition and contains both positive and negative numbers. Show that the difference of any two elements of A also belongs to A.

2. A convex n-gon is divided into triangles by diagonals which do not intersect except at vertices of the n-gon. Each vertex belongs to an odd number of triangles. Show that n must be a multiple of 3.

3. For a vertex X of a quadrilateral, let $h(X)$ be the sum of the distances from X to the two sides not containing X. Show that if a convex quadrilateral $ABCD$ satisfies $h(A) = h(B) = h(C) = h(D)$, then it must be a parallelogram.

Ercole Suppa

68th Eötvös-Kürschák Competition 1968

Organized by János Bolyai Mathematical Society

1. In an infinite sequence of positive integers every element (starting with the second) is the harmonic mean of its neighbors. Show that all the numbers must be equal.

2. There are $4n$ segments of unit length inside a circle radius n. Show that given any line L there is a chord of the circle parallel or perpendicular to L which intersects at least two of the $4n$ segments.

3. For each arrangement X of n white and n black balls in a row, let $f(X)$ be the number of times the color changes as one moves from one end of the row to the other. For each k such that $0 < k < n$, show that the number of arrangements X with $f(X) = n - k$ is the same as the number with $f(X) = n + k$.

Ercole Suppa

69th Eötvös-Kürschák Competition 1969

Organized by János Bolyai Mathematical Society

1. Show that if $2 + 2\sqrt{28n^2 + 1}$ is an integer, then it is a square (for n an integer).

2. A triangle has side lengths a, b, c and angles A, B, C as usual (with b opposite B etc). Show that if
$$a(1 - 2\cos A) + b(1 - 2\cos B) + c(1 - 2\cos C) = 0$$
then the triangle is equilateral.

3. We are given 64 cubes, each with five white faces and one black face. One cube is placed on each square of a chessboard, with its edges parallel to the sides of the board. We are allowed to rotate a complete row of cubes about the axis of symmetry running through the cubes or to rotate a complete column of cubes about the axis of symmetry running through the cubes. Show that by a sequence of such rotations we can always arrange that each cube has its black face uppermost.

Ercole Suppa

70th Eötvös-Kürschák Competition 1970

Organized by János Bolyai Mathematical Society

1. What is the largest possible number of acute angles in an n-gon which is not self-intersecting (no two non-adjacent edges interesect)?

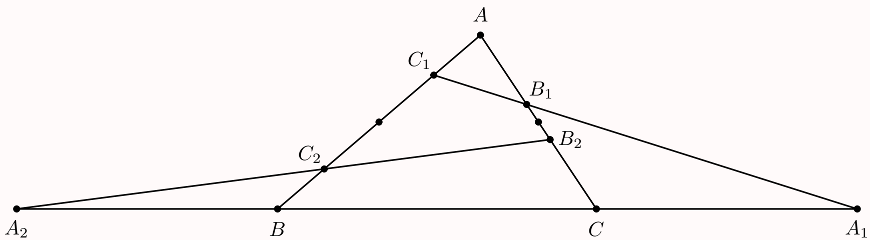
2. A valid lottery ticket is formed by choosing 5 distinct numbers from $1, 2, 3, \ldots, 90$. What is the probability that the winning ticket contains at least two consecutive numbers?

3. n points are taken in the plane, no three collinear. Some of the line segments between the points are painted red and some are painted blue, so that between any two points there is a unique path along colored edges. Show that the uncolored edges can be painted (each edge either red or blue) so that all triangles have an odd number of red sides.

71st Eötvös-Kürschák Competition 1971

Organized by János Bolyai Mathematical Society

1. A straight line cuts the side AB of the triangle ABC at C_1, the side AC at B_1 and the line BC at A_1. C_2 is the reflection of C_1 in the midpoint of AB, and B_2 is the reflection of B_1 in the midpoint of AC. The lines B_2C_2 and BC intersect at A_2.



   Prove that

$$\frac{\operatorname{sen} B_1A_1C}{\operatorname{sen} C_2A_2B} = \frac{B_2C_2}{B_1C_1}$$

2. Given any 22 points in the plane, no three collinear. Show that the points can be divided into 11 pairs, so that the 11 line segments defined by the pairs have at least five different intersections.

3. There are 30 boxes each with a unique key. The keys are randomly arranged in the boxes, so that each box contains just one key and the boxes are locked. Two boxes are broken open, thus releasing two keys. What is the probability that the remaining boxes can be opened without forcing them?

Ercole Suppa

72nd Eötvös-Kürschák Competition 1972

Organized by János Bolyai Mathematical Society

1. A triangle has side lengths a, b, c. Prove that

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 4abc > a^3 + b^3 + c^3$$

2. A class has $n > 1$ boys and n girls. For each arrangement X of the class in a line let $f(X)$ be the number of ways of dividing the line into two non-empty segments, so that in each segment the number of boys and girls is equal. Let the number of arrangements with $f(X) = 0$ be A, and the number of arrangements with $f(X) = 1$ be B. Show that $B = 2A$.

3. $ABCD$ is a square side 10. There are four points P_1, P_2, P_3, P_4 inside the square. Show that we can always construct line segments parallel to the sides of the square of total length 25 or less, so that each P_i is linked by the segments to both of the sides AB and CD. Show that for some points P_i it is not possible with a total length less than 25.

73rd Eötvös-Kürschák Competition 1973

Organized by János Bolyai Mathematical Society

1. For what positive integers n, k (with $k < n$) are the binomial coefficients

$$\binom{n}{k-1} \quad , \quad \binom{n}{k} \quad , \quad \binom{n}{k+1}$$

three successive terms of an arithmetic progression?

2. For any positive real r, let $d(r)$ be the distance of the nearest lattice point from the circle center the origin and radius r. Show that $d(r)$ tends to zero as r tends to infinity.

3. $n > 4$ planes are in general position (so every 3 planes have just one common point, and no point belongs to more than 3 planes). Show that there are at least $\frac{2n-3}{4}$ tetrahedra among the regions formed by the planes.

Ercole Suppa

Ercole Suppa

74th Eötvös-Kürschák Competition 1974

Organized by János Bolyai Mathematical Society

1. A library has one exit and one entrance and a blackboard at each. Only one person enters or leaves at a time. As he does so he records the number of people found/remaining in the library on the blackboard. Prove that at the end of the day exactly the same numbers will be found on the two blackboards (possibly in a different order).

2. S_n is a square side $\frac{1}{n}$. Find the smallest k such that the squares $S_1, S_2, S_3, \cdots$ can be put into a square side k without overlapping.

3. Let
$$p_k(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-x)^{2k}}{(2k)!}$$
Show that it is non-negative for all real x and all positive integers k.

Ercole Suppa

75th Eötvös-Kürschák Competition 1975

Organized by János Bolyai Mathematical Society

1. Transform the equation
$$ab^2\left(\frac{1}{(a+c)^2}+\frac{1}{(a-c)^2}\right)=(a-b)$$
into a simpler form, given that $a > c \geq 0$, $b > 0$.

2. Prove or disprove: given any quadrilateral inscribed in a convex polygon, we can find a rhombus inscribed in the polygon with side not less than the shortest side of the quadrilateral.

3. Let
$$x_0 = 5 \quad , \quad x_{n+1} = x_n + \frac{1}{x_n}$$
Prove that $45 < x_{1000} < 45.1$.

Ercole Suppa

76th Eötvös-Kürschák Competition 1976

Organized by János Bolyai Mathematical Society

1. $ABCD$ is a parallelogram. P is a point outside the parallelogram such that angles $\angle PAB$ and $\angle PCB$ have the same value but opposite orientation. Show that $\angle APB = \angle DPC$.

2. A lottery ticket is a choice of 5 distinct numbers from $1, 2, 3, \ldots, 90$. Suppose that 55 distinct lottery tickets are such that any two of them have a common number. Prove that one can find four numbers such that every ticket contains at least one of the four.

3. Prove that if the quadratic $x^2 + ax + b$ is always positive (for all real x) then it can be written as the quotient of two polynomials whose coefficients are all positive.

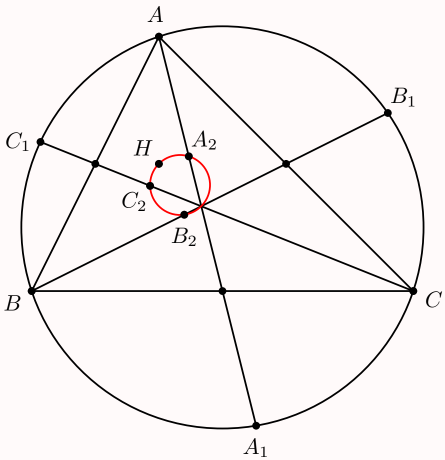
Ercole Suppa

77th Eötvös-Kürschák Competition 1977

Organized by János Bolyai Mathematical Society

1. Show that there are no integers n such that $n^4 + 4^n$ is a prime greater than 5.

2. ABC is a triangle with orthocenter H. The median from A meets the circumcircle again at A_1, and A_2 is the reflection of A_1 in the midpoint of BC. The points B_2 and C_2 are defined similarly. Show that H, A_2, B_2 and C_2 lie on a circle.



3. Three schools each have n students. Each student knows a total of $n + 1$ students at the other two schools. Show that there must be three students, one from each school, who know each other.

78th Eötvös-Kürschák Competition 1978

Organized by János Bolyai Mathematical Society

1. a and b are rationals. Show that if $ax^2 + by^2 = 1$ has a rational solution (in x and y), then it must have infinitely many.

2. The vertices of a convex n-gon are colored so that adjacent vertices have different colors. Prove that if n is odd, then the polygon can be divided into triangles with non-intersecting diagonals such that no diagonal has its endpoints the same color.

3. A triangle has inradius r and circumradius R. Its longest altitude has length H. Show that if the triangle does not have an obtuse angle, then $H \geq r + R$. When does equality hold?

79th Eötvös-Kürschák Competition 1979

Organized by János Bolyai Mathematical Society

1. The base of a convex pyramid has an odd number of edges. The lateral edges of the pyramid are all equal, and the angles between neighbouring faces are all equal. Show that the base must be a regular polygon.

2. f is a real-valued function defined on the reals such that $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all x, y. Prove that $f(x) = x$ for all x.

3. An $n \times n$ array of letters is such that no two rows are the same. Show that it must be possible to omit a column, so that the remaining table has no two rows the same.

Ercole Suppa

80th Eötvös-Kürschák Competition 1980

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. The points of space are coloured with five colours, with all colours being used. Prove that some plane contains four points of different colours.

2. Let $n > 1$ be an odd integer. Prove that a necessary and sufficient condition for the existence of positive integers x and y satisfying
$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}$$
is that n has a prime divisor of the form $4k - 1$.

3. In a certain country there are two tennis clubs consisting of 1000 and 1001 members respectively. All the members have different playing strength, and the descending order of palying strengths in each club is known. Find a procedure which determines, within 11 games, who is in the 1001st place among the 2001 players in these clubs. It is assumed that a stronger player always beats a weaker one.

Ercole Suppa

Eötvös-Kürschák Competitions

81st Eötvös-Kürschák Competition 1981

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Prove that

$$AB + PQ + QR + RP \leq AP + AQ + AR + BP + BQ + BR$$

where A, B, P, Q and R are any five points in a plane.

2. Let $n > 2$ be an even number. The squares of an $n \times n$ chessboard are coloured with $\frac{1}{2}n^2$ colours in such a way that every colour is used for colouring exactly two of the squares. Prove that one can place n rooks on squares of n different colours such that no two of the rooks can take each other.

3. For a positive integer n, $r(n)$ denote the sum of the remainders when n is divided by $1, 2, \ldots, n$ respectively. Prove that $r(k) = r(k-1)$ for infinitely many positive integers k.

Ercole Suppa

Eötvös-Kürschák Competitions

82nd Eötvös-Kürschák Competition 1982

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. A cube of integral dimensions is given in space so that all four vertices of one of the faces are lattice points. Prove that the other four vertices are also lattice points.

2. Prove that for any integer $k > 2$, there exist infinitely many positive integers n such that the least common multiple of $n, n+1, \ldots, n+k-1$ is greater than the least common multiple of $n+1, n+2, \ldots, n+k$.

3. The set of integers is coloured in 100 colours in such a way that all the colours are used and the following is true. For any choice of intervals $[a, b]$ and $[c, d]$ of equal length and with integral endpoints, if a and c as well as b and d, respectively, have the same colour, then the whole intervals $[a, b]$ and $[c, d]$ are identically coloured in that, for any integer x, $0 \leq x \leq b-a$, the numbers $a+x$ and $c+x$ are of the same colour. Prove that -1982 and 1982 are of different colours.

Ercole Suppa

83rd Eötvös-Kürschák Competition 1983

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Let x, y and z be rational numbers satisfying

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0$$

   Prove that $x = y = z = 0$.

2. Prove that $f(2) \geq 3^n$ where the polynomial $f(x) = x^n + a_1 x_{n-1} + \cdots + a_{n-1}x + 1$ has non-negative coefficients and n real roots.

3. Given are $n + 1$ points $P_1, P_2, \ldots, P_n$ and Q in the plane, no three collinear. For any two different points P_i and P_j , there is a point P_k such that the point Q lies inside the triangle $P_i P_j P_k$. Prove that n is an odd number.

Ercole Suppa

84th Eötvös-Kürschák Competition 1984

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Writing down the first 4 rows of the Pascal triangle in the usual way and then adding up the numbers in vertical columns, we obtain 7 numbers as shown above. If we repeat this procedure with the first 1024 rows of the Pascal triangle, how many of the 2047 numbers thus obtained will be odd?

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
\hline
1 & 1 & 4 & 3 & 4 & 1 & 1
\end{array}
$$

2. $A_1B_1A_2$, $B_1A_2B_2$, $A_2B_2A_3$, $\ldots$, $B_{13}A_{14}B_{14}$, $A_{14}B_{14}A_1$ and $B_{14}A_1B_1$ are equilateral rigid plates that can be folded along the edges $A_1B_1, B_1A_2, \ldots, A_{14}B_{14}$ and $B_{14}A_1$ respectively. Can they be folded so that all 28 plates lie in the same plane?

3. Given are n integers, not necessarily distinct, and two positive integers p and q. If the n numbers are not all distinct, choose two equal ones. Add p to one of them and subtract q from the other. If there are still equal ones among the n numbers, repeat this procedure. Prove that after a finite number of steps, all n numbers are distinct.

85th Eötvös-Kürschák Competition 1985

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. The convex $(n+1)$-gon $P_0P_1\dots P_n$ is partitioned into $n-1$ triangles by $n-2$ nonintersecting diagonals. Prove that the triangles can be numbered from 1 to $n-1$ such that for $1 \le i \le n-1$, P_i is a vertex of the triangle numbered i.

2. Let n be a positive integer. For each prime divisor p of n, consider the highest power of p which does not exceed n. The sum of these powers is defined as the power-sum of n. Prove that there exist infinitely many positive integers which are less than their respective power-sums.

3. Let each vertex of a triangle be reflected across the opposite side. Prove that the area of the triangle determined by the three points of reflection is less than five times the area of the original triangle.

Ercole Suppa

86th Eötvös-Kürschák Competition 1986

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Prove that three rays from a given point contain three face diagonals of a rectangular block if and only if the rays include pairwise acute angles such that their sum is $180°$.

2. Let n be an integer greater than 2. Find the maximum value for h and the minimum value for H such that for any positive numbers $a_1, a_2, \ldots, a_n$,

$$h < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \cdots + \frac{a_n}{a_n + a_1} < H$$

3. A and B play the following game. They arbitrarily select k of the first 100 positive integers. If the sum of the selected numbers is even, then A wins. If their sum is odd, then B wins. For what values of k is the game fair?

Ercole Suppa

Eötvös-Kürschák Competitions

87th Eötvös-Kürschák Competition 1987

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Find all 4-tuples (a, b, c, d) of distinct positive integers for which $a + b = cd$ and $ab = c + d$.

2. Does there exist an infinite set of points in space having at least one but finitely many points on each plane?

3. A club has $3n + 1$ members. Every two members play exactly one of tennis, chess and table tennis against each other. Moreover, each member plays each game against exactly n other members. Prove that there exist three members such that every two of the three play a different game.

88th Eötvös-Kürschák Competition 1988

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. P is a point inside a convex quadrilateral $ABCD$ such that the areas of the triangles PAB, PBC, PCD and PDA are all equal. Prove that either AC or BD bisects the area of $ABCD$.

2. From among the numbers $1, 2, \ldots, n$, we want to select triples (a, b, c) such that $a < b < c$ and, for two selected triples (a, b, c) and (a', b', c'), at most one of the equalities $a = a'$, $b = b'$ and $c = c'$ holds. What is the maximum number of such triples?

3. The vertices of a convex quadrilateral $PQRS$ are lattice points. E is the point of intersection of PR and QS and $\angle SPQ + \angle PQR < 180°$. Prove that there exists a lattice point other than P or Q which lies inside or on the boundary of triangle PQE.

Ercole Suppa

Eötvös-Kürschák Competitions

89th Eötvös-Kürschák Competition 1989

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Given are two intersecting lines e and f and a circle having no points of intersection with the lines. Construct a line parallel to f such that the ratio of the lengths of the sections of this line within the circle and between e and the circle is maximum.

2. For any positive integer n, denote by $S(n)$ the sum of its digits in base ten. For which positive integers M is it true that $S(Mk) = S(M)$ for all integers k such that $1 \leq k \leq M$?

3. From an arbitrary point (x, y) in the coordinate plane, one is allowed to move to $(x, y + 2x)$, $(x, y - 2x)$, $(x + 2y, y)$ or $(x, x - 2y)$. However, one cannot reverse the immediately preceding move. Prove that starting from the point $\left(1, \sqrt{2}\right)$, it is not possible to return there after any number of moves.

Ercole Suppa

90th Eötvös-Kürschák Competition 1990

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Given in the plane are two intersecting lines e and f, and a circle C which does not intersect the two lines. For an arbitrary chord AB of C parallel to f, the line AB intersects e at E. Construct AB so as to maximize $\frac{AB}{AE}$.

2. For a positive integer n, let $S(n)$ denote the sum of its digits in base 10. Find all positive integers m such that $S(m) = S(km)$ for all positive integers $k \leq m$.

3. From a point (x, y) in the coordinate plane, we can go north to $(x, y + 2x)$, south to $(x, y - 2x)$, east to $(x + 2y, y)$ or west to $(x - 2y, y)$. No other moves are permitted, and if we move from P to Q, we cannot double back to P on the very next move. Prove that if we start from $\left(1, \sqrt{2}\right)$, we can never return there.

Ercole Suppa

91st Eötvös-Kürschák Competition 1991

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Prove that $(ab+c)^n - c \leq a^n(b+c)^n - a^n c$, where n is a positive integer and $a \geq 1$, $b \geq 1$ and $c > 0$ are real numbers.

2. A convex polyhedron has two triangular faces and three quadrilateral faces. Each vertex of one of the triangular faces is joined to the point of intersection of the diagonals of the opposite quadrilateral face. Prove that these three lines are concurrent.

3. Given are 998 red points in the plane, no three on a line. A set of blue points is chosen so that every triangle with all three vertices among the red points contains a blue point in its interior. What is the minimum size of a set of blue points which works regardless of the positions of the red points?

Ercole Suppa

92nd Eötvös-Kürschák Competition 1992

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Given n positive numbers, define their *strange mean* as the sum of the squares of the numbers divided by the sum of the numbers. Define their *third power mean* as the cube root of the arithmetic mean of their third powers.

   (a) For $n = 2$, determine which of the following assertions is true.

   (i) The strange mean can never be smaller than the third power mean.

   (ii) The strange mean can never be larger than the third power mean.

   (iii) Depending on the choice of numbers, the strange mean might be larger or smaller than the third power mean.

   (b) Answer the same question for $n = 3$.

2. For an arbitrary positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. For $n > 1$, let $f_n(k) = f_1(f_{n-1}(k))$. What is the value of $f_{1992}\left(2^{1991}\right)$?

3. Given a finite number of points in the plane, no three of which are collinear, prove that they can be coloured in two colours so that there is no half-plane that contains exactly three given points of one colour and no points of the other colour.

Ercole Suppa

Eötvös-Kürschák Competitions

93rd Eötvös-Kürschák Competition 1993

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Prove that if a and b are positive integers then there maybe at most a finite number of integers n, such that both $an^2 + b$ and $a(n+1)^2 + b$ are perfect squares.

2. The sides of triangle ABC have different lengths. Its incircle touches the sides BC, CA and AB at points K, L and M, respectively. The line parallel to LM and passing through B cuts KL at point D. The line parallel to LM and passing through C cuts MK at point E. Prove that DE passes through the midpoint of LM.

3. Let $f(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \cdots + (2n+1-k)x^k + \cdots + 2nx + (2n+1)$ where n is a given positive integer. Find the minimum value of $f(x)$ on the set of real numbers.

Ercole Suppa

Eötvös-Kürschák Competitions

94th Eötvös-Kürschák Competition 1994

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Let λ be the ratio of the sides of a parallelogram, with $\lambda > 1$. Determine in terms of λ the maximum possible measure of the acute angle formed by the diagonals of the parallelogram.

2. Consider the diagonals of a convex n-gon.

   (a) Prove that if any $n-3$ of them are omitted, there are $n-3$ remaining diagonals that do not intersect inside the polygon.

   (b) Prove that one can always omit $n-2$ diagonals such that among any $n-3$ of the remaining diagonals, there are two which intersect inside the polygon.

3. For $1 \leq k \leq n$ the set H_k, $k = 1, 2, \ldots, n$, consists of k pairwise disjoint intervals of the real line. Prove that among the intervals that form the sets H_k, one can find $[(n+1)/2]$ pairwise disjoint ones, each of which belongs to a different set H_k.

Ercole Suppa

95th Eötvös-Kürschák Competition 1995

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. A lattice rectangle with sides parallel to the coordinate axes is divided into lattice triangles, each of area 1/2. Prove that the number of right triangles among them is at least twice the length of the shorter side of rectangle. (A lattice point is one whose coordinates are integers. A lattice polygon is one whose vertices are lattice points.)

2. If +1 or −1 is substituted for every variable of a gives polynomial in n variables, its value will be positive if the number of −1's is even, and negative if it is odd. Prove that the degree of the polynomial is at least n. (i.e. it has a term in which the sum of the exponents of the variables is at least n).

3. No three of the points A, B, C and D are collinear. Let E and F denote the intersection points of lines AB and CD resp. lines BC and DA. Circles are drawn with the segments AC, BD and EF as diameters. Show that either the three circles have a common point or they are pairwise disjoint.

96th Eötvös-Kürschák Competition 1996

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. In the quadrilateral $ABCD$, AC is perpendicular to BD and AB is parallel to DC. Prove that $BC \cdot DA \geq AB \cdot CD$.

2. The same numbers of delegates from countries A and B attend a conference. Some pairs of them already know each other. Prove that there exists a non-empty set of delegates from country A such that either every delegate from country B has an even number of acquaintances among them, or every delegate from country B has an odd number of acquaintances among them.

3. For any non-negative integer n mark any $2kn + 1$ diagonals of a convex n-gon. Prove that there exists a poygonal line consisting of $2k + 1$ diagonals marked which does not intersect itself. Show that this is not necessarily true if the number of marked diagonals is kn.

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97th Eötvös-Kürschák Competition 1997

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Let p be an odd prime number. Consider points in the coordinate plane both coordinates of which are numbers in the set $\{0, 1, 2, \ldots, p-1\}$. Prove that it is possible to choose p of these points such that no three are collinear.

2. The incircle of triangle ABC touches the sides at A_1, B_1 and C_1. Prove that its circumcentre O and incentre O' are collinear with the orthocentre of triangle $A_1B_1C_1$.

3. Prove that the edges of a planar graph can be coloured in three colours so that there is no single-colour circuit.

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98th Eötvös-Kürschák Competition 1998

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Does there exist an infinite sequence of positive integers in which no term divides any other terms have a common divisor greater than 1, but there is no integer, greater than 1, which divides each element of the sequence?

2. Prove that, for every positive integer n, there exists a polynomial with integer coefficients whose values at $1, 2, \ldots, n$ are different powers of 2.

3. Determine all integers $N \geq 3$ for which there exist N points in the plane, no 3 collinear, such that a triangle determined by any 3 vertices of their convex hull contains exactly one of the points in its interior.

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99th Eötvös-Kürschák Competition 1999

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Consider the number of positive even divisors for each of the integers $1, 2, \ldots, n$ and form the sum of these numbers. Prepare a similar sum, this time using the odd divisors of the given numbers. Prove that the two sums differ by at most n.

2. Given a triangle in the plane, construct the point P inside the triangle which has the following property: the feet of the perpendicular dropped onto the lines of the sides form a triangle the centroid of which is P.

3. For any natural number k, let there be given more than 2^k different integers. Prove that $k+2$ of these numbers can be selected such that equality

$$x_1 + x_2 + \cdots + x_m = y_1 + y_2 + \cdots + y_m$$

holds for some positive integer m and selected numbers $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$ only if $x_i = y_i$ for each $1 \le i \le m$.

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100th Eötvös-Kürschák Competition 2000

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. For any positive integer n, consider in the Cartesian plane the square whose vertices are $A(0,0)$, $B(n,0)$, $C(n,n)$ and $D(0,n)$. The grid points of the integer lattice inside or on the boundary of this square are coloured either red or green in such a way that every unit lattice square in the square has exactly two red vertices. How many such colourings are possible?

2. Let P be a point in the plane of the non-equilateral triangle ABC different from its vertices. Let the lines AP, BP and CP meet the circumcircle of the triangle at A_P, B_P and C_P, respectively. Prove that there are exactly two points P and Q in the plane that the triangles $A_PB_PC_P$ and $A_QB_QC_Q$ are equilateral. Prove also that the line PQ passes through the circumcentre of triangle ABC.

3. Let k denote a non-negative integer and assume that the integers $a_1, \ldots, a_n$ leave at least $2k$ different remainders when they are divided by $n+k$. Prove that some of them add up to an integer divisible by $n+k$.

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Eötvös-Kürschák Competitions

101st Eötvös-Kürschák Competition 2001

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Given $3n-1$ points in the plane, no three of which are collinear, show that it is possible to select 2n points, such that their convex hull should not be a triangle.

2. Let $k \geq 3$ be an integer, and $n \geq \binom{k}{3}$. Prove that if a_i, b_i, c_i $(1 \leq i \leq n)$ are $3n$ distinct real numbers then there are at least $k+1$ different numbers among the numbers a_i+b_i, a_i+c_i, b_i+c_i. Show that the statement is not necessarily true for $n = \binom{k}{3}$.

3. In a square lattice, consider any triangle of minimum area that is similar to a given triangle. Prove that the centre of its circumscribed circle is not a lattice point.

102nd Eötvös-Kürschák Competition 2002

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. The sides of an acute-angled triangle are pairwise different, its orthocentre is M, the centre of its inscribed circle is K, and the centre of its circumscribed circle is O. Prove that if a circle passes through the points K, O, M and a vertex of the triangle, then it also passes through another vertex.

2. Consider the sequence of the Fibonacci numbers defined by the recursion $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2} (n \geq 3)$. Assume that the fraction $\frac{a}{b}$, where a and b are positive integers, is smaller than one of the fractions $\frac{f_n}{f_{n-1}}$ and $\frac{f_{n+1}}{f_n}$ but is greater than the other. Show that $b \geq f_{n+1}$.

3. Prove that the set of edges formed by the sides and diagonals of a convex 3^n-gon can be partitioned into sets of three edges, such that the edges in each triple form a triangle.

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Eötvös-Kürschák Competitions

103rd Eötvös-Kürschák Competition 2003

Class 9 - 12, Category 1, Round 1

Organized by János Bolyai Mathematical Society

1. Circle k and the circumcircle of the triangle ABC are touching externally. Circle k is also touching the rays AB and AC at the points P and Q, respectively. Prove that the midpoint of the segment PQ is the centre of the excircle touching the side BC of the triangle ABC.

2. Find the smallest positive integer different from 2004 with the property that there exists a polynomial $f(x)$ of integer coefficients such that the equation $f(x) = 2004$ has at least one integer solution and the equation $f(x) = n$ has at least 2004 distinct integer solutions.

3. Some points are given along the circumference of a circle, each of them is either red or blue. The coloured points are subjects to the following two operations:

   (a) a red point can be inserted anywhere along the circle while the colours of its two neighbours are changed from red to blue and vice versa;

   (b) if there are at least three coloured points present and there is a red one among them then a red point can be removed while its two neighbours are switching colours.

   Starting with two blue points is it possible to end up with two red points after an appropriate sequence of the above operations?

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Eötvös-Kürschák Competitions

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