A CLASS OF WYTHOFF-LIKE GAMES

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Abstract

We present a class of two-player Wythoff game variations we dub $\mathrm{Wyt}(f)$ that depends on a given function $f(k)$. In this class a move consists of removing either a positive number of tokens from precisely one of two given piles, or k tokens from one pile and ℓ from the other, subject to the constraint $0 < k \le \ell < f(k)$. We analyze three classes of integer-valued functions $f(k)$: constant, superadditive and polynomial of degree > 1 with nonnegative integer coefficients. The nature of the winning positions in the games is essentially unique for each class.

1. Introduction

We propose and analyze a family of 2-player combinatorial take-away games played on two piles, dubbed $\mathrm{Wyt}(f)$. The two players move alternately by selecting one of the following moves:

- Move of the first type: Take any positive number of tokens from a single pile, possibly the entire pile.
- Move of the second type: Take $k > 0$ tokens from one pile and $\ell > 0$ tokens from the other. This move is restricted by the condition:

$$0 < k \le \ell < f(k), \tag{1}$$

  where $f(k) : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is a function that distinguishes the games from each other.

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A position in the game can be represented by a pair (a, b) $(a, b \in \mathbb{Z}_{\geq 0})$ which denotes the number of tokens in each pile. Without loss of generality we assume throughout $0 \leq a \leq b$. The games we consider here are played under the *normal* play convention, that is, the player making the last move wins the game and the opponent loses. In the *misère* play convention the player making the last move loses and the opponent wins.

Note that these are *impartial* games - the set of possible moves for a player depends only on the game position, not on the player. They are deterministic (no chance moves) and acyclic (the number of tokens decreases at every move until it becomes 0). Therefore we can partition the positions of the game into *Next* player winning positions (denoted N-positions, or by the set $\mathcal{N} := \mathcal{N}(f) \subseteq \mathbb{Z}^2_{\geq 0}$) and *Previous* player winning positions (denoted P-positions, or by the set $\mathcal{P} := \mathcal{P}(f) \subseteq \mathbb{Z}^2_{\geq 0}$).

A game G is *tractable* if:

- For every position, its state (P- or N-position) can be decided in polynomial time.
- The next move from any N-position to a P-position can be computed in polynomial time.
- The winner can consummate a win in at most an exponential number of moves.

To the "run of the mill algorithmicians" the last item dooms the game as intractable. It may be quite a surprise to them that this is not the case: Whereas we dislike computing in exponential time, the human race relishes to observe some of its members being tormented for an exponential length of time! In fact, the easy game Nim and similar take-away games do require that amount of time. For games there are also notions of polynomiality and efficiency. See [12], especially section 4.

In the next sections of this paper, we define sequences A_n and B_n to analyze the P-positions. We use the notation

$$A = \cup_{i=0}^{\infty}\{A_i\}, B = \cup_{i=0}^{\infty}\{B_i\}.$$

In addition, when analyzing the P-positions we frequently use the mex function: Let S be a finite set of nonnegative integers. Then $\text{mex}(S)$ is defined to be the least nonnegative integer not in S. Note that the mex of the empty set is 0.

This paper presents characterizations of the P-positions and tractable winning strategies for the following types of functions:

- $f(k) = t,\ t > 0$ an integer,
- $f(k)$ is a strictly increasing superadditive function,
- $f(k) = \sum_{i=0}^{n} a_i k^i$ is a polynomial of degree $n > 1$ with integer coefficients $a_i \geq 0$ and $a_0 > 0$.

Note that a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is called *superadditive* if it satisfies: $f(k) \geq k$ and $f(k+\ell) \geq f(k) + f(\ell)$ for all $k, \ell \in \mathbb{Z}_{\geq 0}$.

Wyt(f) is a general class of games that encapsulates a number of previously analyzed games and new games. In its simplest case, where $f(k) \in \{0, 1\}$, the move of the second type cannot be done and the game reduces to Nim on 2 piles (in normal play). For the case $f(k) = k + 1$ the game reduces to the classical Wythoff game [24] where in a move of the second type, the player has to take the same positive number of tokens from both piles. An analysis of the constant difference class $f(k) = k + t$, $t \geq 1$ integer and later for the linear class $f(k) = sk + t$ $s, t \geq 1$ integers was done by Fraenkel [8, 11]. For other examples of variations of 2-piles Wythoff, see [13, 3, 15, 10, 18, 20, 5].

In section 2 we deal with the case $f(k) = t$, and in section 3 with superadditivity. In section 4 we handle polynomials, where we resort to real analysis for the proofs. Further possible work is indicated in the final section 5.

2. Constant Function

Considering Wyt(f) for $f(k) = t$, $t > 0$ an integer, using the move of the second type, a player can take k tokens from one pile and ℓ tokens from the other as long as $0 < k \leq \ell < t$. As implied by the penultimate paragraph of the previous section, we may assume $t \geq 2$. The class of games with the restriction $k + \ell < t$ for moves of the second type, is called Cyclic Nimhoff and was settled by Fraenkel and Lorberbom [14] for a general n-piles game; see also [5]. Duchêne and Gravier [7] examined (among other geometrical extensions of Wythoff) a bounded Wythoff game in which it is only possible to take $k < t$ tokens from one pile or $k < t$ tokens from both piles.

Given fixed $t \in \mathbb{Z}_{\geq 2}$, we define $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that for all $m \in \mathbb{Z}_{\geq 0}$,

$$g(m) = tm - (t^2 - 1)\lfloor m/(t+1) \rfloor$$

and

$$A_n = \text{mex}\{A_i, B_i : 0 \leq i < n\}, \qquad B_n = g(A_n), \quad n \geq 0.$$

In this section we show that $\cup_{i=0}^{\infty}\{(A_i, B_i)\}$ are the P-positions of Wyt(f) when $f(k) = t$ for any constant integer $t \geq 2$. To prove this, we begin by showing the relation between the A_n and B_n sequences using a specific numeration system. This numeration system will also help us to derive a tractable strategy for winning the game. The first few P-positions for the games where $t = 3$ and $t = 10$ are shown in Table 1 and Table 2 respectively. After Theorem 12 (section 2.2) we point out in Remark 2, that Table 1 exhibits certain periodicities, which help in exhibiting the structure of the sequences A and B.

We begin with an auxiliary result.

Table 1: $t = 3$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	0	1	2	4	5	8	9	10	12	13	16	17	18	20	21	24	25
B_n	0	3	6	4	7	8	11	14	12	15	16	19	22	20	23	24	27

Table 2: $t = 10$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A_n	0	1	2	3	4	5	6	7	8	9	11	12	13	14	15	16
B_n	0	10	20	30	40	50	60	70	80	90	11	21	31	41	51	61

Lemma 1. (i) *Let $t \geq 2$. For every $m \in \mathbb{Z}_{\geq 0}$, $g(m) \geq m$, with equality if and only if $m \equiv 0 \pmod{(t+1)}$.*
(ii) *The function g is an injection.*

Proof. (i) Write $m = s(t+1) + r$, $0 \leq r < t+1$, $s, r \in \mathbb{Z}_{\geq 0}$. Then

$$g(m) - m = (t-1)(m - (t+1)\lfloor (s(t+1)+r)/(t+1) \rfloor).$$

Thus $(g(m) - m)/(t-1) = m - s(t+1) \geq 0$ with equality if and only $r = 0$.
(ii) An elementary algebra exercise. □

Lemma 1 immediately implies:

Corollary 2. *For every $n \geq 0$, $B_n = A_n$ if and only if $A_n \equiv 0 \pmod{(t+1)}$.*

2.1. A Numeration System

Lemma 3. *Let $t \in \mathbb{Z}_{\geq 2}$. Every nonnegative integer m can be represented uniquely in the form*

$$m = m_2(t^2 - 1) + m_1 t + m_0, \tag{2}$$

where the digits m_0, m_1, m_2 are integers satisfying:

$$0 \leq m_0, m_1 < t, \quad m_2 \geq 0,$$

$$m_0 = m_1 = t - 1 \textit{ is not permitted.} \tag{3}$$

Proof. Theorem 1 of [9] states: Let $1 = u_0 < u_1 < u_2 < \cdots$ be any finite or infinite sequence of integers. Every nonnegative integer m has precisely one representation in the system $S = \{u_0, u_1, u_2, \dots\}$ of the form $m = \Sigma_{i=0}^{n} m_i u_i$, where the m_i are nonnegative integers satisfying:

$$m_i u_i + m_{i-1} u_{i-1} + \cdots + m_0 u_0 < u_{i+1} \text{ for } i \geq 0.$$

By this theorem, every nonnegative integer m can be represented uniquely in the form (2) where $m_0 < t$ and $m_0 + m_1 t < t^2 - 1$. It follows immediately that $m_1 < t$ and that $m_0 = m_1 = t - 1$ is not permitted. □

Throughout this section, we use the convention that for every nonnegative number m, we let m_0, m_1, m_2 be the appropriate digits of m in the above numeration system.

Corollary 4. *With hypotheses as in Lemma 3, $g(m) = m$ if and only if $m_0 = m_1$.*

Proof. $m = m_2(t^2 - 1) + m_1(t + 1) + m_0 - m_1 \equiv 0 \pmod{t + 1}$ if and only if $m_0 = m_1$, since $0 \le m_0, m_1 < t$. The proof is complete by Lemma 1. □

Using our numeration system we can present the g function in a more intuitive way.

Lemma 5. *Let $t \in \mathbb{Z}_{\ge 2}$, and $m \in \mathbb{Z}_{\ge 0}$ which is represented uniquely in the form (2) with digits m_0, m_1, m_2. If $m_1 \le m_0$, then*

$$g(m) = m_2(t^2 - 1) + m_0 t + m_1.$$

Otherwise,

$$g(m) = (m_2 + 1)(t^2 - 1) + m_0 t + m_1.$$

Proof. We have,

$$\begin{aligned} g(m) &= g(m_2(t^2 - 1) + m_1 t + m_0) \\ &= m_2(t^2 - 1)t + m_1 t^2 + m_0 t - (t^2 - 1)(\lfloor (m_1 t + m_0)/(t + 1) \rfloor + m_2(t - 1)) \\ &= m_2(t^2 - 1) + m_1 t^2 + m_0 t - (t^2 - 1)\lfloor (m_1 t + m_0)/(t + 1) \rfloor. \end{aligned}$$

If $m_1 \le m_0$, then $\lfloor (m_1 t + m_0)/(t + 1) \rfloor = \lfloor (m_1(t + 1) + (m_0 - m_1))/(t + 1) \rfloor = m_1 + \lfloor (m_0 - m_1)/(t + 1) \rfloor = m_1$ since $0 \le m_0, m_1 < t$. Therefore, the g function for this case becomes:

$$\begin{aligned} g(m) &= m_2(t^2 - 1) + m_1 t^2 + m_0 t - m_1(t^2 - 1) \\ &= m_2(t^2 - 1) + m_0 t + m_1. \end{aligned}$$

Otherwise, $\lfloor (m_1 t + m_0)/(t + 1) \rfloor = m_1 - 1$ and the g function becomes:

$$\begin{aligned} g(m) &= m_2(t^2 - 1) + m_1 t^2 + m_0 t - (m_1 - 1)(t^2 - 1) \\ &= (m_2 + 1)(t^2 - 1) + m_0 t + m_1. \end{aligned}$$

□

Remark 1. Notice that in the two displayed formulas of Lemma 5 the digits m_0, m_1 of the unique representation (2) of m have been switched. Furthermore, m has a unique representation of the form (2) if and only if $m' := m_2(t^2 - 1) + m_0 t + m_1$ has a unique representation. Their value difference is $m - m' = (m_1 - m_0)(t - 1)$.

We presented the relation between A_n and B_n using the special numeration system. In the next couple of lemmas we show that an independent simple characterization of the $\{A_i\}$ and $\{B_i\}$ sets follows from the use of this numeration system. This characterization will enable us to prove that the A_n and B_n pairs constitute the P-positions of the game.

Lemma 6. *Let $t \in \mathbb{Z}_{\geq 2}$, and $m \in \mathbb{Z}_{\geq 0}$ which is represented uniquely in the form (2) with digits m_0, m_1, m_2. Then $m \in A$ if and only if $m_1 \leq m_0$.*

Proof. Induction on m. Assume that the statement is true for all n in $K = \{0, 1, \ldots, m-1\}$. Suppose first that $m_1 \leq m_0$. Using the induction hypothesis, we show that then $m \notin \{B_n = g(n) : n \in K \cap A\}$. It then follows from the mex definition that $m \in A$.

Let $n \in K \cap A$. Then $n_1 \leq n_0$ by the induction hypothesis. If $n_1 = n_0$, then $n = g(n) < m$ by Corollary 4. So we may assume $n_1 < n_0$. Let $r = r_2(t^2-1)+r_1t+r_0$ be the unique representation of $g(n)$. Lemma 5 and Remark 1 then imply that $r_1 = n_0$ and $r_0 = n_1$. Thus $r_0 < r_1$. Assuming $m = r$, the uniqueness of the representation (Lemma 3) and Remark 1 then imply $r_1 = m_1$ and $r_0 = m_0$, contradicting $m_1 < m_0$. Therefore, $m \neq g(n)$.

Secondly, suppose that $m \in A$. Let $m' := m_2(t^2-1) + m_0t + m_1$. Assume that $m_1 > m_0$. Then $m - m' = (m_1 - m_0)(t-1) > 0$, so $m' \in K \cap A$ by the induction hypothesis. By Lemma 5, $m = g(m') \in B$. In fact, if, say, $m = A_n$, then $m' = A_i$ for some $i < n$, and $m = g(m') = B_i$. But this contradicts the definition of the mex function. Hence $m_1 \leq m_0$. □

Consequently we have:

Lemma 7. $m = m_2(t^2-1) + m_1t + m_0 \in B$ *if and only if* $m_0 \leq m_1$.

Proof. If $m_1 > m_0$ then $m \notin A$ by Lemma 6 and therefore $m \in B$ by the mex property. If $m_1 = m_0$ then $m \in A$ by Lemma 6 and $m = g(m) \in B$ by Corollary 4.

Conversely, if $m \in B$, then by definition there exists $a \in A$ such that $m = g(a)$. Let a_0, a_1, a_2 be the digits of a represented in form (2). Then by Lemma 6, $a_1 \leq a_0$; and by Lemma 5, $m_0 = a_1$ and $m_1 = a_0$, so $m_0 \leq m_1$. □

Corollary 8. (i) *If $A_n = m_2(t^2-1)+m_1t+m_0$ then $B_n = m_2(t^2-1)+m_0t+m_1$.*
(ii) $B_n - A_n = (m_0-m_1)(t-1) \geq 0$. *In particular, $B_n = A_n$ if and only if $m_0 = m_1$.*

Proof. (i) By Lemma 6, $m = A_n$ if and only if $m_1 \leq m_0$; and if $m_1 \leq m_0$, then Lemma 5 implies that $g(m) = B_n = m_2(t^2-1) + m_0t + m_1$.
(ii) This also follows from $m_1 \leq m_0$. □

Theorem 9. *For $f(k) = t$ a constant, the set of P-positions of* Wyt(f) *is given by $W := \cup_{i=0}^{\infty}\{(A_i, B_i)\}$.*

Proof. Since Wyt(f) is an acyclic game, it suffices to show two things:

I. Every move from any position in W lands in a position outside W.

II. For every position outside W, there is a move to some position in W.

I. A move of the first type from $(A_n, B_n) \in W$ leads to a position not in W, because there are no repeating terms in A, nor in B: (i) The mex property implies that the sequence A is strictly increasing thus has no repeating terms. (ii) There are also no repeating terms in B, since the function g is an injection (Corollary 1 (ii)) and there are no repeating terms in A.

Suppose that a move of the second type from $(A_n, B_n) \in W$ produces a position $(A_m, B_m) \in W$. Then $m < n$. This move can be made either in the form

$$\text{(i) } A_n \to A_m,\ B_n \to B_m; \text{ or in the form } \text{ (ii) } A_n \to B_m,\ B_n \to A_m.$$

(i) $A_n \to A_m$, $B_n \to B_m$. Then

$$B_n - B_m = t(A_n - A_m) - (t^2-1)\left(\left\lfloor \frac{A_n}{t+1} \right\rfloor - \left\lfloor \frac{A_m}{t+1} \right\rfloor\right).$$

If $\lfloor A_n/(t+1) \rfloor = \lfloor A_m/(t+1) \rfloor$ then $B_n - B_m = t(A_n - A_m) \geq t$, contradicting the move rule (1). Otherwise, $\lfloor A_n/(t+1) \rfloor - \lfloor A_m/(t+1) \rfloor \geq 1$. Since $A_n - A_m = k < t$, we get

$$B_n - B_m \leq t(t-1) - (t^2-1) = 1 - t < 0,$$

again contradicting (1).

(ii) $A_n \to B_m$, $B_n \to A_m$. Let $r := A_m$. Then $r_1 \leq r_0$ by Lemma 6. We may assume $r_0 > r_1$, because $r_0 = r_1$ implies $B_m = A_m$ (Corollary 8(ii)), whence case (ii) reverts back to case (i). The move (ii) can only be made if $A_n > B_m$. Then $B_n - A_m = g(A_n) - A_m \geq A_n - A_m > B_m - A_m = (r_0 - r_1)(t-1) \geq t-1$, contradicting move rule (1).

II. Let (x, y) with $x \leq y$ be any position not in W. The construction of A_n by the mex rule implies that the set $\cup_{i=0}^{\infty}\{A_i, B_i\} = \mathbb{Z}_{\geq 0}$. Therefore, $x = B_n$ or $x = A_n$ for some $n \geq 0$. We consider several cases.

(i) $x = B_n$. Then move $y \to A_n$ using the first move rule.

(ii) $x = A_n$ and $y > B_n$, then move $y \to B_n$ using the first move rule.

(iii) $x = A_n \leq y < B_n$. Recall that we can represent x and y uniquely in our numeration system: $x = x_2(t^2-1) + x_1 t + x_0$ and $y = y_2(t^2-1) + y_1 t + y_0$.

Let

$$x' := x_2(t^2-1) + x_1 t + s, \quad y' := x_2(t^2-1) + st + x_1,$$

where

$$s := y_1 + \lfloor (y_0 - x_1)/t \rfloor.$$

We show the following:

(a) (a1) If $y_0 \ge x_1$ then $s = y_1$, $x_0 > y_1$; if $y_0 < x_1$ then $s = y_1 - 1$, $x_0 \ge y_1$.
(a2) $0 \le x_1 \le s < x_0 < t$.

(b) There exists m such that $x' = A_m$ and $y' = B_m$.

(c) The move $x \to x' = A_m$, $y \to y' = B_m$ is a legal move, that is, $0 < x - x' < t$, $0 \le y - y' < t$.

(a) Notice that $s = y_1 + \lfloor (y_0 - x_1)/t) \rfloor \in \{y_1 - 1, y_1\}$, since $0 \le x_1, y_0, y_1 < t$. By Corollary 8(i), A_n and B_n have the same coefficient (digit) multiplier of $t^2 - 1$, so a fortiori $y_2 = x_2$. Hence $x = A_n \le y < B_n = g(A_n)$ implies:

$$x_1 t + x_0 \le y_1 t + y_0 < x_0 t + x_1. \tag{4}$$

The right-hand side of (4) is equivalent to $(x_0 - y_1)t > y_0 - x_1$. If $y_0 \ge x_1$, then $s = y_1$ and $x_0 > y_1$, so $s < x_0 < t$. If $y_0 < x_1$, then $s = y_1 - 1$, yet $x_0 \ge y_1$, and we have again $s < x_0 < t$. The left-hand side of (4) is equivalent to $y_0 - x_0 \ge (x_1 - y_1)t$. By Lemma 6, $x_1 \le x_0$, since $x = A_n$. Hence $s \ge y_1 + \lfloor (y_0 - x_0)/t) \rfloor \ge x_1 \ge 0$.

(b) By (a2), $x_1 \le s < t$, so by Lemma 6, $x' = A_m$ for suitable m. Again by (a2), $0 \le s < t$, so $y' = B_m$ by Corollary 8(i).

(c) From (a), $0 < x - x' = x_0 - s < t$. Now $y - y' = (y_1 - s)t + y_0 - x_1$, since $y_2 = x_2$. By (a1), if $y_0 \ge x_1$ then $s = y_1$, so $0 \le y - y' = y_0 - x_1 < t$; and if $y_0 < x_1$ then $s = y_1 - 1$, so $0 < y - y' = t - (x_1 - y_0) < t$. $\square$

2.2. Strategy Tractability and Structure of the P-Positions

Theorem 10. *Given game position (x, y) with $0 \le x \le y$, there is a tractable linear-time algorithm to decide whether or not $(x, y) \in \mathcal{P}$.*

Proof. We wish to compute whether or not $(x, y) \in \mathcal{P}$ in time linear in the input size $\Theta(\log xy)$. Expand x and y in our numeration system, which can be done linearly by using the simple greedy algorithm from [9]:

$$\begin{aligned} x_2 &= \lfloor x/(t^2 - 1) \rfloor \\ x_1 &= \lfloor (x - x_2(t^2 - 1))/t \rfloor \\ x_0 &= x - x_2(t^2 - 1) - x_1 t. \end{aligned}$$

Then, check whether $x_0 \ge x_1$. If negative, $(x, y) \in \mathcal{N}$. If positive, check whether $y_2 = x_2$ and $y_1 = x_0$ and $y_0 = x_1$. If positive, then $(x, y) \in \mathcal{P}$ (Corollary 8(i)); otherwise, $(x, y) \in \mathcal{N}$. $\square$

Corollary 11. *Let $t \in \mathbb{Z}_{\ge 2}$ and $f(k) = t$ a constant function. Then there is a tractable strategy for winning Wyt(f).*

Proof. Using the instructions in the second part of the proof of Theorem 9, the greedy algorithm in Theorem 10, and Lemmas 6 and 7, it is clear how to construct a tractable strategy for winning Wyt(f) for any given N-position. $\square$

We now turn to the structure of the sequences A and B.

A sequence $\{s_n\}_{n\geq 0}$ with $\lim_{n\to\infty} s_n = \infty$ is called *arithmetically periodic* if there exist integers $1 < q \leq p$ (the *periods*), and distinct nonnegative integers $r_1, \ldots, r_q \in [0, p-1]$, such that $s_i \equiv r_i \pmod p$ whenever $n \equiv i \pmod q$ $(i = 1, \ldots, q)$.

This is a variation of the definitions of arithmetic (or *additive*) periodicity given or implied in [1], [4], [17]. It accommodates sequences that are not necessarily monotonically increasing.

Theorem 12. *Let $t \geq 2$.* (i) *Each of the sequences A and B is arithmetically periodic with periods $q = (t-1)(t+2)/2$ and $p = t^2 - 1$.*
(ii) *Each of the sequences A and B contains $(t-1)(t+2)/2$ distinct residues* mod p*, $t-1$ of which are common to both sequences.*
(iii) *$A \cup B$ contains all the $t^2 - 1$ residues* mod p*, $t-1$ of which appear in both sequences.*
(iv) *For all $n \geq 0$, $0 \leq B_n - A_n \leq (t-1)^2$.*

Proof. (i) Lemmas 6 and 7 and Corollary 8 imply that the sequences A and B have residues that are periodic mod p. Indeed, membership of x in A and B depends only on the relative size of x_0 and x_1, not on the size of x_2.

(ii) Using (i) we can assume $x_2 = 0$. By the mex property, all the residues of A in $[0, p-1]$ are increasing, so they are distinct. Their number is precisely the number of x_0, x_1 satisfying $0 \leq x_1 \leq x_0 \leq t-1$. Since $x_0 = x_1 = t-1$ is forbidden, this number is precisely $(t-1)(t+2)/2$, $t-1$ of which are common to both sequences, namely those for which $0 \leq x_0 = x_1 < t-1$. The proof for the residues of B is similar.

(iii) Follows immediately from (ii).

(iv) Follows immediately from Corollary 8(ii). □

Remark 2. Notice that the preceding propositions (i)-(iv) of Theorem 12 can be seen in Table 1.

Almost all game positions (x, y), $0 \leq x \leq y$ are N-positions – see [22], [23]. Therefore $(x, y) \notin \mathcal{P}$ with high probability. Theorem 12 can help to dispose of them quickly by computing the residues of x and y mod p and searching for them in a pre-constructed table of the $p = t^2 - 1$ residue pairs of (A_n, B_n) mod p.

3. Superadditive Functions

In this section we examine Wyt(f), when f is a strictly increasing superadditive function. We show that the P-positions of this game have a simple recursive formula. As an example, the first few P-positions for the game for which $f(x) = x^2$ are shown in Table 3.

Table 3: $f(x) = x^2$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	1	2	3	5	6	7	8	10	11	12	13	14	15
B_n	0	1	4	9	25	36	49	64	100	121	144	169	196	225

Remark 3. The function f in this example is a polynomial, and polynomials are considered in the next section. However, a polynomial of degree $t \geq 1$ of the form

$$f(x) = \sum_{i=0}^{t} a_i x^i, \ a_i \in \mathbb{Z}_{\geq 0} \text{ for } 0 \leq i \leq t, \ a_t > 0, \tag{5}$$

is clearly superadditive if and only if $a_0 = 0$. In the next section we consider the case $a_0 > 0$.

Remark 4. Notice that if $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is strictly increasing, then $f(k) \geq k$.

Let

$$A_n = \operatorname{mex}\{A_i, B_i \,:\, 0 \leq i < n\}, \qquad B_n = f(A_n), \quad n \geq 0. \tag{6}$$

Theorem 13. *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a strictly increasing superadditive function. Then the set of P-positions of* $\mathrm{Wyt}(f)$ *is given by $W := \cup_{i=0}^{\infty}\{(A_i, B_i)\}$.*

Proof. As in the proof of Theorem 9, it suffices to show:

I. Every move from any position in W lands in a position outside W.

II. For every position outside W, there is a move to some position in W.

I. The mex definition implies that A_n is a strictly increasing sequence. Thus, for every $i > j$ we have $A_i > A_j$. Hence $B_i = f(A_i) > f(A_j) = B_j$ since f is a strictly increasing function. Therefore, there are no repeating terms in A_n nor in B_n. It follows that a move of the first type from $(A_n, B_n) \in W$ leads to a position $\notin W$.

Suppose that a move of the second type from $(A_n, B_n) \in W$ produces a position $(A_m, B_m) \in W$. Then $m < n$. Let $k := A_n - A_m$ and $\ell := B_n - B_m$. Since A_n is a strictly increasing sequence we have $k > 0$. The superadditivity of f then implies,

$$\ell = f(A_n) - f(A_m) \geq f(A_n - A_m) = f(k).$$

By Remark 4 we have $\ell \geq k$. Thus $0 < k \leq \ell \geq f(k)$, contradicting move rule (1). The move $(A_n, B_n) \to (B_m, A_m)$ where $m < n$ is impossible too: Let $k' := A_n - B_m$ and $\ell' := B_n - A_m$. Since $B_n = f(A_n) \geq A_n$, we have: $0 < k' \leq k \leq \ell \leq \ell'$. As was shown above, $\ell \geq f(k)$, so $\ell' \geq \ell \geq f(k) \geq f(k')$ since f is a strictly increasing function. Hence $0 \leq k' \leq \ell' \geq f(k')$, again contradicting move rule (1).

II. Let (x, y) with $0 \le x \le y$ be any position not in W. It follows from the mex definition that $\cup_{i=0}^{\infty}\{A_i, B_i\} = \mathbb{Z}_{\ge 0}$. Therefore, $x = B_n$ or $x = A_n$ for some $n \ge 0$.
Case (i) $x = B_n$. Then move to the position (A_n, B_n) by subtracting $y - A_n$ tokens from one pile, using a move of the first type.
Case (ii) $x = A_n$ and $y > B_n$. Then subtract $y - B_n$ tokens from one pile, using a move of the first type.
Case (iii) $x = A_n \le y < f(A_n) = B_n$. Then move to $(0, 0)$ using a move of the second type. □

The time complexity for deciding whether a given position is a P-position or N-position depends on the time complexity of the function f. Assume f to be a polynomial time computable function. The naïve algorithm for deciding whether a position (x, y) is a winning position consists of calculating A_n and B_n using their recursive definition (6) until it is known whether $x \in A_n$ or not. This method takes exponential time and space since the input position is given in succinct form. In the previous section and in other Wythoff-like games (e.g. [11]) constructing a special numeration system assisted in building a tractable strategy to win the game. We show here that this is not necessary for this case.

Remark 5. For all superadditive functions f, $f(0) = 0$. This follows from $f(0) = f(0 + 0) \ge f(0) + f(0)$.

Lemma 14. *Let $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ be a strictly increasing superadditive function. For every $x \in \mathbb{Z}_{\ge 0}$, if $f(x) = x$ then $x = A_x = B_x$.*

Proof. Induction on x. Since $A_0 = 0$ by the mex property, $B_0 = f(0) = 0$ (Remark 5). Assume that the statement is true for all $m < x$, $m \in \mathbb{Z}_{\ge 0}$. If $f(x) = x$ then since f is strictly increasing, for every $m < x$, $f(m) = m$. The induction hypothesis then implies $\{A_0, \ldots A_{x-1}, B_0, \ldots B_{x-1}\} = \{0, 1, \ldots, x - 1\}$. Therefore, $x = A_x$ by the mex property. If $f(x) = x$ then also $B_x = f(x) = x$. □

Lemma 15. *Let $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ be a strictly increasing superadditive function. If f is not the identity function $f(x) = x$, then there exist $N \in \mathbb{Z}_{\ge 1}$ and $a > 1$ such that for every integer $n \ge N$, $f(n)/n \ge a$ and for every integer $n < N$, $f(n) = n$.*

Proof. Since f is not the identity function, there exists a minimum integer $N \in \mathbb{Z}_{\ge 1}$ such that $f(N) \ne N$. Thus for every integer $n < N$, $f(n) = n$. Notice that $N \ne 0$ because $f(0) = 0$ (Remark 5). By Remark 4, we have $f(N) \ge N + 1$.

Let $a := 1 + 1/(2N - 1)$. Since $N \ge 1$ it is clear that $a > 1$. We can write every integer $n \ge N$ in the form $n = kN + r$, where $0 \le r < N$ and $k \in \mathbb{Z}_{\ge 1}$. The superadditivity of f and $f(N) \ge N + 1$ then imply that

$$f(n) = f(kN + r) \ge f(kN) + f(r) \ge kf(N) + f(r) \ge k(N + 1) + r.$$

Since $r < N$ and $k \ge 1$ we have

$$\frac{f(n)}{n} \ge \frac{k(N + 1) + r}{kN + r} = 1 + \frac{k}{kN + r} \ge 1 + \frac{k}{kN + kr} \ge 1 + \frac{1}{N + r} \ge a.$$

□

Theorem 16. *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a polynomial time computable strictly increasing superadditive function. Let A_n, B_n be given by (6), and let (x, y) with $0 \leq x \leq y$ be any game position. Then there is a polynomial-time algorithm to decide whether or not $x \in A$.*

Proof. We apply the following recursive

Procedure 1. Input: x. Output: 1 if $x \in A$; 0 otherwise.

Find z such that $f(z)$ is as close as possible to x, beginning with $z = x/2$ and then proceeding in the form of a binary search on the interval $[0, x]$ ($O(\log x)$ operations). There are three possible outcomes:

1. There is no z such that $x = f(z)$. Then return 1 since $x \in A$ by the mex definition.
2. There exists z such that $x = f(z)$ and $z = x$. Then return 1 since $x \in A$ by Lemma 14.
3. There exists z such that $x = f(z)$ and $z < x$. Then $x \in A$ if and only if $z \notin A$. So apply Procedure 1 with z as input and return 1 minus the returned value from this new call of Procedure 1.

Suppose the recursive search algorithm stops with x_t, for which $f^t(x_t) = x$, where f^t denotes f composed with itself t-times. We show that $t = O(\log x)$. Consequently, the algorithm is polynomial because each recursive call is polynomial in $\log x$.

By Lemma 15 there exist $a > 1$ and minimal $N \in \mathbb{Z}_{\geq 1}$, such that for every integer $n \geq N$, $f(n)/n \geq a$. In particular, $f(N)/N \geq a$. Since $f(N) \geq N$ it follows that $f(f(N))/f(N) \geq a$. Hence,

$$\frac{f^2(N)}{N} = \frac{f^2(N)}{f(N)} \frac{f(N)}{N} \geq a^2;$$

and by induction we have:

$$\frac{f^t(N)}{N} \geq a^t.$$

Therefore $f^t(N) \geq a^t N$. We assume $f(x_n) \neq x_n$. Otherwise, $f(x) = x = f^t(x_t) = x_t$ and then $t = 0$. By Lemma 15, $x_t \geq N$. Since f is an increasing function, $f(x_t) \geq f(N)$, and by simple induction, $f^t(x_t) \geq f^t(N)$. Hence $x = f^t(x_t) \geq f^t(N) \geq a^t N$. Then $\log_a(x) \geq t \log_a(aN) \geq t$, since $a > 1$ and $N \geq 1$. Consequently, $t = O(\log x)$. □

Corollary 17. *Let f be a polynomial time computable strictly increasing superadditive function. Then there is a tractable strategy for winning Wyt(f).*

Proof. Using the algorithm in Theorem 16 and the instructions in the second part of the proof of Theorem 13, it is clear how to construct a tractable strategy for winning Wyt(f) for any given N-position. Note that by using the binary search described in the beginning of Theorem 16, it is possible to calculate $f^{-1}(x)$, if it exists, in polynomial time. □

4. Polynomial

In this section we consider the game Wyt(f) for which $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a polynomial of the form (5) with $t \geq 2$ and $a_0 > 0$.

Thus f is convex although not superadditive (Remark 3). The analysis of this class of games is similar to that of the previous section, since the function which defines the B sequence is superadditive, as we show presently.

We define here, as before, the sequences that compose the P-positions of this class of games. For $x \in \mathbb{R}_{\geq 0}$, we define

$$g(x) = \begin{cases} \max\{f(x),\, f(1)x\} & \text{if } x \in [1, \infty) \\ f(1)x & \text{if } x \in [0, 1), \end{cases}$$

and

$$A_n = \text{mex}\{A_i, B_i \,:\, 0 \leq i < n\}, \qquad B_n = g(A_n), \quad n \geq 0. \tag{7}$$

The function g is defined on $\mathbb{R}_{\geq 0}$ in order to enable us to use basic calculus for proving that g is a superadditive function. Note that $g(\mathbb{Z}_{\geq 0}) \subseteq \mathbb{Z}_{\geq 0}$.

As an example, the first few P-positions for the game for which $f(x) = x^2 + 9$ are displayed in Table 4.

Table 4: $f(x) = x^2 + 9$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	1	2	3	4	5	6	7	8	9	11	12	13	14
B_n	0	10	20	30	40	50	60	70	80	90	130	153	178	205

Lemma 18. *The function $g(x)$ is a strictly increasing superadditive function.*

Proof. If $g(x)$ is a nonnegative continuous convex function which vanishes at the origin, then $g(x)$ is superadditive – see [2], Theorem 5. Since $g(x)$ is a nonnegative continuous function and $g(0) = 0$, it suffices to show that $g(x)$ is a convex function in $[0, \infty)$. By [16] Corollary 1.1.9, if the right derivative $g'_+(x)$ is nondecreasing in $[0, \infty)$ then $g(x)$ is a convex function in this interval. We show here that $g'_+(x)$ is nondecreasing by showing: (i) $g'_+(x)$ is nondecreasing in $[0, 1)$. (ii) $g'_+(x)$ is nondecreasing in $(1, \infty)$. (iii) $g'_+(1) \geq g'_+(x)$ for $x \in [0, 1)$.

(i) In $[0, 1)$, $g'_+(x) = g'(x) = f(1)$. Hence, $g'_+(x)$ nondecreasing in $[0, 1)$.

(ii) In $[1, \infty)$, $g(x)$ is a convex function, since a maximum of convex functions is convex ([16], Theorem 1.1.3). Therefore by [16] Corollary 1.1.6, $g'_+(x)$ is nondecreasing in $[1, \infty)$.

(iii) For $h > 0$,

$$\frac{g(1+h) - g(1)}{h} \geq \frac{f(1)(1+h) - f(1)}{h} = f(1).$$

Hence $g'_+(1) \geq f(1) = g'_+(x)$ for $x \in [0, 1)$.

Therefore, $g'_+(x)$ is nondecreasing in $[0, \infty)$ and thus $g(x)$ is superadditive there. Since $g(x)$ is continuous and $g'_+(x) \geq f(1) > 0$ for all $x \in [0, \infty)$, $g(x)$ is strictly increasing there. $\square$

Let

$$S = \{n \in \mathbb{Z}_{\geq 0} : f(A_n) \leq f(1)A_n\}.$$

Remark 6. Since $t \geq 2$, $f(A_n) > f(1)A_n$ for all sufficiently large n, so S is finite, containing at most the first few nonnegative integers. The next lemma throws some light on the set S.

Lemma 19. *The set S is nonempty, and for every $n \in S$ we have $A_n < f(1)$ and $A_n = n$.*

Note 1. For the example $f(x) = x^2 + 9$ we have $A_n < f(1)$ and $f(A_n) \leq f(1)A_n$ for $0 \leq n \leq 9$, but $f(A_n) > f(1)A_n$ for all $n > 9$. Thus $A_n = n$ for $0 \leq n \leq 9$, but $A_n > n$ for all $n > 9$ (see Table 4).

Proof. By the mex property, $A_0 = 0$. Hence $B_0 = g(0) = f(1)0 = 0$. Again by mex, $A_1 = 1$. Thus $f(A_1) = f(1)A_1$, so $S \neq \emptyset$. Also, $B_n = g(A_n) = \max\{f(A_n), f(1)A_n\}$ for all $n \geq 1$. In particular, $B_1 = f(1) \geq 2$.

Next we show $A_n < f(1)$ for all $n \in S$. Suppose $A_n \geq f(1)$ for some $n \in S$. We note that $n \geq 1$, since $0 \notin S$: $f(0) > f(1)0$. Let $k := A_n - f(1) \geq 0$. Since $t \geq 2$ and $a_0 > 0$ we have,

$$f(A_n) = f(f(1)+k) \geq a_t(f(1)+k)^t + a_0 > (f(1)+k)^2 \geq f(1)(f(1)+k) = f(1)A_n.$$

Thus, $f(A_n) > f(1)A_n = B_n = g(A_n)$, contradicting the definition of g. Therefore $A_n < f(1) = B_1$ for all $n \in S$.

We already verified that $A_n = n$ for $n \in \{0, 1\}$. Suppose that $A_m = m$ for all $m < n$ for which $A_n < f(1) = B_1$. Then $A_n = \text{mex}\,\{A_0, \ldots, A_{n-1}, B_1, \ldots, B_{n-1}\} = \text{mex}\,\{0, \ldots, n-1\} = n$, since $B_{n-1} > \cdots > B_1 = f(1) > A_n$. $\square$

Theorem 20. *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a polynomial of the form* (5) *with $t \geq 2$ and $a_0 > 0$. Then the set of P-positions of* $\text{Wyt}(f)$ *is given by $W := \cup_{i=0}^{\infty}\{(A_i, B_i)\}$, where A_i, B_i are defined in* (7).

Proof. As in the proof of Theorems 9 and 13, it suffices to show:

I. Every move from any position in W lands in a position outside W.

II. For every position outside W, there is a move to some position in W.

I. The proof is the same as in part I of Theorem 13, using g instead of f and using $g(x) \geq f(x)$.

II. The proof for this statement is as in part II of Theorem 13 up to and including Case (iii) $x = A_n \leq y < f(A_n) = B_n$. There are two additional cases:
Case (iv) $x = A_n$, $f(A_n) \leq y < B_n$ and there exists m for which $y = B_m < B_n$. Then move to the position (A_m, B_m) by subtracting $A_n - A_m$ tokens from the first pile, using a move of the first type.
Case (v) $x = A_n$, $f(A_n) \leq y < B_n$ and $y = B_m$ for no m. Since $B_n = g(A_n) > f(A_n)$ it follows from the definition of g and Remark 6 that:

$$B_m = g(A_m) = f(1)A_m \text{ for all } 0 \leq m \leq n. \tag{8}$$

Further, by Lemma 19,

$$A_m = m < f(1) \text{ for all } 0 \leq m \leq n. \tag{9}$$

Write

$$y = f(1)s + r,\ 0 \leq r < f(1),\ s, r \in \mathbb{Z}_{\geq 0}, \text{ and } u := A_n - A_s.$$

We claim that the move $(A_n, y) \to (A_s, B_s)$ is a valid move of the second type. For proving this claim, it suffices to show:

(a) $0 \leq s < n$.

(b) $y > B_s$.

(c) Either $0 < u \leq r < f(u)$ or $0 < r \leq u < f(r)$. Thus, condition (1) of the move of the second type is satisfied by subtracting u tokens from the first pile and r tokens from the second pile.

(a) We have $y = f(1)s + r < B_n = f(1)n$ by (9). Thus $s < n$, since $r < f(1)$. Clearly $s \geq 0$.

(b) By (8) and (a), $r = y - f(1)s = y - B_s$. Therefore $r > 0$ since $y \neq B_s$.

(c) By (9) and (a), $u = A_n - A_s = n - s > 0$ and $n - s \leq n < f(1)$. Thus if $u \leq r$, then $0 < u \leq r < f(1) \leq f(u)$. Otherwise, if $u > r$, then $0 < r < u < f(1) \leq f(r)$. □

Corollary 21. *Let f be a polynomial of the form* (5) *with $t \geq 2$ and $a_0 > 0$. Then there is a tractable strategy for winning* Wyt(f).

Proof. By Lemma 18, g is a strictly increasing superadditive function, and clearly polynomial-time computable. Using the algorithm in Theorem 16 and the instructions how to move from any N-position to a P-position in the second part of the proof of Theorem 20, it is clear how to construct a tractable strategy for winning Wyt(f) for any given N-position. □

5. Further Work

In this paper we studied the P-positions and winning strategies of the game $\mathrm{Wyt}(f)$ for three classes of functions: constant, superadditive and polynomial with nonnegative coefficients. There are two main direction in extending this work. One is to examine other classes of functions and possibly to generalize the results presented here to other classes of functions. For example, $\mathrm{Wyt}(f)$ where $f(k) = sk + t$, for $s, t \in \mathbb{R}_{>0}$ (The case $s, t \in \mathbb{Z}_{>0}$ was studied in [11]), or for an arbitrary polynomial function f.

Another direction is to study deeper properties in each class. One of the important properties of impartial games is the Sprague-Grundy function, which enables playing *sums* of games. Some results on the Sprague-Grundy function for Wythoff's game can be found in [21, 17, 4]. An additional direction is to study the sets of restrictions and extensions of the game, in the sense of a subset or superset of the set of possible moves, that preserve its P-positions. These directions were studied for Wythoff and other Wythoff-like games in [14, 6, 19].

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