Annales Mathematicae et Informaticae
41 (2013) pp. 57–62

Proceedings of the
15th International Conference on Fibonacci Numbers and Their Applications
Institute of Mathematics and Informatics, Eszterházy Károly College
Eger, Hungary, June 25–30, 2012

On h-perfect numbers

Heiko Harborth

Diskrete Mathematik, Technische Universität Braunschweig
38023 Braunschweig, Germany
h.harborth@tu-bs.de

Abstract

Let $\sigma(x)$ denote the sum of the divisors of x. The diophantine equation $\sigma(x)+\sigma(y)=2(x+y)$ equalizes the abundance and deficiency of x and y. For $x=n$ and $y=hn$ the solutions n are called h-perfect since the classical perfect numbers occur as solutions for $h=1$. Some results on h-perfect numbers are determined.

Keywords: perfect numbers, amicable numbers

MSC: 11A25

1. Introduction

Let $\sigma(n)$ denote the sum of the divisors of n, that is,

$$\sigma(n)=\prod_{i=1}^{r}\frac{p_i^{\alpha_i+1}-1}{p_i-1} \quad \text{for} \quad n=\prod_{i=1}^{r}p_i^{\alpha_i}.$$

Since the classical antiquity there exist two famous problems for $\sigma(n)$.

At first it is asked for perfect numbers n fulfilling

$$\sigma(n)=2n.$$

All even perfect numbers are of the form $n=(2^p-1)2^{p-1}$ where p is a prime number and where 2^p-1 is a so-called Mersenne prime number, too. Nearly 50 such prime numbers are known. The existence of odd perfect numbers is still unknown.

Secondly, it is asked for amicable number pairs x,y such that

$$\sigma(x)-x=y \quad \text{and} \quad \sigma(y)-y=x.$$

Several thousand pairs are known. It remains unknown whether there are infinitely many pairs.

Nonperfect numbers n are called abundant if $\sigma(n) > 2n$ and called deficient if $\sigma(n) < 2n$. Then it may be asked for perfect number pairs x, y fulfilling the diophantine equation

$$\sigma(x) + \sigma(y) = 2(x + y), \tag{1.1}$$

that is, x and y equalize abundance and deficiency.

There exist many solutions x, y of (1.1). For fixed d let X and Y be the sets of solutions x and y of $\sigma(x) = 2x + d$ and $\sigma(y) = 2y - d$, respectively. The sets X and Y are finite (see [1], p. 169). Then all pairs x, y with $x \in X$ and $y \in Y$ are solutions of (1.1).

It may be remarked that perfect and amicable numbers are special cases of (1.1): Perfect numbers for $x = y$ and amicable numbers for $\sigma(x) = \sigma(y)$.

Here it is proposed to consider the special class of solutions of (1.1) when y is a multiple of x, that is,

$$\sigma(n) + \sigma(hn) = 2(n + hn) = 2n(h + 1). \tag{1.2}$$

If $h = 1$ then n is a perfect number. Therefore solutions n of (1.2) may be called h-perfect numbers. Some results on h-perfect numbers are determined in the following.

2. Powers of two

For $h = 2^t$ all h-perfect numbers are dependent on a sequence of certain prime numbers being similar to Mersenne prime numbers.

Theorem 2.1. *A number n is 2^t-perfect, $t \geq 1$, if and only if it holds $n = 2^\alpha((2^t + 1)2^\alpha - 1)$ where $(2^t + 1)2^\alpha - 1$ is a prime number.*

Proof. Suppose that n is 2^t-perfect, $t \geq 1$.

If $(n, 2) = 1$ then equation (1.2) implies

$$\sigma(n) + \sigma(n2^t) = \sigma(n)(1 + 2^{t+1} - 1) = \sigma(n)2^{t+1} = 2n(1 + 2^t).$$

Since the left term of (1.2) is divisible by 2^{t+1} whereas the right term of (1.2) is divisible by 2 only, odd 2^t-perfect numbers do not exist.

If $n = s2^\alpha$, $\alpha \geq 1$, $(s, 2) = 1$ then equation (1.2) yields

$$\sigma(s2^\alpha) + \sigma(s2^{t+\alpha}) = 2(s2^\alpha + s2^{t+\alpha}).$$

This is equivalent to

$$\sigma(s)((2^t + 1)2^\alpha - 1) = (2^t + 1)2^\alpha s \quad \text{with} \quad s = v((2^t + 1)2^\alpha - 1),\ v \geq 1, \tag{2.1}$$

since $((2^t + 1)2^\alpha - 1, (2^t + 1)2^\alpha) = 1$.

If $v > 1$ then equation (2.1) determines

$$v((2^t+1)2^\alpha - 1) + v + 1 \leq \sigma(v((2^t+1)2^\alpha - 1)) = v(2^t+1)2^\alpha,$$

a contradiction.

If $v = 1$ and if $s = (2^t+1)2^\alpha - 1$ is a composite number then equation (2.1) yields

$$(2^t+1)2^\alpha < \sigma((2^t+1)2^\alpha - 1) = (2^t+1)2^\alpha,$$

again a contradiction.

If $v = 1$ and if $s = (2^t+1)2^\alpha - 1$ is a prime number then equations (2.1) and (1.2) are fulfilled and $n = s2^\alpha$ is 2^t-perfect. □

In [2] the first 16 and 12 prime numbers $p = (2^t+1)2^\alpha - 1$ are listed for $t = 1$ and $t = 2$, respectively. Thus 10, 44, 184, 752, 12224, 49024,… are the first 2-perfect numbers. The question for odd 2^t-perfect numbers, $t \geq 1$, is completely answered by nonexistence whereas it is still open in the classical case of perfect numbers.

3. Nonexistence

For some classes of values of h it can be proved that h-perfect numbers do not exist.

Theorem 3.1. *For $h = c2^t$, $(c,2) = 1$, $c \geq 3$, there are no even h-perfect numbers if $c + 2 < 2^{t+2}$ and there are no h-perfect numbers if $c + 2 < 2^{t+1}$.*

Proof. For even n let $n = r2^\alpha$, $\alpha \geq 1$, $(r,2) = 1$. Now suppose that n is $c2^t$-perfect for $c + 2 < 2^{t+2}$. Equation (1.2) implies

$$(2^{\alpha+1} - 1)\sigma(r) + (2^{\alpha+t+1} - 1)\sigma(cr) = r2^{\alpha+1}(c2^t + 1).$$

Using $\sigma(cr) \geq cr + \sigma(r)$ it follows

$$\sigma(r)(2^{\alpha+1} - 1 + 2^{\alpha+t+1} - 1) \leq (2^{\alpha+1} + c)r.$$

Then $\sigma(r) \geq r$ together with $\alpha \geq 1$ determines

$$2^{t+1} \leq 2^{\alpha+t+1} \leq c + 2,$$

a contradiction.

For odd n suppose that n is $c2^t$-perfect for $c + 2 < 2^{t+1}$. Equation (1.2) implies

$$\sigma(n) + (2^{t+1} - 1)\sigma(cn) = 2n(1 + c2^t).$$

With $\sigma(cn) \geq cn + \sigma(n)$ it follows

$$2^{t+1}\sigma(n) \leq (c+2)n$$

and with $\sigma(n) \geq n$ the contradiction

$$2^{t+1} \leq c + 2$$

is obtained. □

For $h < 100$ by Theorem 3.1 no h-perfect numbers occur if $h = 12$, 20, 24, 40, 48, 56, 72, 80, 88, or 92.

The following theorem presents another example of partial nonexistence.

Theorem 3.2. *There is no even 3^t-perfect number, $t \geq 1$.*

Proof. Suppose that $n = r2^\alpha$ is an h-perfect number for $h = 3^t$, $t \geq 1$, $\alpha \geq 1$, $(r, 2) = 1$. Equation (1.2) yields

$$\sigma(r)(2^{\alpha+1} - 1) + \sigma(r3^t)(2^{\alpha+1} - 1) = r2^{\alpha+1}(1 + 3^t). \tag{3.1}$$

Case I: $(r, 3) = 1$. It follows

$$\sigma(r)(2^{\alpha+1} - 1)(1 + (3^{t+1} - 1)/2) = r2^{\alpha+1}(1 + 3^t)$$

and equivalently

$$\sigma(r)(2^{\alpha+1} - 1)(1 + 3^{t+1}) = r2^{\alpha+2}(1 + 3^t).$$

With $\sigma(r) \geq r$ the inequality

$$(2^{\alpha+1} - 1)(1 + 3^{t+1}) \leq 2^{\alpha+2}(1 + 3^t)$$

is obtained being equivalent to

$$(3^t - 1)2^{\alpha+1} \leq 1 + 3^{t+1}.$$

This is a contradiction for $\alpha, t \geq 1$ excluded $\alpha = t = 1$. Then, however, the left term of (3.1) is divisible by 3 and, in the contrary, 3 does not divide the right term of (3.1) due to $(r, 3) = 1$.

Case II: $r = s3^\beta$, $\beta \geq 1$, $(s, 3) = 1$, and $(s, 2) = 1$ since $(r, 2) = 1$. By equation (3.1) it follows

$$\sigma(s)(2^{\alpha+1} - 1)(3^{\beta+1} + 3^{\beta+t+1} - 2) = s2^{\alpha+2}3^\beta(1 + 3^t)$$

and with $\sigma(s) \geq s$

$$2^{\alpha+1}3^{\beta+1} + 2^{\alpha+1}3^{t+\beta+1} - 2^{\alpha+2} - 3^{\beta+1} - 3^{t+\beta+1} + 2 \leq 2^{\alpha+2}3^{t+\beta} + 2^{\alpha+2}3^\beta.$$

This inequality is equivalent to

$$(3^\beta(1 + 3^t) - 2)(2^{\alpha+1} - 3) \leq 4$$

yielding a contradiction for $\alpha, \beta, t \geq 1$. $\square$

4. Even perfect-perfect numbers

For some values of h there exist only a small number of h-perfect numbers.

Theorem 4.1. *For $h = 6$ only 13 is h-perfect and for any other even perfect number h there are no h-perfect numbers.*

Proof. Let $h = (2^p - 1)2^{p-1}$ be an even perfect number, that is, p and $2^p - 1$ both are prime numbers. Suppose that n is an h-perfect number.

For even n, that is, $n = r2^\alpha$, $\alpha \geq 1$, $(r, 2) = 1$, Theorem 3.1 implies the condition $2^p + 1 \geq 2^{p+1}$ being impossible.

For odd n two cases are distinguished.

Case I: $n = r(2^p - 1)^\alpha = rq^\alpha$, $\alpha \geq 1$, $(r, 2^p - 1) = (r, q) = 1$. By equation (1.2),

$$\sigma(rq^\alpha) + \sigma(r2^{p-1}q^{\alpha+1}) = 2rq^\alpha(1 + q2^{p-1})$$

and hence

$$\sigma(r)(q^{\alpha+1} - 1 + (2^p - 1)(q^{\alpha+2} - 1)) = r(q - 1)(2q^\alpha + 2^p q^{\alpha+1}).$$

With $\sigma(r) \geq r$ and $2^p - 1 = q$ this yields

$$q^{\alpha+1} - 1 + q^{\alpha+3} - q \leq 2q^{\alpha+1} + q^{\alpha+3} + q^{\alpha+2} - 2q^\alpha - q^{\alpha+2} - q^{\alpha+1}$$

and thus the contradiction

$$2q^\alpha \leq q + 1.$$

Case II: $(n, 2^p - 1) = (n, q) = 1$. Equation (1.2) yields

$$\sigma(n) + \sigma(nq2^{p-1}) = 2n(1 + q2^{p-1}),$$

$$\sigma(n) + \sigma(n)(2^p - 1)(q + 1) = n(2 + q2^p),$$

and thus

$$\sigma(n)(1 + q(q + 1)) = n(2 + q(q + 1)).$$

Since $(1 + q(q + 1), 2 + q(q + 1)) = 1$ it is necessary that

$$\sigma(n) = v(2 + q(q + 1)) \quad \text{with} \quad n = v(1 + q(q + 1)),\ v \geq 1. \tag{4.1}$$

If $v > 1$ in equation (4.1) then

$$v(1 + q(q + 1)) + v + 1 \leq \sigma(n) = v(2 + q(q + 1))$$

is a contradiction.

If $v = 1$ in equation (4.1) and if $1 + q(q + 1)$ is a composite number then

$$2 + q(q + 1) < \sigma(n) = 2 + q(q + 1)$$

is a contradiction.

It remains that $v = 1$ in equation (4.1) and $1 + q(q + 1)$ is a prime number. This, however, is impossible for odd prime numbers p since 3 divides $1 + q(q + 1) = 1 + (2^p - 1)2^p$ due to $2^p \equiv -1 \pmod 3$. Thus $p = 2$ determines $1 + q(q + 1) = 13$ as the unique solution of equations (4.1) and (1.2) for $h = (2^2 - 1)2^{2-1} = 6$. $\square$

5. Small values of h

For $h \leq 16$ the discussion is completed for $h = 2$, 4, 6, 8, 12, and 16. For $h = 3$, 9, and 10 even h-perfect numbers do not exist. So far no h-perfect numbers are known for $h = 3$, 9, 10, and 13. The numbers $n = 14$ and $n = 7030$ are 5-perfect, $n = 135$ and $n = 1365$ are 7-perfect, $n = 182$ is 11-perfect, $n = 5$ and $n = 118$ are 14-perfect, and $n = 455$ is 15-perfect.

Finally, there are two corollaries for the Fibonacci number $F_7 = 13$ as consequences of Theorems 3.1 and 4.1.

Corollary 5.1. *Only 13 is an h-perfect number for any even perfect number h.*

Corollary 5.2. *Only 13 is a* $3 \cdot 2^t$*-perfect number for any* $t \geq 1$.

References

[1] Sierpinski, W., Elementary Theory of Numbers. Warszawa 1964.

[2] Online Ecyclopedia of Integer Sequences (OEIS), A007505 and A050522.