PERFECT NUMBERS CONCERNING
FIBONACCI SEQUENCE

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Abstract: We proved that there are no perfect numbers in the set

$$\left\{ \frac{F_{nm}}{F_m} \Big| n, m \in I\!N \right\},$$

where $F = \{F_n\}_{n=0}^{\infty}$ is the Fibonacci sequence.

1. Results and auxiliary lemmas

Let $I\!N$ and $\mathcal{P}$ denote the set of all positive integers and the set of all prime numbers, respectively. (m, n) denotes the greatest common divisor of the integers m and n. The notation $m \parallel n$ means that m is a unitary divisor of n, i.e. that $m|n$ and $(\frac{n}{m}, m) = 1$.

A positive integer N is called perfect if it is equal to the sum of all its proper divisors, i.e., if $\sigma(N) = 2N$, where $\sigma(N)$ denotes the sum of all positive divisors of N. Such integers were considered already by Euclid, who proved that i f the number $1 + 2 + \cdots + 2^n$ happens to be a prime then its product by 2^n is perfect. Euler was the first to prove that Euclid's method gives all even perfect numbers:

Euler's Theorem. *If N is an even perfect number, then it can be written in the form $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. Conversely, if p and $2^p - 1$ are prime numbers, then the product $2^{p-1}(2^p - 1)$ is perfect.*

For odd perfect numbers the situation is much worse since it is not known whether such numbers exist at all. This question forms one of the oldest problems in number theory. It is well-known that every odd perfect number is of the form $p^a x^2$, where p is a prime and $p \equiv a \equiv 1 \pmod 4$, furthermore all prime divisors of x is congruent to $-1 \pmod 4$.

Let $F = \{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2} \quad \text{for all integers} \quad n \geq 2.$$

We denote the Lucas sequence by $L = \{L_n\}_{n=0}^{\infty}$, which is given by $L_0 = 2$, $L_1 = 1$ and by the relation $L_n = L_{n-1} + L_{n-2}$ for all integers $n \geq 2$.

Recently, F. Luca [3] proved that there are no perfect Fibocacci or Lucas numbers. Our purpose in this note is to improve this result by proving the following

Theorem. *Let*

$$\mathcal{F} := \left\{ \frac{F_{nm}}{F_m} \ \middle| \ n, \, m \in I\!N \right\}.$$

Then there are no perfect numbers in the set $\mathcal{F}$.

We note that all numbers of the set $\mathcal{F}$ are positive integers, furthermore $F_n \in \mathcal{F}$ and $L_n \in \mathcal{F}$ for all $n \in I\!N$. Thus, there are no perfect Fibocacci or Lucas numbers. The following 51 numbers belong to $\mathcal{F}$ which are ≤ 10000:

1, 2, 3, 4, 5, 7, 8, 11, 13, 17, 18, 21, 29, 34, 47, 48, 55, 72, 76, 89, 122, 123, 144, 199, 233, 305, 322, 323, 329, 377, 521, 610, 842, 843, 987, 1292, 1353, 1364, 1597, 2207, 2208, 2255, 2584, 3571, 4181, 5473, 5777, 5778, 5796, 6765, 9349.

Our proof will make use of the Ribenboim's result about the square-classes of the Fibonacci and Lucas sequences. For a sequence $X = \{X_n\}_{n=0}^{\infty}$ we say that the terms X_n and X_m are square equivalent if there exist non-zero integers u and v such that

$$u^2 X_n = v^2 X_m$$

or equivalently

$$X_n X_m = t^2 \quad \text{with a suitable non-zero integer} \quad t.$$

The equivalent classes are called square-classes of X. A square-class is say trivial if it contains only one element.

Lemma 1. ([4]) *The square-class of a Fibonacci number F_k is trivial, if $k \neq 1, 2, 3, 6$ or 12 and the square-class of a Lucas number L_k is trivial, if $k \neq 0, 1, 3$ or 6.*

It is known that for each positive integer M there exists the smallest postive integer $f = f(M)$ such that $F_f \equiv 0 \pmod{M}$. This number $f = f(M)$ is called the rank of apparition of M in the Fibonacci sequence F.

We shall recall some properties of the Fibonacci sequence, which will be used at the proofs of our theorems.

Lemma 2. *We have*

(a) $F_k \equiv 0 \pmod{M}$ *if and only if* $f(M) \mid k$ $(k, \, M \in I\!N)$,

(b) $(F_i, \, F_j) = F_{(i, \, j)}$ *for all* $i, \, j \in I\!N$,

(c) $f(p) \mid p - (5/p)$ *for all odd primes p,*
        where $(5/p)$ is the Legendre symbol with $(5/5) = 0$,

(d) $f(p) \mid \frac{p-(5/p)}{2}$ *if and only if* $p \equiv 1 \pmod{4}$,

(e) $p^{e+w} \parallel F_{mtp^w}$ *if* $p \in \mathcal{P}$ *and* $e,\ w,\ m,\ t \in I\!N$ *with* $p^e \parallel F_m,\ p \nmid t,$

(f) $f(2^e) = \begin{cases} 3, & \text{if } e = 1 \\ 6, & \text{if } e = 2 \\ 3 \cdot 2^{e-2}, & \text{if } e \geq 3. \end{cases}$

Proof . The proof of Lemma 2 may be found in [1], [2], [5], [6].

2. The proof of the theorem

The proof of our theorem follows from following Lemma 3–4.

Lemma 3. *There are no even perfect numbers in the set* $\mathcal{F}$.

Proof. Assume that there is an even perfect number N in the set $\mathcal{F}$. Then by Euler's Theorem, we have

$$(1) \qquad\qquad N = \frac{F_{nm}}{F_m} = 2^{p-1}(2^p - 1),$$

for some positive integers $n \geq 2$ and m, where both p and $2^p - 1$ are primes.

Let
$$\alpha := \frac{1 + \sqrt{5}}{2}.$$

It is clear to check that

$$\alpha^{k-1} \geq F_k \geq \alpha^{k-2} \quad \text{for all} \quad k \in I\!N,$$

consequently

$$(2) \qquad\qquad \frac{F_{nm}}{F_m} = 2^{p-1}(2^p - 1) \geq \alpha^{(n-1)m-1}.$$

It is obvious from (1) that $2^{p-1}(2^p - 1)$ is the divisor of F_{nm}, therefore Lemma 2(a) implies

$$(3) \qquad\qquad f\left(2^{p-1}(2^p - 1)\right) \mid nm \ \text{ and } \ nm \geq f\left(2^{p-1}(2^p - 1)\right).$$

Since $2^{2p-1} > 2^{p-1}(2^p - 1)$, we deduce from (2) and (3) that

$$(2p - 1)\frac{\log 2}{\log \alpha} > (n-1)m - 1 = nm - m - 1 \geq f\left(2^{p-1}(2^p - 1)\right) - m - 1,$$

and so

$$m > f\left(2^{p-1}(2^p - 1)\right) - (2p - 1)\frac{\log 2}{\log \alpha} - 1.$$

Hence, in view of $n \geq 2$ we have

$$(2p-1)\frac{\log 2}{\log \alpha} > (n-1)m - 1 \geq m - 1 > f\left(2^{p-1}(2^p - 1)\right) - (2p-1)\frac{\log 2}{\log \alpha} - 2,$$

and so

(4)
$$2(2p-1)\frac{\log 2}{\log \alpha} + 2 > f\left(2^{p-1}(2^p - 1)\right).$$

It is clear to check that

$$f\left(2^{p-1}(2^p - 1)\right) = \begin{cases} 12, & \text{if } p = 2 \\ 24, & \text{if } p = 3 \\ 60, & \text{if } p = 5 \end{cases},$$

which with (4) shows that $p \geq 7$, because

$$2(2p-1)\frac{\log 2}{\log \alpha} + 2 < \begin{cases} 12, & \text{if } p = 2 \\ 24, & \text{if } p = 3 \\ 60, & \text{if } p = 5 \end{cases}.$$

Thus we have proved that (1) implies $p \geq 7$.

Assume that (1) is satisfied for a suitable prime $p \geq 7$ and positive integers n, m. Then from Lemma 2 (f) we get

$$f\left(2^{p-1}(2^p - 1)\right) \geq f(2^{p-1}) = 3 \cdot 2^{p-3},$$

which together with (4) leads to

$$2(2p-1)\frac{\log 2}{\log \alpha} + 2 > 3 \cdot 2^{p-3}.$$

This inequality is impossible for all prime $p \geq 7$, thus Lemma 3 is proved.

Lemma 4. *There are no odd perfect numbers in the set $\mathcal{F}$.*

Proof. Assume that there exists an odd perfect number N in the set $\mathcal{F}$. Then

$$N = \frac{F_{nm}}{F_m} \quad \text{for some positive integers } \ n \geq 2, \ m.$$

It well-known that in this case we can write N as in the form

(5)
$$N = \frac{F_{nm}}{F_m} = p^a (q_1^{a_1} \cdots q_s^{a_s})^2$$

with distinct primes $p, q_1, \ldots, q_s$ and positive integers $a, a_1, \ldots, a_s$, furthermore

(6) $\qquad p \equiv a \equiv 1 \pmod 4 \ \text{ and } \ q_1 \equiv \cdots \equiv q_s \equiv -1 \pmod 4.$

First we prove that

(7) $$(nm, 2) = 1.$$

Assume that n is even. Then

$$N = \frac{F_{nm}}{F_m} = \frac{F_{\frac{n}{2}m}}{F_m} \cdot L_{\frac{n}{2}m} = p^a (q_1^{a_1} \cdots q_s^{a_s})^2.$$

By using the fact

(8) $$(F_k, \ L_k) = \begin{cases} 2, & \text{if } 3|k \\ 1, & \text{if } (3, k) = 1, \end{cases}$$

we have $(F_{\frac{n}{2}m}, \ L_{\frac{n}{2}m}) = 1$. Thus, the last relation shows that one of the numbers $\frac{F_{\frac{n}{2}m}}{F_m}$, $L_{\frac{n}{2}m}$ is a square. This, using Lemma 1, implies that

(9) $$\frac{n}{2}m \in \{1, \ 2, \ 3, \ 6, \ 12\}.$$

Thus, we have

(10) $\quad nm \in \{2, \ 4, \ 6, \ 12, \ 24\} \ \text{ and } \ F_{nm} \in \{1, \ 3, \ 2^3, \ 2^4 \cdot 3^2, \ 2^5 \cdot 3^2 \cdot 7 \cdot 23\}.$

Since N is an odd divisor of F_{nm}, one can check from (10) that any odd divisor of F_{nm} is not a perfect number.

Now assume that n is odd and m is even. Then

$$\frac{F_{nm}}{F_m} = \frac{F_{n\frac{m}{2}}}{F_{\frac{m}{2}}} \cdot \frac{L_{n\frac{m}{2}}}{L_{\frac{m}{2}}}$$

and

$$\left(\frac{F_{n\frac{m}{2}}}{F_{\frac{m}{2}}}, \ \frac{L_{n\frac{m}{2}}}{L_{\frac{m}{2}}} \right) = 1.$$

Thus, we infer from (5) that one of the numbers $\frac{F_{n\frac{m}{2}}}{F_{\frac{m}{2}}}$, $\frac{L_{n\frac{m}{2}}}{L_{\frac{m}{2}}}$ is a square. This, using Lemma 1, implies that (9) and (10) are satisfied. As we shown above, these are impossible.

Thus, we have proved that nm is odd.

Now we complete the proof of Lemma 4.

If $s \geq 1$, then we infer from (5), (7) and Lemma 2(a) that

$$f(q_1) \mid nm, \quad \text{i.e.} \quad f(q_1) \text{ is odd.}$$

This implies that

$$f(q_1) \mid \frac{q_1 - (\frac{5}{q_1})}{2}.$$

From Lemma 2(d), we have $q_1 \equiv 1 \pmod 4$, but this contradicts to (6). Thus we have proved that the odd perfect number N has the form $N = \frac{F_{nm}}{F_m} = p^a$, with a prime p and a positive integer a. In this case, we have

$$2 = \frac{\sigma(N)}{N} = 1 + \frac{1}{p} + \cdots + \frac{1}{p^a} < \frac{p}{p-1},$$

which gives $p < 2$. This is impossible.

The proof of Lemma 4 is complete and the theorem is proved.

References

[1] P. BUNDSCHUH and J. S. SHIUE, Generalization of a paper by D. D. Wall, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* Ser II **56** (1974), 135–144.

[2] D. H. LEHMER, An extended theory of Lucas' function, *Ann. of Math.* **31** (1930), 419–448.

[3] F. LUCA, Perfect Fibonacci and Lucas numbers, submitted.

[4] P. RIBENBOIM, Square classes of Fibonacci and Lucas sequences, *Portugaliae Math.*, **48** (1991), 469–473.

[5] P. RIBENBOIM, The book of prime number records, Springer Verlag, New York-Berlin, 1991.

[6] O. WYLER, On second order recurrences, *Amer. Math. Monthly* **72** (1965), 500–506.

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