EXACT ENUMERATION OF ACYCLIC AUTOMATA

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Abstract. A linear recurrence relation is derived for the number of unlabeled initially connected acyclic automata. The coefficients of this relation are determined by another, alternating, recurrence relation. The latter determines in particular the number of acyclic automata with labeled states. Certain simple enumerative techniques developed by the author long ago for counting initially connected automata and acyclic digraphs are combined and applied. Calculations show that the obtained results improve recent upper bounds for the number of minimal automata (with accepting states) recognizing finite languages. Various related questions are also discussed.

Résumé. Une relation linéaire de recurrence est derivée pour le nombre d'automates acycliques non-étiquetés connexes depuis l'état initial. Les coefficients de cette relation sont détermines par une autre relation de récurrence qui est alternante. Celle-ci détermine, en particulier, le nombre d'automates acycliques aux états étiquetés. Certaines techniques énumératives, développées il y a longtemps par l'auteur pour l'énumération d'automates acycliques connexes depuis l'état initial et de graphes acycliques, sont combinées et appliquées. Des calculs montrent que les résultats ainsi obtenus améliorent certains bornes supérieures récentes pour le nombre d'automates minimaux (avec des états acceptants) qui reconnaissent des langages finis. En plus, quelques problèmes divers y reliés sont discutés.

1. Introduction

Recently Domaratzki [Do01] and [Do02] (see also [DoKS01]) obtained some lower and upper bounds for the number of minimal n-state automata recognizing *finite* languages. In particular one of the upper bounds is based upon the enumeration of initially connected acyclic automata with numbered states, where the transitions between states are compatible with the state numbers (from lesser to greater). These automata proved to be enumerated by the familiar (unsigned) Genocchi numbers [St99, ex.5.8(d)] (close to the Bernoulli numbers) in the case of 2 input letters and by certain generalized Genocchi numbers for $k > 2$ inputs. The author noted that a better bound should follow from the enumeration of such automata as unlabeled ones. It is this problem, natural and interesting by itself, which is solved here. The idea is to combine two approaches which we developed long ago for counting labeled acyclic digraphs [Li75] and arbitrary initially connected automata [Li69]. The point is that in the latter case, automata *do not* have non-trivial automorphisms; so that the problems of counting them as having labeled or unlabeled states are equivalent. As

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an intermediate step we count labeled acyclic automata and, more generally, quasi-acyclic automata with a given number of absorbing states (see the precise definitions in Section 2).

Numerical calculations suggest that our formulae indeed provide a significantly better upper bound for the number of minimal n-state automata with accepting states recognizing finite languages (acceptors). This assertion remains, however, unproven since we have not extracted any asymptotics or tight estimates from the formulae obtained. Nor could we express the results in terms of generating functions. We discuss these and other related questions, including some conjectures and old results, in the second half of the paper.

Initially connected acyclic automata with a unique "pre-dead" state can also be enumerated in a similar way, and these numbers provide a somewhat better upper bound for the numbers of minimal automata.

The present research is motivated by abstract automata theory and is represented in terms of automata. However our main results can be considered independently of automata theory as the enumeration of some rather natural types of directed graphs.

2. Definitions. Preliminaries

2.1. Initial automata. Generally, for automata theory we refer to [HoU79]. For the reader's convenience, together with terms adopted in the present paper we point out some of their synonyms which often appear in the current literature. Throughout the paper we consider deterministic *initial finite completely defined* automata without outputs. Thus, an (initial) *automaton* is a quadruple $\mathcal{A} = (Q, q_0, X, \delta)$, where $Q = Q_\mathcal{A}$ is the set of states, $q_0 \in Q$ is the initial state, X is the input alphabet and $\delta = \delta_\mathcal{A} : (Q, X) \to Q$ is the transition function. δ extends naturally to the set X^* of all finite words over X : if $w = x_1x_2 \dots x_s$, then $\delta(q, w) := \delta((\delta(\delta(q, x_1), x_2) \dots), x_s)$. By definition, $\delta(q, \epsilon) = q$ for any state q, where ϵ is the *empty* word. If $\delta(q, w) = q'$, we say that the automaton $\mathcal{A}$ goes or passes from the state q to q' under the action of the input word $w \in X^*$ and that q' is *reachable* (accessible) from q in $\mathcal{A}$. The number of states $m = |Q_\mathcal{A}|$ is called the *size* of $\mathcal{A}$. Any input letter x determines a mapping δ^x from the set of states to itself, and δ can be identified with the set of mappings $\{\delta^x\}_{x \in X}$.

Sometimes we admit *non-initial* automata; these are triples (Q, X, δ) in which no initial state is distinguished.

2.2. Acceptors recognizing languages. An *acceptor* means an automaton with accepting states, i.e. a pair $(\mathcal{A}, F)$, where $\mathcal{A}$ is an automaton and $F \subseteq Q_\mathcal{A}$ is a nonempty set of states called *accepting*, or final. The other states are called *non-accepting*. In the literature, acceptors are often called simply automata or recognizers.

$L(\mathcal{A}, F)$ denotes the language *recognized* (or accepted) by the acceptor $(\mathcal{A}, F)$, i.e. the set of all words under which $\mathcal{A}$ goes from the initial state to an accepting state: $L(\mathcal{A}, F) := \{w \mid w \in X^*,\ \delta(q_0, w) \in F\}$. $(\mathcal{A}, F)$ is called *minimal* recognizing a given language $L \subseteq X^*$ if $L = L(\mathcal{A}, F)$ and $\mathcal{A}$ is of minimal size (number of states) among all the acceptors recognizing L.

2.3. Recurrent and transient states. We call a state q of an automaton $\mathcal{A}$ *recurrent* if there exists a nonempty word w which returns $\mathcal{A}$ from q to itself: $\delta(q, w) =$

q. Such states are also known as cyclic or looping. Non-recurrent states are called *transient*. Evidently, any (completely defined) finite automaton has recurrent states. Moreover, for any state, there is a recurrent state reachable from it. It follows that finite automata cannot be acyclic in the strict sense of this term; so we have to relax the restrictions.

2.4. Acyclic automata. Dead and pre-dead states. An automaton is called *acyclic* if it has a unique recurrent state. The recurrent state of an acyclic automata is called its *dead state* (or sink).

It is convenient to distinguish the dead state of acyclic automata, and we will designate it separately by the letter "D" possibly with a subscript. By 2.3, the dead state D of an acyclic automaton is *absorbing* (a trap), i.e. $\delta(D, x) = D$ for any $x \in X$. In a sense, the dead state is a formal element of an acyclic automata. From now on, n will denote the number of *transient* states of acyclic automata, so that $n = m - 1$. The transient states are usually labeled by $q_0, q_1, \ldots, q_{n-1}$, where q_0 is the initial state. It is important to stress that we do *not* demand that the transition function δ be compatible with the numbers (or any other order) of the labels $q_1, q_2, \ldots$.

Call a state q of an acyclic automata *pre-dead* if only the dead state D is reachable from q by all inputs. Such states always exist: these are the sinks of the partial automata obtained after the deletion of the dead state and all transitions to it.

2.5. Initially connected automata. A state of an automaton is referred to as a *source* (or maximal) if there are no transitions to it. It is easy to see that any non-empty acyclic automaton has at least one source.

An automaton $\mathcal{A}$ is called *initially connected* if all of its states are reachable from the initial state. An acyclic automaton is initially connected if and only if q_0 is its unique source. In the current literature, initially connected automata are sometimes referred to as start-useful automata (or automata with no start-useless state).

The transition diagram of an acyclic automata is an acyclic (multi)digraph up to loops in the dead state, and in the case of initially connected acyclic automata this is an acyclic digraph with a unique sink and a unique source (a two-pole acyclic network).

2.6. Subautomata. Let $\mathcal{A}$ be an automaton with the set of states Q. If R is a subset of Q and if $\delta(q, x) \in R$ for any $q \in R$ and $x \in X$, then R and the restriction of δ to R form an automaton called a *subautomaton*. In other words, a subautomaton absorbs all transitions: it admits transitions to it from the outside, but all the transitions from it lead again to it. This notion is naturally extended to acceptors: $F \cap R$ serves as the set of accepting states.

2.7. Lemma. *For any state $q \in Q$, all states reachable from it in an automaton $\mathcal{A}$ form an initially connected subautomaton $\mathcal{A}^{(q)}$ with the initial state q.* □

$\mathcal{A}^{(q)}$ is a minimal subautomaton containing q and is said to be *generated* by the state q.

The subautomaton $\mathcal{A}^{(q_0)}$ is called the *initially connected component* of $\mathcal{A}$.

By definition, subautomata generated by states satisfy the following heredity property: if q' is reachable from q, then $\mathcal{A}^{(q')}$ is a subautomaton of $\mathcal{A}^{(q)}$ and $\mathcal{A}^{(q)(q')} = \mathcal{A}^{(q')}$.

2.8. Isomorphism. Two automata $\mathcal{A} = (Q, q_0, X, \delta)$ and $\mathcal{A}' = (Q', q_0', X, \delta')$ with the same input alphabet X are called *isomorphic* (by states) if there is a one-to-one correspondence (isomorphism) between their sets of states $\rho : Q' \to Q$ such that $\rho(q_0') = q_0$ and $\rho(\delta'(q', x)) = \delta(\rho(q), x)$ for all states $q' \in Q'$ and all $x \in X$. An isomorphism of acceptors must additionally preserve the property of states to be or not to be accepting.

Isomorphisms from $\mathcal{A}$ to $\mathcal{A}$ are called *automorphisms*. All automorphisms of $\mathcal{A}$ form a group.

Two states q and q' of an automaton $\mathcal{A}$ are called *similar* if the subautomata $\mathcal{A}^{(q)}$ and $\mathcal{A}^{(q')}$ generated by them are isomorphic. $\mathcal{A}$ is referred to as a *primitive* automaton if all its subautomata generated by a single state are pairwise non-similar.

The following assertion is well known (see, e.g., [Li69]) and easily provable since any automorphism preserves the initial state and all paths from it:

2.9. Lemma. *The group of automorphisms of an initially connected automaton is trivial.*

2.10. Finite languages and minimal acceptors. Consider an acceptor $(\mathcal{A}, F)$. If there is a recurrent state q in it reachable from the initial state q_0 and an accepting state q' reachable from q, then it is evident that the language $L = L(\mathcal{A}, F)$ recognized by $(\mathcal{A}, F)$ is infinite. Conversely, if $\mathcal{A}$ is acyclic and $D \notin F$, then $L(\mathcal{A}, F)$ is finite. These facts explain a particular interest of researchers to acyclic automata and acceptors, which prove to be efficient tools for a formal representation and processing of artificial and natural languages; see, in particular, [Re92, DaMWW00].

The following important claim is valid (see, e.g., [Mi99]):

2.11. Proposition.
1. *For any finite language L, there exists a unique (up to isomorphism) minimal acceptor $(\mathcal{A}, F) = (\mathcal{A}_L, F_L)$ recognizing it. Moreover:*
2. *$\mathcal{A}_L$ is an initially connected acyclic automaton.*
3. *For any state, there is an accepting state reachable from it.*

The first assertion is a direct corollary of the famous Myhill – Nerode theorem [HoU79]; the second and third assertions are evident. In the literature, automata satisfying properties 2 and 3 are sometimes called stripped or trim, and automata satisfying properties 3 are said to have no final-useless state.

In fact, the minimal acceptors are known to be completely characterized by one more property. Call $(\mathcal{A}, F)$ *reduced* if $L_{q'} \neq L_q$ for any two different states q' and q, where L_q denotes the set of all words recognizable by the subautomaton $\mathcal{A}^{(q)}$ (more exactly, by the corresponding acceptor): $L_q := \{w \mid \delta(q, w) \in F\}$. In particular, $L_{q_0} = L$. If $L_{q'} = L_q$, the states q' and q are said to be *equivalent*, and if such $q' \neq q$ exist, the acceptor $(\mathcal{A}, F)$ is called *reducible.*

2.12. Lemma. *$(\mathcal{A}, F)$ is the minimal acceptor (for the language recognized by it) if and only if it satisfies assertions* 2 *and* 3 *of Proposition* 2.11 *and is reduced.*

Two elementary facts concerning acceptors should also be mentioned: if $L = L(\mathcal{A}, F)$ is finite, then $D \notin F$; $\epsilon \in L$ if and only if $q_0 \in F$.

2.13. Enumerators. Now we can obtain some upper bounds for the number $M_k(n)$ of minimal $(n+1)$-state acceptors recognizing finite languages. Denote by $C_k(n)$ the number of initially connected acyclic automaton, counted up to isomorphism (that is, *unlabeled*), with n transient states and k inputs. It is clear from assertion 3 of Proposition 2.11 that in any minimal acceptor $(\mathcal{A}, F)$ recognizing a finite language, F must contain all the pre-dead states. Consequently, in any automaton $\mathcal{A}$ there are no more than 2^{n-1} ways to choose F. Therefore

$$M_k(n) \leq 2^{n-1} C_k(n). \tag{1}$$

Moreover, we can strengthen this bound. As we have just seen, if a minimal acceptor had two or more pre-dead states, then all of them would be accepting. But then they would be equivalent, which is impossible for minimal acceptors by Lemma 2.12. Thus we obtain the following (cf. [Ma00]).

2.14. Corollary. *Any minimal acceptor recognizing a finite nonempty language has only one pre-dead state q_*. The state q_* is accepting, and it is reachable from any transient state.*

Therefore, to estimate the number of minimal acceptors, we may restrict ourselves to initially connected acyclic automata with a unique pre-dead state. Denoting by $C_k^{(1)}(n)$ their number, we obtain instead of (1) a tighter upper bound:

$$M_k(n) \leq 2^{n-1} C_k^{(1)}(n). \tag{2}$$

This inequality, however, does not strengthen (1) very significantly; see Table 5 below, conjectured formula (15) in Subsect. 7.2 and the discussion therein.

Our main aim is to obtain a formula for the number of unlabeled initially connected acyclic automata $C_k(n)$. To derive it we first count labeled acyclic automata; let $a_k(n)$ denote the number of them with $n+1$ states including D.

2.15. Quasi-acyclic automata. We need also a generalization of acyclic automata called *quasi-acyclic*: these are automata in which all recurrent states are absorbing, i.e. they are dead ones (sinks). This natural class is not too much popular in automata theory since an acceptor with more than one dead states cannot be minimal.

$a_k(n,r)$ will denote the number of quasi-acyclic automata with $r \geq 1$ dead states $D_1, D_2, \ldots, D_r$ and n transient labeled states. Thus, $a_k(n,1) = a_k(n)$. It is important that instead of being the dead states, $D_1, D_2, \ldots, D_r$ may form an arbitrary subautomaton: $a_k(n,r)$ counts also the number of all automata with such a fixed absorbing subautomaton (*"black hole"*) and n other, transient, states. Later on, we will make use of this fruitful treatment, in particular in the formula for the number of labeled initially connected acyclic automata $c_k(n)$.

3. Main results

We begin with quasi-acyclic automata, not necessarily initially connected.

3.1. Theorem. $a_k(0,r) = 1$, *and for $n \geq 1$ the quantity $a_k(n,r)$ is determined by the following recursion:*

$$a_k(n,r) = \sum_{t=0}^{n-1} \binom{n}{t} (-1)^{n-t-1} (t+r)^{k(n-t)} a_k(t,r), \quad r \geq 1. \tag{3}$$

Proof. We reason as in the case of acyclic digraphs [Li75]. Consider arbitrary quasi-acyclic automata with k inputs, n (labeled) transient states and r dead states. Let $Y \subseteq Q$ be a set of $n-t$ transient states ($0 \le t \le n$). Introduce the property Π_Y of an automaton to have Y as a subset of its sources. There are $(t+r)^{k(n-t)}a_k(t,r)$ such automata: we take an arbitrary quasi-acyclic automaton with the set $Q \backslash Y$ of transient states, add Y to it and define the $k(n-t)$ transitions from Y to $(Q\backslash Y) \cup Z$ in an arbitrary way, where $Z = Z_r$ denotes the set of dead states. Now by the inclusion–exclusion method we can count the number of automata possessing none of these properties, and it should be equated to 0, since any nonempty acyclic automaton possesses a source. Thus, we obtain the formula

$$\sum_{t=0}^{n} \binom{n}{t} (-1)^{n-t}(t+r)^{k(n-t)}a_k(t,r) = 0, \quad n \ge 1,\ r \ge 1, \tag{4}$$

which is equivalent to (3). □

3.2. Theorem. $c_k(1) = 1$, *and for* $n > 1$, *the number of labeled initially connected acyclic automata* $c_k(n)$ *is determined by the following recursion:*

$$\sum_{t=1}^{n} \binom{n-1}{t-1} a_k(n-t, t+1)c_k(t) = a_k(n), \tag{5}$$

where $a_k(n) = a_k(n,1)$.

Proof. In [Li69] (see also [Li69a, Li69b]) we used a simple enumerative method, which we call the "injection method", in order to count arbitrary labeled initially connected automata (see formula (11) below). This method generalizes the well-known method of counting connected graphs of various types and related objects ("exponential structures" by Stanley [St99, 5.5]). In practice it is applicable fruitfully to digraphs possessing a generalized connectivity. Briefly, the idea is to "inject" the (connected) digraphs under consideration $\mathfrak{C}$ into an appropriate class of digraphs $\mathfrak{A}$ in such a way that any graph $\Delta \in \mathfrak{A}$ contains a uniquely determined subgraph Γ (its "connected" component) belonging to $\mathfrak{C}$. And, conversely, we require that the number $\alpha(n,t)$ of graphs $\Delta \in \mathfrak{A}$ with a given component $\Gamma \in \mathfrak{C}$ depend only on the sizes t of Γ and n of Δ (see [Li77] for a more general and abstract description of this method, which covers Theorem 3.1 as well). If these properties hold, we obtain immediately a linear recurrence relation of form

$$\sum_{t=1}^{n} \binom{n}{t} \alpha(n,t)c(t) = a(n), \tag{6}$$

where $a(n)$ and $c(n)$ stand for the number of graphs with n nodes, resp., in $\mathfrak{A}$ and $\mathfrak{C}$. The factor $\binom{n}{t}$ corresponds to the case when the component can contain any t-element set of nodes; for graphs with a distinguished root this factor is replaced by $\binom{n-1}{t-1}$, and so on. We called $\alpha(n,t)$ the *kernel* of equation (6).

In the problem under consideration, $\mathfrak{C}$ is the class of initially connected acyclic automata, and we take the set of acyclic automata as $\mathfrak{A}$. In any acyclic automaton (or, equivalently, automaton transition diagram) $\Delta \in \mathfrak{A}$ we select its initially connected component $\Gamma = \Delta^{(q_0)}$. Now, given an initially connected acyclic component Γ with

t labeled transient states, we consider the possible acyclic automata Δ with n transient states over it. Following the idea formulated in Subsect. 2.15 we may interpret these automata as the quasi-acyclic ones with $t+1$ dead and $n-t$ transient states. Therefore, regardless of a particular choice of Γ, there are $\alpha(n,t) = a_k(n-t, t+1)$ such Δ, and the injection method is applicable here. To complete the proof of (5) we need only to add that t states of the component Γ including q_0 can be chosen in $\binom{n-1}{t-1}$ ways. □

Now, according to Lemma 2.9, for unlabeled initially connected acyclic automata we have the formula

$$C_k(n) = \frac{c_k(n)}{(n-1)!}. \tag{7}$$

It is interesting to note that we do not know formal, purely *analytical* reasons which would explain why the solution of equation (5) is divisible by $(n-1)!$ for any n. The same remark concerns also formulae (7′) and (11) below.

3.3. Automata with one pre-dead state. Similar arguments can be applied to acyclic automata with a unique pre-dead state.

Consider labeled automata which have q_1 as the pre-dead state. Let $b_k(n,r)$ denote the number of quasi-acyclic automata which have n transient states *different* from q_1, r dead states including D and the property that q_1 is the unique state such that all transitions from it go to D. Reasoning as in the proof of Theorem 3.1 with $Y \subseteq Q\backslash\{q_1\}$ we obtain the recurrence relation

$$b_k(n,r) = \sum_{t=0}^{n-1} \binom{n}{t} (-1)^{n-t-1} [(t+r+1)^k - 1]^{n-t} b_k(t,r), \quad r \geq 1, \tag{3′}$$

which together with the initial conditions $b_k(0,r) = 1$ determines the function $b_k(n,r)$ for all $r \geq 1$. In particular $b_k(n,1) = b_k(n)$ is the number of acyclic automata with q_1 as the unique pre-dead state and n other transient states. The factor $[(t+r+1)^k - 1]^{n-t}$ in equation (3′) is the number of admissible transitions from Y, where $|Y| = n-t$, to the other $t+r+1$ states including q_1: for every state $q \in Y$, there is only one inadmissible set of transitions, all to D.

Let $\bar{c}_k(n)$ denote the number of the corresponding initially connected automata. Take an acyclic automaton Δ with q_1 as the unique pre-dead state. Its initially connected component Γ contains q_1, for otherwise a pre-dead state of Γ would be another pre-dead state of Δ. Let Γ contain $t \geq 0$ other transient states. Then reasoning just as in the proof of Theorem 3.2 we obtain

$$\sum_{t=1}^{n} \binom{n-1}{t-1} b_k(n-t, t+1) \bar{c}_k(t) = b_k(n), \quad n \geq 1. \tag{5′}$$

Finally, due to Lemma 2.9 we have the following (cf. (7)).

3.4. Theorem. *The number of unlabeled initially connected acyclic automata with a unique pre-dead state satisfies the following equation*

$$C_k^{(1)}(n+1) = \frac{\bar{c}_k(n)}{(n-1)!}, \quad n \geq 1, \quad \text{and} \quad C_k^{(1)}(1) = 1, \tag{7′}$$

where $\bar{c}_k(n)$ *is determined by formulae* (5′) *and* (3′), *and* $n+1$ *is the number of transient states* including *the pre-dead state.* □

3.5. Remark. There are some reasons to rescale formulae (3′) and (5′) replacing $b_k(n,r)$ and $\bar{c}_k(n)$ by new quantities which are closer to $a_k(n,r)$ and $c_k(n)$, namely, by $a_k^{(1)}(n,r) := nb_k(n-1,r)$, the number of labeled quasi-acyclic automata with r dead states, and $c_k^{(1)}(n) := (n-1)\bar{c}_k(n-1)$, $n>1$, the number of initially connected acyclic automata. In both cases, n is the total number of transient states, one of them is distinguished, and the distinguished state is a unique to have all transitions going to D. The distinguished state is an *arbitrary* state, not necessarily q_1 (but it is clearly different from q_0 in the case of $c_k^{(1)}(n)$).

4. Autonomous automata

Consider the particular case of automata with one input: $k=1$. Such automata are usually called *autonomous* (or unary). It is evident that autonomous acyclic n-state automata are equinumerous with labelled trees on $n+1$ nodes. So, $a_1(n)=(n+1)^{n-1}$. More generally, quasi-acyclic automata are in one-to-one correspondence with forests of rooted labeled trees, and there are

$$r(n+r)^{n-1} = a_1(n,r)$$

of them with $n+r$ nodes and $r \geq 1$ trees, where every dead state serves as the distinguished root of a tree [St99, 5.3.2]. Substituting these values into formula (3) we obtain the following identity:

$$\sum_{t=0}^{n} \binom{n}{t} (-1)^{n-t}(t+r)^{n-1} = 0. \tag{8}$$

Of course, it is not new, see, e.g., [Go72, (1.13)].

An autonomous acyclic automaton is initially connected if and only if it is a chain starting at q_0 and finishing at D. There are $c_1(n)=(n-1)!$ such labeled chains (hence $C_1(n)=1$ for all n). Therefore formula (5) for $k=1$ turns into the following variation of the familiar Riordan identity [Ri62] (cf. also [Li69a]):

$$\sum_{t=1}^{n} \binom{n-1}{t-1} (n+1)^{n-t}(t+1)(t-1)! = (n+1)^n. \tag{9}$$

5. Minimal acceptors

The exact enumeration of minimal acceptors recognizing finite languages remains an open problem (cf. [CaP02, DoKS01]). Here we are interested in the relationship between initially connected acyclic automata and minimal acceptors corresponding to them. We begin with several new definitions.

5.1. Rank and diameter. By the *rank* of a state q of an acyclic automaton we understand the number equal to 1 less than the maximal length of (simple) paths from q to the dead state. For automata with a unique pre-dead state q_*, this is the maximal length of paths (words) leading from q to q_*. In particular, the rank of q_* is 0. States of rank 1 are also called here *pre-pre-dead*: these are the states becoming sinks after the deletion of the dead and pre-dead states. In the literature, rank is also known under other names such as height or layer.

The maximal rank of states is called the *diameter* of an acyclic automaton. The diameter of an initially connected acyclic automaton is equal to the rank of the initial state, and for the minimal acceptor recognizing a finite language L it is equal to the maximal length of words in L.

5.2. "Useless" automata. There exist initially connected acyclic automata with a unique pre-dead state which cannot become minimal acceptors for *any* choice of the set of accepting states, for instance, such are automata with 3 or more pre-pre-dead states in which all transitions from them lead to the pre-dead state q_*. Indeed, for any choice of F, at least two pre-pre-dead states are both accepting or both non-accepting. Consequently, they are equivalent and may be merged together.

More generally, minimal acceptors recognizing finite languages can have no more than $2(2^k - 1)$ pre-pre-dead states. Indeed, all transitions from a pre-pre-dead state lead to the dead or pre-dead states. Hence there are $2^k - 1$ possible sets of transitions (we must exclude the only case where all transitions lead to D : it would create one more pre-dead state.) Now, any such set of transitions may be implemented no more than twice, once in an accepting state and once in a non-accepting state, and the estimate follows by Lemma 2.12.

There are similar constraints, though less restrictive, concerning states of rank 2 or more.

5.3. Primitive automata. At the opposite extreme, there are initially connected acyclic automata with a unique pre-dead state for which any F containing the pre-dead state gives rise to minimal acceptors. Such automata can be easily characterized.

5.4. Proposition. *Let $\mathcal{A}$ be an initially connected acyclic automaton with a unique pre-dead state q_*. Any F containing q_* gives rise to minimal acceptors $(\mathcal{A}, F)$ if and only if $\mathcal{A}$ is primitive.*

Proof. If $\mathcal{A}$ contains two similar states q' and q (see the definition in Subsect. 2.8) then we can easily find a subset F such that the acceptor $(\mathcal{A}, F)$ is reducible. For example, if F contains all transient states both of $\mathcal{A}^{(q')}$ and $\mathcal{A}^{(q)}$, then $L_{q'} = L_q$. By Lemma 2.12, $(\mathcal{A}, F)$ is not minimal.

On the contrary, suppose that $(\mathcal{A}, F)$ is reducible. This means that there are different equivalent states q' and q, i.e. states such that $L_{q'} = L_q$. It is evident that for any F containing q_*, the rank of q is equal to the maximal length of words in the language L_q : the longest path from q to F terminates in q_*. The same is valid for q', therefore q' and q are of the same rank. Take now equivalent q' and q of minimal rank. We have $L_q = L(\mathcal{A}^{(q)}, \bar{F})$ and $L_{q'} = L(\mathcal{A}^{(q')}, \bar{F}')$, where $\bar{F}$ and $\bar{F}'$ denote the sets of all accepting states reachable from q and q' resp. Now, if the acceptors $(\mathcal{A}^{(q)}, \bar{F})$ and $(\mathcal{A}^{(q')}, \bar{F}')$ are reduced (minimal), then by Proposition 2.11 (1), they coincide up to isomorphism; so that $\mathcal{A}$ is not primitive.

Suppose conversely that the acceptor $(\mathcal{A}^{(q)}, \bar{F})$ is reducible. Hence there are different states q'' and q''' in it such that $L_{q''} = L_{q'''}$. Now $L_{q''} = L(\mathcal{A}^{(q)(q'')}, \bar{F}'') = L(\mathcal{A}^{(q'')}, \bar{F}'')$ and similarly for $L_{q'''}$, what means that q'' and q''' are equivalent in

$(\mathcal{A}, F)$. But q'' and q''' are of rank smaller than q, which contradicts the choice of q and q'. □

6. Calculations and estimates

6.1. Tables. We restrict our calculations mainly to automata with 2 inputs. We used Maple in all computations. Tables 1 and 2 contain data for quasi-acyclic, acyclic and initially connected acyclic automata with labeled states.

In Table 3 we give numerical values for unlabeled initially connected acyclic automata and compare them with known lower and upper bounds. Inequality (1) together with a lower bound for $M_2(n)$ obtained in [DoKS01] give rise to the inequality

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \le C_2(n). \tag{10}$$

These numbers, as well as the ratios $C_2(n)/(2n-1)!!$, are also contained in Table 3.

$C_2(n)$ are compared with the numbers Genocchi(n) which count, by [Do01], initially connected acyclic automata in which states are properly ordered. Accordingly, the last column of Table 3 represents

the average number of numberings (orderings) compatible with the transition functions in initially connected acyclic automata.

Table 4 contains intermediate data for quasi-acyclic automata with a distinguished pre-dead state (formula (3′)). Numerical data for $C_2^{(1)}(n)$ and $C_3^{(1)}(n)$, and their ratios with $C_2(n)$ and $C_3(n)$ are contained in Table 5.

The upper bounds by inequalities (1) and (2) for the number of minimal automata are provided in Table 6; these data are compared with the exact values and bounds published in [DoKS01, Do02].

6.2. "Cyclic" automata. For comparison and completeness, we also calculate all initially connected automata, not necessarily acyclic. $h_k(n)$ denotes the number of such labeled automata with n states and k inputs. Then $h_k(1) = 1$ and by [Li69],

$$h_k(n) = n^{kn} - \sum_{t=1}^{n-1} \binom{n-1}{t-1} n^{k(n-t)} h_k(t). \tag{11}$$

Now $H_k(n) = h_k(n)/(n-1)!$ is the number of unlabeled initially connected automata. Numerical data for them with $k = 2, 3$ are given in Table 7.

As follows easily from formula (11), $H_k(n)$ is divisible by n^k, see [Li69]. Remark, incidentally, that some similar observations are valid for $C_k^{(1)}(n)$; in particular, (2^k-1) divides $C_k^{(1)}(n)$.

7. Further discussion

7.1. Possible generalizations. Instead of completely defined automata we could consider partial deterministic automata, that is automata for which the transition function is defined not necessarily for all pairs (q, x). In this case we could exclude the dead state and consider genuine acyclic automata. This class does not introduce anything substantially new, since we can transform it bijectively into the class of completely defined automata considered above by adding a new dead state and all

TABLE 1. The number of labeled quasi-acyclic automata $a_2(n, r)$ with n transient and r dead states

n/r	1	2	3	4	5
0	1	1	1	1	1
1	1	4	9	16	25
2	7	56	207	544	1175
3	142	1780	9342	32848	91150
4	5941	103392	709893	3142528	10682325
5	428856	9649124	82305144	440535696	1775027000
6	47885899	1329514816	13598786979	85529171200	398824865275
7	7685040448	254821480596	3046304952000	22041805076944	116816612731200

$$\begin{aligned}
a_2(0,r) &= 1 \\
a_2(1,r) &= r^2 \\
a_2(2,r) &= 2r^2 + 4r^3 + r^4 \\
a_2(3,r) &= 21r^2 + 60r^3 + 48r^4 + 12r^5 + r^6 \\
a_2(4,r) &= 568r^2 + 1920r^3 + 2160r^4 + 1040r^5 + 228r^6 + 24r^7 + r^8 \\
a_2(5,r) &= 29705r^2 + 111400r^3 + 150400r^4 + 97160r^5 + 33190r^6 + 6280r^7 \\
&\quad +680r^8 + 40r^9 + r^{10} \\
a_2(6,r) &= 2573136r^2 + 10379520r^3 + 15778080r^4 + 12160800r^5 + 5330520r^6 \\
&\quad +1406592r^7 + 231360r^8 + 24240r^9 + 1590r^{10} + 60r^{11} + r^{12}
\end{aligned}$$

TABLE 2. The number of labeled acyclic and initially connected acyclic automata

n	$a_2(n)$	$c_2(n)$	$a_2(n)/c_2(n)$
1	1	1	1.000
2	7	3	2.333
3	142	32	4.438
4	5941	762	7.797
5	428856	32712	13.110
6	47885899	2235360	21.422
7	7685040448	224100000	34.293
8	1681740027657	31115906640	54.048
9	482368131521920	5733129144960	84.137
10	175856855224091311	1356239286057600	129.665
11	79512800815739448576	401263604225164800	198.156
12	43701970591391787395197	145349590736723788800	300.668
13	28714779850695689959247872	63331019483788869120000	453.408
14	22239820866807621347245261875	32702367239716877602099200	680.068
15	20060586399267989706814051311616	9760224335684945097034649600	1015.200

undefined transitions as leading to it. If necessary, we could enumerate partial acyclic automata specified additionally by the number of transitions between states (or, equivalently, complete acyclic automata specified by the number of transitions to the dead states).

There is a less trivial generalization of automata under consideration which often appears in the literature; the class of *multi-initial* automata, that is deterministic automata with a distinguished set of initial states. By a slight modification of the proofs given in Section 3, the formulae for *labeled* initial acyclic automata can be generalized to multi-initial as well as to *multi-initially connected* automata (automata

TABLE 3. The number of unlabeled initially connected acyclic automata $C_2(n)$

	I	II	III	II/I	III/II
n	(2n-1)!!	$C_2(n) = \frac{c_2(n)}{(n-1)!}$	Genocchi(n)		
1	1	1	1	1.000	1.000
2	3	3	3	1.000	1.000
3	15	16	17	1.067	1.063
4	105	127	155	1.210	1.220
5	945	1363	2073	1.442	1.521
6	10395	18628	38227	1.792	2.052
7	135135	311250	929569	2.303	2.987
8	2027025	6173791	28820619	3.046	4.668
9	34459425	142190703	1109652905	4.126	7.804
10	654729075	3737431895	51943281731	5.708	13.898
11	13749310575	110577492346	2905151042481	8.042	26.273
12	316234143225	3641313700916	191329672483963	11.515	52.544
13	7905853580625	132214630355700	14655626154768697	16.724	110.847
14	213458046676875	5251687490704524	1291885088448017715	24.603	245.994
15	6190283353629375	226664506308709858	129848163681107301953	36.616	572.865

TABLE 4. The number of labeled quasi-acyclic automata $b_2(n,r)$ with a distinguished pre-dead state, $n+1$ transient and r dead states

n/r	1	2	3	4	5
0	1	1	1	1	1
1	3	8	15	24	35
2	39	176	495	1104	2135
3	1206	7784	29430	84600	204470
4	69189	585408	2791125	9841728	28569765
5	6416568	67481928	389244600	1627740504	5518006200
6	881032059	11111547520	75325337235	364616173440	1413735254155
7	168514815360	2483829653544	19371055651200	106576788695352	465181963908480

in which every state is reachable from an initial state). Note however that multi-initially connected automata can have non-trivial automorphisms (preserving the property of states to be initial); so that the enumeration of such unlabeled automata is an additional non-trivial problem.

One more useful generalization concerns non-deterministic automata; it is possible to apply the enumerative technique of Section 3 to non-deterministic automata with labeled states (cf. also [Ge96]).

7.2. Asymptotics. Asymptotics of $a_k(n), C_k(n)$ and $C_k^{(1)}(n)$ remain open problems. As Table 1 suggests (and as is typical for deterministic automata), only a small fraction of acyclic automata are initially connected. The data in two last columns of Table 3 suggest that $C_2(n)$ is closer to the lower bound. Note that the Genocchi numbers grow much faster than $(2n-1)!! = \dfrac{(2n)!}{n!2^n}$: asymptotically as $n \to \infty$, $\text{Genocchi}(n) \sim \dfrac{4(2n)!}{\pi^{2n}}$.

TABLE 5. The number of unlabeled initially connected acyclic automata with a unique pre-dead state $C_k^{(1)}(n),\ k = 2, 3$

	I	II	I/II	III	IV	III/IV
n	$C_2^{(1)}(n)$	$C_2(n)$		$C_3^{(1)}(n)$	$C_3(n)$	
1	1	1	1.000000	1	1	1.000000
2	3	3	1.000000	7	7	1.000000
3	15	16	0.937500	133	139	0.956835
4	114	127	0.897638	5362	5711	0.938890
5	1191	1363	0.873808	380093	408354	0.930793
6	15993	18628	0.858546	42258384	45605881	0.926599
7	263976	311250	0.848116	6830081860	7390305396	0.924195
8	5189778	6173791	0.840614	1520132414241	1647470410551	0.922707
9	118729335	142190703	0.835001	447309239576913	485292763088275	0.921731
10	3104549229	3737431895	0.830664	...	...	0.921060
20			0.813154			0.919137
40			0.805872			0.918746
60			0.803707			0.918682
80			0.802679			0.918661
100			0.802082			0.918652
150			0.801310			
200			0.800935			
250			0.800715			

TABLE 6. Upper bounds for the number of minimal acceptors $M_2(n)$

	I	II	III	IV	II/I	III/I	IV/I
n	$M_2(n)$: [DoKS01]	$2^{n-1}C_2^{(1)}(n)$: formula (2)	$2^{n-1}C_2(n)$: formula (1)	Up.Bound: [Do02]			
1	1	1	1	1	1.000	1.000	1.000
2	6	6	6	6	1.000	1.000	1.000
3	60	60	64	64	1.000	1.067	1.067
4	900	912	1016	1120	1.013	1.129	1.244
5	18480	19056	21808	26432	1.031	1.180	1.430
6	487560	511776	596096	889216	1.050	1.223	1.824
7		16894464	19920000				
8		664291584	790245248				
9		30394709760	36400819968				

For arbitrary k, more generally (cf. (10)),

$$\prod_{i=1}^{n} \left(i^k - (i-1)^k\right) \leq C_k(n), \tag{12}$$

which follows easily from the enumeration of chain-like initially connected acyclic automata. These are $(n+1)$-state automata of diameter $n-1$. It follows that all of them are reduced since there is only one state of each possible rank, whereas similar states of an initially connected acyclic automaton are necessarily of the same rank.

TABLE 7. The number of unlabeled initially connected automata $H_k(n), k = 2, 3$

n	$H_2(n) = h_2(n)/(n-1)!$	$H_2(n)/n^2$	$H_3(n) = h_3(n)/(n-1)!$
1	1	1	1
2	12	3	56
3	216	24	7965
4	5248	328	2128064
5	160675	6427	914929500
6	5931540	164765	576689214816
7	256182290	5228210	500750172337212
8	12665445248	197897582	572879126392178688
9	705068085303	8704544263	835007874759393878655
10	43631250229700	436312502297	1510492370204314777345000
11	2970581345516818	24550259053858	3320470273536658970739763334
12	220642839342906336	1532241939881294	8718034433102107344888781813632

Hence by Proposition 5.4,

$$2^{n-1} \prod_{i=1}^{n} \left(i^k - (i-1)^k \right) \leq M_k(n), \tag{13}$$

a result of [Do02].

We assume that a significant (i.e. not tending to 0 as n tends to infinity) fraction of initially connected acyclic automata with a unique pre-dead state are primitive. This fraction increases with k but, presumably, it does not tend to 1 as n tends to infinity taking into account the arguments given in Subsect. 5.2: there is a significant fraction of initially connected acyclic automata with 3 or more pre-pre-dead states, and in a significant fraction of them at least 2 such states are similar. Thus these automata give rise to no acceptors (note, incidentally, that pre-dead states are similar; so that an initially connected acyclic automaton with several pre-dead states is not primitive). If this assumption is valid, by Proposition 5.4 we get the following hypothetical relationship (cf. (2)): $M_k(n) = \Theta(2^{n-1} C_k^{(1)}(n)), n \to \infty$. Moreover, we assume the validity of the following asymptotic formula

$$M_k(n) \sim \gamma_k \, 2^{n-1} C_k^{(1)}(n), \quad n \to \infty, \tag{14}$$

where γ_k is a constant depending on k, $0 < \gamma_k < 1$ for $k > 1$, and $\gamma_k \to 1$ as $k \to \infty$.

The similarity of formulae (3′), (5′) and (7′) to, respectively, (3), (5) and (7) suggests that the numbers $C_k^{(1)}(n)$ should be close to $C_k(n)$ for large n. As extensive calculations show, this is apparently the case; moreover, the fraction of automata with a unique pre-dead state among all initially connected automata decreases monotonically and tends to a positive limit as n grows. So we conjecture that

$$C_k^{(1)}(n) \sim \beta_k \, C_k(n), \quad n \to \infty, \quad 0 < \beta_k < 1, \quad k > 1. \tag{15}$$

From our calculations we conclude that if (15) is valid, then $\beta_2 \approx 0.800$, $\beta_3 \approx 0.918$, $\beta_4 \approx 0.963$, $\beta_5 \approx 0.982$ and $\beta_6 \approx 0.991$. The corresponding data for $k = 2$ and 3 are represented in Table 5; for $k = 2$ see also Figure 1.

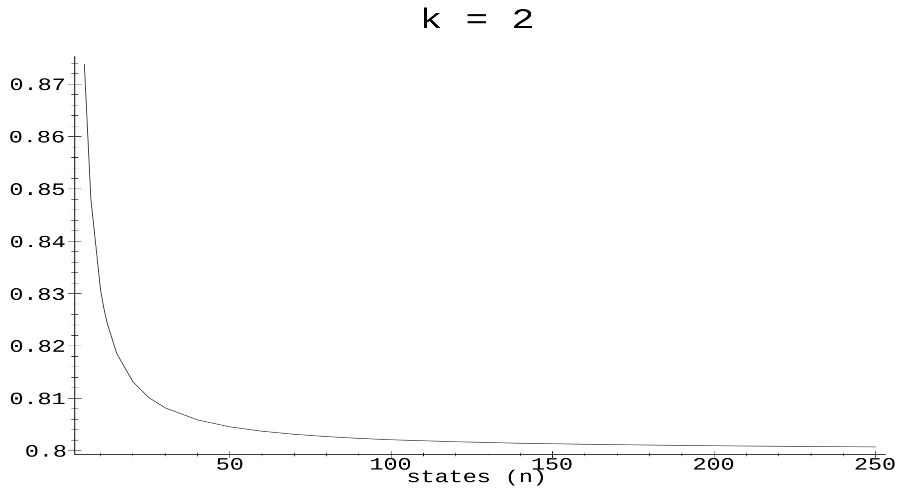

FIGURE 1. Fraction of initially connected automata with a unique pre-dead state

If both conjectures are valid, then

$$M_k(n) \sim \beta_k \, \gamma_k \, 2^{n-1} C_k(n), \quad n \to \infty. \tag{16}$$

There are other intriguing questions, in particular the distribution of the diameter and the number of pre-dead states in acyclic and initially connected acyclic automata. For comparison, according to McKay [McK89], the diameter of a random acyclic (labeled) digraph has an asymptotically normal distribution with mean $\mu \, n$, where $\mu \approx 0.764$. In a random acyclic digraph, the mean number of sinks tends to $1.488\ldots$ and the mean number of pre-sink nodes tends to $1.326\ldots$ (see [Li75, Li77a]). Almost all acyclic digraphs are connected [BeR88].

7.3. Splittable kernels. We return to the general linear recurrence relation of form (6). Its kernel $\alpha(n,t)$ is said to be *splittable* if it can be represented as the product of single-variable functions of n, t and $n-t$:

$$\alpha(n,t) = f(n)g(t)h(n-t) \tag{17}$$

for all non-negative n, t and $n-t$ (we might consider $\binom{n}{t}$ as a part of the kernel as well, and this factor is clearly splittable). If (17) is valid, then (6) turns into the convolution

$$\sum_{t=1}^{n} \frac{h(n-t)}{(n-t)!} \frac{c(t)g(t)}{t!} = \frac{a(n)}{f(n)n!}, \tag{18}$$

which can be easily represented in terms of appropriate generating functions. Such formulae facilitate extracting asymptotics (see, e.g., [Ro73, Li75] for the case of acyclic digraphs).

The kernels of recurrence relations for (initially connected) automata are typically unsplittable (unlike the case of general (di)graphs). There is a simple necessary condition:

7.4. Lemma [Li77]. *If $\alpha(n,t)$ is splittable, then there exist numbers U and V, not both equal to 0, such that for all $n > 2$,*

$$\begin{vmatrix} \alpha(n,n-1) & U\alpha(n-1,n-1) \\ \alpha(n,n-2) & V\alpha(n-1,n-2) \end{vmatrix} = 0. \tag{19}$$

By (19) it is easy to see that the kernel of formula (3) is unsplittable.

7.5. Asymptotics of general initially connected automata. The kernel of formula (11) for all initially connected automata is also unsplittable, and this simple recurrent formula is not very suitable for obtaining asymptotics (numerical experiments show, however, that it is not so bad for *approximate* calculations, contrary to what we expected formerly). For fixed $k > 1$ we managed only to extract the asymptotics $h_k(n) = y_k^{-n} n^{kn+O(\sqrt{n \log n})}$, where

$$y_k = z_k e^k (1 - z_k)^{k-1} \tag{20}$$

and z_k is the real root of the equation

$$ze^{k(1-z)} = 1 \tag{21}$$

different from 1 (thus, $y_2 \approx 1.196$); see [Li69b]. Later on, Korshunov [Ko78] developed a strong technique which enabled him to prove that

$$h_k(n) \sim \nu_k y_k^{-n} n^{kn+1}, \quad n \to \infty, \tag{22}$$

(where ν_k is a complicated constant) and which has nothing to do with the exact enumeration. Hopefully his technique can be modified so as to cover the case of acyclic automata.

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References

[BeR88] E. A. Bender and R. W. Robinson, The asymptotic number of acyclic digraphs. II, *J. Combin. Th.* Ser. **B**, **44**:3 (1988), 363–369. MR90a:05098.

[CaP02] C. Campeanu and A. Paun, The number of similarity relations and the number of minimal deterministic finite cover automata. In: Proc. 7th Int. Conf. *Implementation and Application of Automata*, July 2002, University of Tours (to appear).

[DaMWW00] J. Daciuk, S. Mihov, B. W. Watson and R. E. Watson, Incremental construction of minimal acyclic finite-state automata, *Comput. Linguistics*, **26**:1 (2000), 3–16. MR2001d:68123.

[Do01] M. Domaratzki, A generalization of the Genocchi numbers with applications to enumeration of finite automata. Submitted (2001). Preliminary version available at `http://www.cs.queensu.ca/TechReports/Reports/2001-449.ps`.

[Do02] M. Domaratzki, Improved bounds on the number of automata accepting finite languages. In: Proc. 6th Int. Conf. *Developments in Language Theory*, Sept. 2002, Kyoto (to appear).

[DoKS01] M. Domaratzki, D. Kisman and J. Shallit, On the number of distinct languages accepted by finite automata with n states. In: Proc. 3rd Int. Workshop *Descriptional Complexity of Automata, Grammars and Related Structures* (2001), J. Dassow and D. Wotschke eds, 2001, 66–78.

[Ge96] I. M. Gessel, Counting acyclic digraphs by sources and sinks, *Discr. Math.*, **160** (1996), 253–258. MR97i:05058.

[Go72] H. W. Gould, *Combinatorial Identities. A standardized set of tables listing 500 binomial coefficient summations.* Morgantown, W.Va., 1972. MR50#6879.

[HoU79] J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, Reading, Mass., 1979. MR83j:68002.

[Ko78] A. D. Korshunov, Enumeration of finite automata, *Problemy Kibernetiki*, **34** (1978), 5–82 (in Russian). MR80a:68050.

[Li69] V. A. Liskovets, The number of initially connected automata, *Kibernetika* (Kiev), No 3 (1969), 16–19 (in Russian; Engl. transl.: *Cybernetics*, **4** (1969), 259–262). MR46#1509.

[Li69a] V. A. Liskovets, On a recurrent method for the enumeration of graphs with labelled vertices, *Dokl. AN SSSR*, **184**:6 (1969), 1284–1287 (in Russian; Engl. transl.: *Soviet Math. Dokl.*, **10**:1 (1969), 242–246). MR39#5426.

[Li69b] V. A. Liskovets, *Some Enumerative Problems of Graph Theory.* Ph.D. Thesis, Institute of Mathematics, Academy of Sciences of BSSR, Minsk, 1969, 160 p. (in Russian).

[Li75] V. A. Liskovets, On the number of maximal vertices of a random acyclic digraph, *Th. Probab. Appl.*, **20**:2 (1975), 401–409 (Engl. transl.). MR52#1822.

[Li77] V. A. Liskovets, On a general enumeration scheme for labelled graphs, *Dokl. AN BSSR*, **21**:6 (1977), 496–499 (in Russian). MR58#21797.

[Li77a] V. A. Liskovets, Unsolved problem 17. In: *Comb. and Asympt. Analysis* (coll. papers), No 2 (1977), 176, Krasn. State Univ., Krasnoyarsk (in Russian).

[Ma00] D. Maurel, Pseudo-minimal transducer, *Th. Comput. Sci.*, **231**:1 (2000), 129–139. MR2000j:68107.

[McK89] B. D. McKay, On the shape of a random acyclic digraph, *Math. Proc. Cambridge Philos. Soc.*, **106**:3 (1989), 459–465. MR90f:05067.

[Mi99] S. Mihov, Direct construction of minimal acyclic finite states automata, *Annuaire Univ. Sofia, Fac. Math. Inform.*, **92**:1-2 (1998), 87–104. MR2001j:68072.

[Re92] D. Revuz, Minimisation of acyclic deterministic automata in linear time, *Th. Comput. Sci.*, **92**:1 (1992), 181–189. MR92m:68046.

[Ri62] J. Riordan, Enumeration of linear graphs for mappings of finite sets, *Ann. Math. Statist.*, **33** (1962), 178–185. MR24#A3084.

[Ro73] R. W. Robinson, Counting labeled acyclic digraphs. In: Proc. Third Ann Arbor Conf. *New directions in the theory of graphs* (Univ. Michigan, Ann Arbor, Mich., 1971), Academic Press, New York, 1973, 239–273. MR51#249.

[St99] R. P. Stanley, *Enumerative Combinatorics*, v. 2. Cambridge Univ. Press, Cambridge, 1999. MR2000k:05026.

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