MEANDER ALGEBRAS

by Roland BACHER

1. Introduction and main results

The aim of this paper is to introduce and study some algebras related to meanders. The reader can for instance consult the paper [**LZ**] for a brief account of the (brief) history of meanders. The paper [**LZ1**] contains further motivations coming from physics.

DEFINITION 1.1. — A *system of meandercurves* is a finite disjoint union of differentiable Jordan curves (curves without self-intersections enclosing topological discs) in the complex plane **C**. We require moreover that each of the above Jordan curves intersects **R** non-trivially and transversally (see Figure 1.1 for an example with 3 components).

Two systems of meandercurves σ and $\sigma' \subset \mathbf{C}$ are *equivalent* if there exists an orientation preserving diffeomorphism φ of **C** which restricts to an order preserving diffeomorphism of **R** and satisfies $\varphi(\sigma) = \sigma'$.

An *n-meander with k component* is an equivalence class of a system of meandercurves having k components and intersecting the real line in exactly $2n$ points.

A *connected* meander is a meander consisting of only one connected component (Jordan curve).

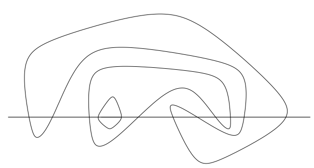

Figure 1.1. A system of meandercurves representing a 6-meander with 3 components.

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For the sake of concision we will henceforth identify systems of meanders with the corresponding meanders.

We would also like to warn the reader that most authors use the word meander only for connected meanders.

It is clear that the number $m_{n,k}$ of n-meanders having k components is always finite. The following table displays the first values for the numbers $m_{n,1}$ of connected n-meanders.

TABLE 1.2. — *The first numbers $m_{1,1}, m_{2,1}, \ldots$ of connected n-meanders are given by*

$$\begin{aligned}
m_{1,1} &= 1\,,\ 2\,,\ 8\,,\ 42\,,\ 262\,,\ 1828\,,\ 13820\,,\ 110954\,,\ 933458\,,\\
m_{10,1} &= 8152860\,,\ 73424650\,,\ 678390116\,,\ 6405031050\,,\ 61606881612\,,\\
m_{15,1} &= 602188541928\,,\ 5969806669034\,,\ 59923200729046\,,\\
m_{18,1} &= 608188709574124
\end{aligned}$$

The computation of the values $m_{n,k}$ needs about the same amount of work.

Tables up to $n = 14$ of these numbers have also appeared in [**FGG1**] and in [**LZ**].

DEFINITION 1.3. — An increasing positive sequence $t_1, t_2, \ldots$ has *exponential growth* if $\liminf(t_n)^{1/n} > 1$. The limit τ (if it exists) of the sequence $\left((t_n)^{1/n}\right)_{n\in\mathbf{N}}$ is then called the *exponential growth rate* of the sequence $t_1, t_2, \ldots$

In section 2 we will prove the following result.

THEOREM 1.4. — *There exists a constant λ independent of k such that all the sequences $\left(m_{n,k}\right)_{n\in\mathbf{N}}$ have exponential growth of rate λ (i.e. $\lim_{n\to\infty}(m_{n,k})^{1/n} = \lambda$).*

It has been conjectured (see for instance [**FGG1**]) that one has

$$m_{n,1} \sim \text{constante}\ \frac{(7/2)^{2n}}{n^{7/2}}$$

which would imply $\lambda = 12.25$.

Denote by $\mathbf{H}^+ = \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}$ and by $\mathbf{H}^- = \{z \in \mathbf{C} \mid \operatorname{Im}(z) < 0\}$ the upper and lower open halfplanes of $\mathbf{C}$.

DEFINITION 1.5. — Given a meander μ, call a connected component of $\mu \cap \mathbf{H}^+$ an *upper arch* and a connected component of $\mu \cap \mathbf{H}^-$ a *lower arch*. An upper arch without any arch above it (ie. contained in the unbounded component of $\mathbf{H}_+ \setminus \mu$) or a lower arch without arches below it is *exterior*.

A meander μ is *tame* if every connected component of μ contains at least one exterior arch.

Every connected meander is tame. Figure 1.1 shows a meander which is not tame. It has 1 exterior upper arch and 3 exterior lower arches.

A n-meander has always exactly n upper and n lower arches. The number a of exterior upper arches and the number b of exterior lower arches are generally smaller. For $n > 1$ any pair of integers $(a, b) \neq (1, 1)$ with $1 \leq a, b \leq n$ can occur.

Denote by m_n^t the number of all (non-isomorphic) tame n-meanders.

TABLE 1.6. — *The first terms of the sequence $m_1^t, m_2^t, \ldots$ are*

$m_1^t = 1\ ,\ 3\ ,\ 15\ ,\ 93\ ,\ 657\ ,5063\ ,\ 41535\ ,\ 357205\ ,\ 3187599\ ,$
$m_{10}^t = 29303687\ ,\ 276062807\ ,\ 2654603987\ .$

THEOREM 1.7. — *For $n \geq 1$ the numbers m_n^t of tame n-meanders satisfy the inequalities*

$$(m_{n,1})^2 \leq (m_n^t)^2 \leq \frac{n(n+1)}{2} m_{2n-1,1} \quad .$$

In particular, the sequence m_n^t has also exponential growth of rate λ.

In order to gain more information on λ, we introduce in section 3 the graded algebra $\mathcal{MS}$ of meander-slices. As an abstract algebra $\mathcal{MS}$ is not very interesting since it is the free algebra on four generators of degree 1. It contains however an interesting subalgebra $\mathcal{S}$ and an interesting quotient algebra $\mathcal{Q}$ which have both a natural grading. The subalgebra $\mathcal{S} \subset \mathcal{M}$ injects into $\mathcal{Q}$ (respecting the grading) showing that

$$s_n = \dim(\mathcal{S}_n) \leq q_n = \dim(\mathcal{Q}_n)$$

(where $\mathcal{A}_n$ denotes the finite-dimensional vector space spanned by all homogenous elements of degree n in a graded algebra $\mathcal{A}$).

TABLE 1.8. — *The first terms of the sequence $s_0, s_1, s_2, \ldots$ are*

$s_0 = 1\ ,\ s_1 = 1\ ,\ 3\ ,\ 7\ ,\ 23\ ,\ 63\ ,\ 213\ ,\ 627\ ,\ 2149\ ,\ 6597\ ,$
$s_{10} = 22787\ ,\ 71883\ ,\ 249523\ ,\ 802291\ ,\ 2794365\ ,\ 9111917\ ,$
$s_{16} = 31814061\ ,\ 104862813\ ,\ 366796437\ ,\ 1219313185 \quad .$

TABLE 1.9. — *The first terms of the sequence $q_0, q_1, q_2, \ldots$ are*

$q_0 = 1\ ,\ 4\ ,\ 15\ ,\ 56\ ,\ 207\ ,\ 764\ ,\ 2805\ ,\ 10288\ ,\ 37609\ ,\ 137380\ ,$
$q_{10} = 500655\ ,\ 1823440\ ,\ 6629423\ ,\ 24090332\ ,\ 87418221\ ,\ 317085352\ ,$
$q_{16} = 1148825185\ ,\ 4160744164\ ,\ 15054719697\ ,\ 54454345624\ ,$
$q_{20} = 196805925995\ ,\ 711077858188\ ,\ 2567375653681\ ,\ 9267176552040\ ,$
$q_{24} = 33430012251123\ ,\ 120565130387572\ ,\ 434578910451203$

The numbers s_{2n} and q_{2n} are closely related to tame n-meanders.

The study of the algebras $\mathcal{S}$ and $\mathcal{Q}$ is motivated by the following result.

THEOREM 1.10. — *The sequences $(s_n)^{1/n}$ and $(q_n)^{1/n}$ have a common limit $\gamma \geq \sqrt{\lambda}$ where λ is the growth rate of connected meanders. Moreover, $(s_n)^{1/n} \leq \gamma \leq (q_n)^{1/n}$ for all n.*

The above tables suggest the asymptotics

$$s_n \sim \text{constante } n^{-1}(7/2)^n \quad \text{and} \quad q_n \sim \text{constante } n\,(7/2)^n$$

(the sequence s_n is asymptotically of the form $A\ n^\alpha\ \gamma^n$ if and only if q_n is asymptotically of the form $B\ n^{\alpha+2}\ \gamma^n$). These data seem to indicate an equality between γ^2 and λ. I was however unable to prove (or disprove) this.

DEFINITION 1.11. — Given a meander μ, its *interior* $I(\mu)$ is the open subset $I(\mu) = \{z \in \mathbf{C} \mid \mathrm{Ind}_\mu(z) \equiv 1 \pmod 2\} \subset \mathbf{C} \setminus \mu$ consisting of all connected components of $\mathbf{C} \setminus \mu$ which are enclosed by an odd number of Jordan curves in μ. The interior $I(\mu)$ is hence homeomorphic to a finite disjoint union of open discs with a finite number of holes in them (some discs may be sitting in holes of bigger discs).

The *zeroth Betti number* b_0 of μ is the total number of connected components (discs having perhaps holes) in $I(\mu)$ and the *first Betti number* is the total number of holes in these discs. (The Betti numbers of μ are the Betti numbers of the open submanifold $I(\mu) \subset \mathbf{C}$.)

A meander is *a forest-meander* if its first Betti number b_1 is zero. (The terminology originates in the following observation: Every meander is the boundary of an ϵ-neighbourhood of an essentially unique planar bipartite graph $\subset \mathbf{C}$. Vertices of this graph are the connected components of $I(\mu) \cap \mathbf{H}_\pm$ and edges are the connected components of $I(\mu) \cap \mathbf{R}$. Forest-meanders correspond then to graphs which are forests.)

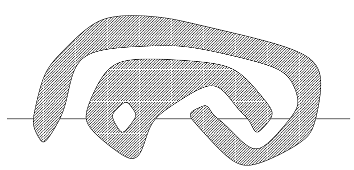

Figure 1.2. The interior of the meander in Figure 1.1 (with $b_0 = 2$ and $b_1 = 1$).

TABLE 1.12. — *The first terms of the sequence m_n^f of n-forest meanders are given by*

$m_1^f = 1\ ,\ 3\ ,\ 15\ ,\ 97\ ,\ 733\ ,\ 6147\ ,\ 55541\ ,\ 530773\ ,\ 5298723\ ,$
$m_{10}^f = 54780831\ ,\ 582817337\ ,\ 6350647873\ ,\ 70614662303\ ,\ 798935833885$

THEOREM 1.13. — *The sequence m_n^f counting n-forest meanders has exponential growth of rate $\lambda^f \geq \lambda$.*

The proof of Theorem 1.13 is analogous to the proof of Theorem 1.4 and is left to the reader. (The inequality $\lambda^f \geq \lambda$ is simply the observation that every tame meander is also a forest meander.)

2. Proofs of Theorems 1.4 and 1.7

LEMMA 2.1.

(i) One has for all $a, b > 1$

$$m_{a+b,1} \geq \frac{9}{2}\, m_{a,1}\, m_{b,1} \quad .$$

(ii) One has for all $n > 1$

$$m_{n+1,1} \geq 3\, m_{n,1} \quad .$$

LEMMA 2.2. — *One has for all $n, k \geq 1$*

$$m_{n-k+1,1} \leq m_{n,k} \leq \binom{n+k-1}{k-1} m_{n+k-1,1} \quad .$$

Proof of Theorem 1.4. — Let us first consider the case $k = 1$. In this case, assertion (i) of Lemma 2.1. shows by standard arguments that the sequence $\left(m_{n,1}\right)^{1/n}$ admits a limit $\lambda \in \mathbf{R}_{\geq 0} \cup \infty$ and assertion (ii) implies that $\lambda \geq 3$.

Let us show that $\lambda \leq 16$: At each intersection of an n-meander μ with the real line, we get an upper and a lower strand which bend either to the left or to the right (here bend to the left means that a traveller following this strand returns first to the real line at the left from its starting point).

If both strands bend to the left, write the letter) ,

If both strands bend to the right, write the letter (,

If the lower strand bends to the left and the upper strand bends to the right write the letter / ,

If the lower strand bends to the right and the upper strand bends to the left write the letter \ .

Reading the symbols encoding the type of the $2n$ strands of μ from left to right, we get a word of length $2n$ in the alphabet $\{\ )\ ,\ (\ ,\ /\ ,\ \backslash\ \}$ which encodes the meander μ uniquely. For instance, the meander of Figure 1.1 is encoded by the word

$$(/ (( ) / (\backslash \backslash))) \quad .$$

This shows that there are at most 16^n different n-meanders (and even less connected n-meanders) and hence the growth-rate λ of connected meanders cannot exceed 16.

Lemma 2.2 implies now easily that the same constant λ works for all values of k. ∎

Remarks 2.3.

(i) The number 16^n is of course an upper bound for the number of all n-meanders. This last number can be exactly computed by remarking that the upper and lower part of an n-meander both define *n-arch systems* or *parenthesis systems consisting of n parentheses.* This shows that the number $\sum_k m_{n,k}$ of all n-meanders equals the square C_n^2 of the n-th Catalan number $C_n = \binom{2n}{n}\frac{1}{n+1}$ which counts the number of distinct parenthesis systems consisting of n parentheses (cf. for instance [**LZ**] or description (o) in Exercice 19 on Catalan numbers in [**S**]). This refinement leads however to the the same (bad) upper bound $\lambda \leq 16$ for the growth-rate λ of connected meanders.

(ii) Lemma 2.2 (together with the fact that $m_{n,1}$ is submultiplicative thus implying $m_{n,1} \leq \lambda^n$ for all n) can be combined with Remark (i) above in order to get a lower bound on λ:

$$C_n^2 = \sum_k m_{n,k} \leq \sum_{k=1}^{n} \binom{n+k-1}{k-1} \lambda^{n+k-1} .$$

The biggest real root of the polynomial equation $\leq \sum_{k=1}^{n} \binom{n+k-1}{k-1} x^{n+k-1} = \left(\binom{2n}{n}\frac{1}{n+1}\right)^2$ yields hence for any natural integer n a lower bound for λ. Unvortunately the inequalities of Lemma 2.2 are generally very far from sharp and the obtained lower bound (about 1.89 with $n = 100$) is not interesting.

(iii) The inequality $m_{n+1,1} \geq 3m_{n,1}$ can be sharpened to $m_{n+1,1} \geq 6m_{n,1}$ by elementary considerations involving the leftmost interior arch of connected meanders.

Proof of Lemma 2.1. — Let μ be a connected a-meander and let ν be a connected b-meander. Draw the meanders μ and ν in the complex plane $\mathbf{C}$ such that $\mu \subset \{z \in \mathbf{C} \mid \mathrm{Re}(z) < 0\}$ and $\nu \subset \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Choose now an exterior upper arch A in

μ and an exterior upper arch B in ν. Cutting them open in the midpoint of A and B and gluing them together as suggested by figure 2.1 yields a connected $(a+b)$-meander.

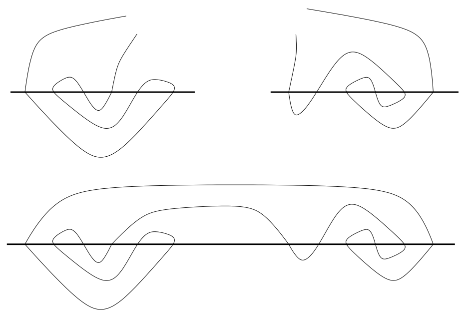

Figure 2.1. Two connected 3-meanders μ, ν cutted open in exterior upper arches and the associated connected 6-meander.

The same construction works of course also using exterior lower arches. Moreover, two connected $(a+b)$-meanders ρ, ρ' constructed from connected a-meanders μ, μ' and b-meanders ν, ν' are equivalent if and only if $\mu = \mu'$, $\nu = \nu'$ and using the same choice of exterior arches.

One has

$$\sharp(\{\text{exterior upper arches of } \mu\} \cup \{\text{exterior lower arches of } \mu\}) \geq 3$$

if μ is an a-meander with $a > 1$. This shows that there are at least $4 = 1 \cdot 2 + 2 \cdot 1$ different possibilities for the above construction applied to two connected meanders μ and ν. Moreover, if there are only 4 possiblities for μ and ν then there are $5 = 1 \cdot 1 + 2 \cdot 2$ possiblities with μ and $\overline{\nu}$ where $\overline{\nu}$ denotes the b-meander obtained by reflecting ν with respect to the real line. This proves the inequality

$$m_{a+b,1} \geq m_{a,1}\,(4+5)\,\frac{m_{b,1}}{2}$$

thus proving assertion (i) of the lemma. The proof of assertion (ii) is similar and left to the reader. ∎

Proof of Lemma 2.2. — The first inequality is trivial: Any connected $(n-k+1)$-meander can be embedded into an n-meander with k components by adding $(k-1)$ different 1-meanders.

In order to prove the second inequality take an n-meander μ_k having k components. Suppose that μ intersects the real line $\mathbf{R} \subset \mathbf{C}$ in the points $p_1, \dots, p_{2n}$. Let $i \in \{1, 2, \dots, 2n\}$ be the smallest integer such that the two points p_i and p_{i+1} belong to two different components of μ_k. Transform the n-meander μ_k into an $(n+1)$-meander μ_{k-1} according to figure 2.2 (the meander μ_k is only partially drawn; the ommitted parts remain unchanged).

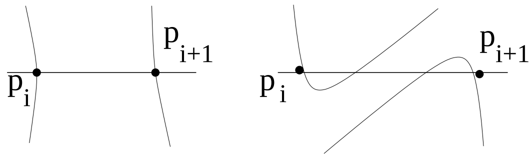

Figure 2.2. A part of μ_k and the corresponding part of the associated meander μ_{k-1}

It is easy to check that the meander μ_{k-1} is an $(n+1)$-meander having $(k-1)$ components. Iterating this construction yields finally a connected $(n+k-1)$-meander μ_1.

In order to recover the initial meander μ_k from μ_1 one needs to remember the $k-1$ places where the construction of figure 2.2 took place. Encoding the meander as int the proof of Theorem 1.4 one remarks that all the involved places are encoded by a four-letter word starting either with \ or with (and there are exactly $(n+k-1)$ such letters in the word encoding μ_1 (the remaining $(n+k-1)$ letters belong to the set $\{ \ /\ ,)\ \}$). This shows that there are at most $\binom{n+k-1}{k-1}$ different n-meanders μ_k with k components giving rise to the same connected $(n+k-1)$-meander μ_1. Hence the second inequality. ∎

Proof of Theorem 1.7. — The tameness of connected meanders implies the first inequality.

Let T now denote the set of tame n-meanders. For each meander μ in T let C_0 denote the connected component of μ containing the leftmost exterior upper and lower arch (which belong obviously to the same connected component). Since μ is tame every connected component contains at least one exterior arch. Choose now in every connected component of μ different from C_0 an exterior arch. This choice fixes a partition of the set T into subsets $T(a,b)$ where $T(a,b)$ consists of all those tame n-meanders in which we have choosen a exterior upper arches and b exterior lower arches.

To each ordered pair μ, μ' of elements in $T(a,b)$ we associate a unique connected $(2n-1)$-meander as follows: Draw μ at the left of μ'. Cut the first (leftmost) intersection of μ and μ' with the real line $\mathbf{R}$ open and separate the two ends by pulling the upper strands up and by lowering the lower strands. Cut also all choosen exterior arches in μ and μ' in their midpoints. Glue all open ends using "rainbows" ie. a configuration of $(2a+1)$ concentric upper halfcircles respectively of $(2b+1)$ concentric lower halfcircles (Figure 2.2 shows this construction: The two boxes represent schematically (after cutting) tame n-meanders μ and μ' of $T(3,2)$. The rainbows are dotted).

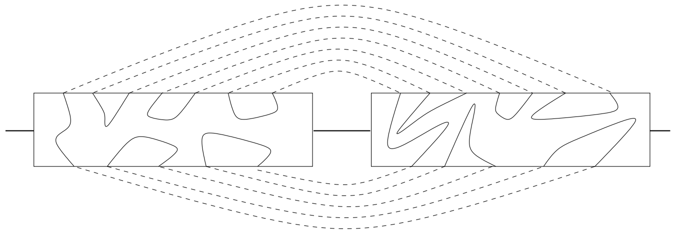

Figure 2.2.

This construction is injectif and the tameness of the meanders μ and μ' implies that the result is a connected meander. We get hence an injective application (this constructions cannot produce equivalent meanders from two pairs $(\mu_1, \mu_1') \in T(a_1, b_1) \times T(a_1, b_1)$ and $(\mu_2, \mu_2') \in T(a_2, b_2) \times T(a_2, b_2)$ if $(a_1, b_1) \neq (a_2, b_2)$) from the set

$$\cup_{a,b} T(a, b) \times T(a, b)$$

into the set of all connected $(2n-1)$-meanders. Setting $t_{a,b} = \sharp(T(a, b))$ we have hence

$$\sum_{a,b} t_{a,b}^2 \leq m_{2n-1,1} \quad .$$

But since $a, b \geq 0$ are integers such that $a + b \leq n - 1$ there are at most $\binom{n+1}{2}$ non-zero terms on the left.

Applying the Cauchy-Schwartz inequality to the $\binom{n+1}{2}$-dimensional vector $(t_{a,b})$, $0 \leq a, b,\ a + b \leq n - 1$ and to the vector $(1, 1, 1, \ldots, 1, 1)$ of the same dimension we get

$$(m_n^t)^2 = \Big(\sum_{a,b} t_{a,b}\Big)^2 \leq \Big(\sum_{a,b} t_{a,b}^2\Big) \binom{n+1}{2}$$

(with equality if and only if all the $\binom{n+1}{2}$ terms of the sums are equal). ∎

3. The algebra $\mathcal{MS}$ of meanderslices

Definition 3.1. — A *meanderslice* is a planar graph Γ contained in $\{z \in \mathbf{C} \mid -1 \leq \mathrm{Re}(z) \leq 1\}$ having the following properties:

(i) The sets $U_- = \{-1 + j\sqrt{-1}\}_{j=1,2,3,\ldots}$, $U_+ = \{1 + j\sqrt{-1}\}_{j=1,2,3,\ldots}$, $L_- = \{-1 - j\sqrt{-1}\}_{j=1,2,3,\ldots}$ and $L_+ = \{1 - j\sqrt{-1}\}_{j=1,2,3,\ldots}$ are the vertices of Γ and all vertices of Γ are of degree exactly one.

(ii) There exist integers $k_+,\ k_-$ and a natural integer N such that for all $j \geq N$ the vertices $-1 + j\sqrt{-1}$ and $1 + (j + k_+)\sqrt{-1}$ are joined by an edge and similarly the vertices

$-1-j\sqrt{-1}$ and $1-(j+k_-)\sqrt{-1}$ are joined by an edge and none of these edges intersects the real intervall $[-1,1]$.

(iii) All edges of Γ are either ordinary edges or closed loops and the set of loops in Γ is finite. All intersections of edges and loops with the real interval $[-1,1]$ are transversal. The number of such intersections is the *degree* of Γ.

(iv) The interior of all edges and loops of Γ is contained in $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\}$.

(v) Finiteness condition: Every loop and every edge not meeting both boundary components intersect the real interval $[-1,1]$ non-trivially.

There is of course an obvious notion of equivalence for meanderslices (orbits of orientation-preserving isotopies of $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\}$ which fix the boundary and preserve the real points) and we will consider only equivalence classes.

A meanderslice of degree n will often be called an n-meanderslice.

Although a meanderslice is defined as an infinite graph, condition (ii) allows a finite description using a finite planar graph as illustrated in Figure 3.1: all omitted edges are above or below the shown edges and join vertices on the two distinct boundary components in the obvious way. Such finite planar pictures are of course obtained by chopping out the slice contained in the strip $\{z \in \mathbf{C} \mid -1 \leq \mathrm{Re}(z) \leq 1\}$ of suitable meanders (hence the terminology).

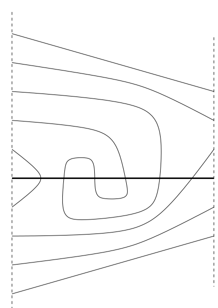

Figure 3.1. Part of a meanderslice of degree 6

The free vector space $\mathcal{MS}$ (over an arbitrary commutative field) on the set of all meanderslices is endowed with an associative product defined as the obvious composition (as for instance for braids in the braid group or for elements in the Temperley-Lieb algebra) given by juxtaposition.

The finiteness condition (v) implies then that $\mathcal{MS}$ is a finitely generated algebra and more precisely we have the following result.

Theorem 3.2. — *The algebra $\mathcal{MS}$ is a graded free algebra with unit on four generators of degree one.*

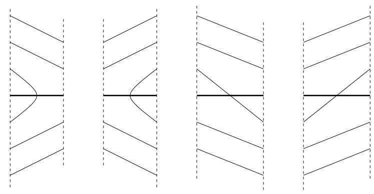

Figure 3.2. The four generators) , (, \ and / of $\mathcal{MS}$

The meanderslice of figure 3.1 represents for instance the element) (\ )) / of $\mathcal{MS}$.

Corollary 3.3. — *The subalgebra $\mathcal{MS}^{\mathrm{e}}$ obtained by considering only meanderslices of even degree is a free algebra with unit on 16 generators (which are all of degree 2).*

We leave the easy proofs of Theorem 3.2 and Corollary 3.3 to the reader.

The subalgebra $\mathcal{MS}^{\mathrm{e}}$ has the following interesting extra feature: Given a meanderslice $\Gamma \in \mathcal{MS}^{\mathrm{e}}$, the connected components of $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\} \setminus \Gamma$ can be oriented in such a way that adjacent components have always different orientations and such that the point $-1+\epsilon$ is contained in a positively oriented component for ϵ small enough. This extra structure is then compatible with the product. One can hence also orient the edges by requiring that their orientations induce the orientation on the positively oriented connected components of $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\} \setminus \Gamma$. We will have a closer look at $\mathcal{MS}^e$ in section 4.

Remarks 3.4.

(i) Dropping the finiteness condition (v) one can still construct an algebra which is however no longer graded (there subsists however a $\mathbf{Z}/2\mathbf{Z}$ grading) and no more finitely generated.

(ii) Allowing Reidemeister II moves of edges with the real line (ie. forgetting the special role of the real open intervall $(-1,1)$) one gets a quotient of $\mathcal{MS}$ which is very usefull for many computations involving meanders. This quotient is a close relative of the Temperley-Lieb algebras (and one can of course use get rid of the loops by introducing a parameter). Most tables of this paper were computed by programming a computer to do computations in this quotient.

Let $\mathcal{R}$ denote the vectorspace spanned by all meanderslices σ which have the property that all edges of σ which cross the real segment $[-1,1]$ are either loops or have both endpoints on the right component $\{z \in \mathbf{C} \mid \mathrm{Re}(z) = 1\}$ of the boundary. Figure 3.3 shows a basis of the homogeneous vectorspace $\mathcal{R}_3 \subset \mathcal{R}$ spanned by all elements of degree 3 in $\mathcal{R}$.

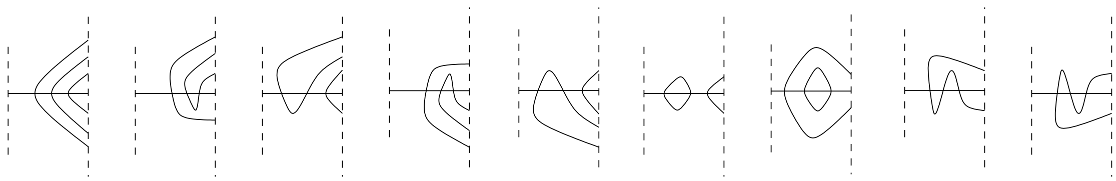

Figure 3.3. All meanderslices of degree 3 in $\mathcal{R}$.

It is easy to check that $\mathcal{R} \subset \mathcal{MS}$ is a graded subalgebra of $\mathcal{MS}$ which is however no longer finitely generated (in fact, $\mathcal{R}$ is a free algebra on an infinite set of homogeneous generators:

$$1\ ,\ 3\ ,\ 2\ ,\ 13\ ,\ 16\ ,\ 106\ ,\ 166\ ,\ 1073\ ,\ 1934\ ,\ 12142$$

(for instance only the last two elements of figure 3.3 are not products of elements of lower degree in $\mathcal{R}$) are the numbers of homogenous generators in degree $1, 2, \ldots, 10$ in such a free basis of the algebra $\mathcal{R}$.

Proposition 3.5. — *The dimensions r_n of the vector space spanned by homogenous elements of degree n in $\mathcal{R}$ are given by*

$$r_{2n} = \binom{2n}{n}^2 \qquad \text{and} \qquad r_{2n-1} = \binom{2n-1}{n-1}^2 \quad .$$

This yields the asymptotics

$$r_n \sim \frac{2}{\pi}\frac{4^n}{n} \quad .$$

For any meanderslice $\sigma \in \mathcal{R}$ of degree n there exist uniquely defined integers $\alpha = d^+(\sigma), \beta = d^-(\sigma) \geq 0$ such that none of the edges ending at $1 + (\alpha + 1)\sqrt{-1}$ and $1 - (\beta + 1)\sqrt{-1}$ intersects the real segment $[-1, 1]$ and all edges of σ ending at $1 + \alpha\sqrt{-1}$ and $1 - \beta\sqrt{-1}$ intersect $[-1, 1]$.

Definition 3.6. — Given a meanderslice $\sigma \in \mathcal{R}$, the pair of numbers $(\alpha, \beta) = (d^+(\sigma), d^-(\sigma))$ defined as above is called the *bidegree* of σ.

Sketch of proof for Proposition 3.5. — Let $r_n(\alpha, \beta)$ denote the number of n-meanderslices $\sigma \in \mathcal{R}$ with $d^+(\sigma) = \alpha$ and $d^-(\sigma) = \beta$. We have obviously $r_0(\alpha, \beta) = 0$ with the exception $r_0(0, 0) = 1$. Setting $r_n(\alpha, \beta) = 0$ if $\alpha < 0$ or $\beta < 0$ one shows for all $\alpha, \beta \geq 0$ the recursion relation

$$r_{n+1}(\alpha, \beta) = r_n(\alpha - 1, \beta - 1) + r_n(\alpha - 1, \beta + 1) + r_n(\alpha + 1, \beta - 1) + r_n(\alpha + 1, \beta + 1) \quad .$$

From this one proves the equalities

$$r_n(\alpha,\beta) = r_n(\beta,\alpha) = r_n(\alpha,n)\ r_n(n,\beta)$$

which implies $r_n = \big(\sum_k r_n(n,k)\big)^2$. Moreover one checks by induction that

$$r_n(n,n-2k) = \binom{n}{k} - \binom{n}{k-1}$$

which shows that $\sum_k r_n(n,k) = \binom{n}{[n/2]}$ (with $[n/2]$ denoting the integer part of $n/2$) and implies the result.

The asymptotics follow of course from Stirlings formula $n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}$. ∎

Remark 3.7. — Using generating series, the equality $r_{2n} = \binom{2n}{n}^2$ can be shown to be equivalent to the identity

$$\Big(1 - 2x\sum_{k=0}^{\infty} C_k\, x^k\Big)^{-1} = \sum_{k=0}^{\infty}\binom{2k}{k}\, x^k$$

involving the Catalan numbers $C_k = \binom{2k}{k}\frac{1}{k+1}$.

Let us consider the subspace $\mathcal{L} \subset \mathcal{MS}$ generated by all meanderslices containing at least one loop. It is obvious that $\mathcal{L}$ is an ideal of $\mathcal{MS}$. The quotient $\mathcal{Q} = \mathcal{MS}/\mathcal{L}$ is still graded and the dimension q_n in degree n of $\mathcal{Q}$ is exactly the number of all meanderslices of degree n which have no loops.

We introduce also the quotient $\mathcal{S} = \mathcal{R}/(\mathcal{L}\cap\mathcal{R})$ which is easily seen to be a subalgebra of both $\mathcal{Q}$ (this is obvious) and of $\mathcal{MS}$ (this comes from the observation that the product in $\mathcal{MS}$ of two loopless meanderslices in $\mathcal{R}$ never produces a loop).

Let $s_n = \dim(\mathcal{S}_n)$ and $q_n = \dim(\mathcal{Q}_n)$ denote the dimensions of the homogeneous vectorspaces of degree n in $\mathcal{S}$ and in $\mathcal{Q}$.

PROPOSITION 3.8. — *We have the inequalities*

$$s_n \le q_n \le (n+1)^2 s_n$$

and

$$s_a s_b \le s_{a+b} \quad , \quad q_a q_b \ge q_{a+b} \quad .$$

Proof. — The inclusion $\mathcal{S} \subset \mathcal{Q}$ shows the inequality $s_n \le q_n$.

Let $\sigma \in \mathcal{S}$ be a meanderslice of degree n of bidegree $(\alpha,\beta) = (d^+(\sigma), d^-(\sigma))$. Choosing two integers $0 \le a \le \alpha$, $0 \le b \le \beta$ we can connect the $(\alpha-a)$ "highest strands" (extremities of edges ending at $1+(a+1)\sqrt{-1},\dots,1+\alpha\sqrt{-1}$) and the $(\beta-b)$ "lowest strands" with the left boundary component of σ thus getting an ordinary meanderslice of degree n. This construction is injective and surjective (between the two obvious bases of $\mathcal{S}_n$ and $\mathcal{Q}_n$) and since the non-negative integers α, β are at most equal to n we get the inequality $q_n \leq (n+1)^2 s_n$.

Figure 3.4 illustrates this by showing a meanderslice of bidegre $(3,1)$ in $\mathcal{S}$ and all meanderslices in $\mathcal{Q}$ obtained by applying the above construction (with $0 \leq \alpha \leq 3$ and $0 \leq \beta \leq 1$).

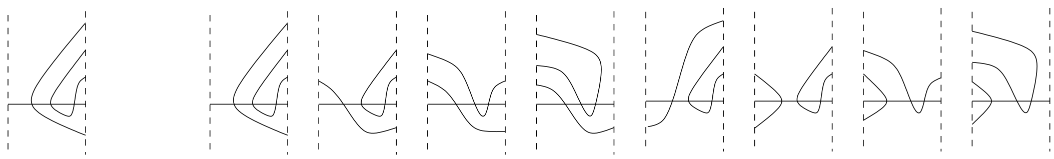

Figure 3.4.

The inequality $s_a s_b \leq s_{a+b}$ follows from the fact that $\mathcal{S}$ is a graded free algebra.

The inequality $q_a q_b \geq q_{a+b}$ follows from the fact that every meanderslice $\sigma \in \mathcal{Q}$ of degree $a+b$ factorizes as $\sigma = \sigma_a \sigma_b$ where σ_k is a meanderslice of degree k in $\mathcal{Q}$. ■

Remark 3.9. — Denoting by $s_n(\alpha, \beta)$ the dimension of the subspace spanned by all n-meanderslices in $\mathcal{S}$ which have bidegree (α, β), the above proof shows in fact the equality

$$q_n = \sum_{\alpha,\beta} (\alpha+1)(\beta+1) s_n(\alpha,\beta) \quad .$$

Let μ be a tame n-meander having $k(\mu)$ connected components $C_1, \dots, C_{k(\mu)}$. Denote by $c_i(\mu)$ the number of exterior arches in the component C_i. Given a subset $I \subset \{1, \dots, k(\mu)\}$ we denote by $c_I^+(\mu)$ (respectively $c_I^-(\mu)$) the number of exterior upper (respectively lower) arches contained in the union $\cup_{i \in I} C_i$.

We denote furthermore by $\mathcal{T}_n$ the set of all tame n-meanders.

Proposition 3.10.

(i) One has

$$s_{2n} = \sum_{\mu \in \mathcal{T}_n} \prod_{i=1}^{k(\mu)} \left(2^{c_i(\mu)} - 1\right) .$$

(ii) For the numbers q_{2n} we have

$$q_{2n} = \sum_{\mu \in \mathcal{T}_n} (-1)^{k(\mu)} \sum_{I \subset \{1,2,\dots,k(\mu)\}} (-1)^{\sharp(I)} \left(1 + c_I^+(\mu)\right) 2^{c_I^+(\mu)} \quad \left(1 + c_I^-(\mu)\right) 2^{c_I^-(\mu)} .$$

Proof. — Given a tame n-meander μ contained in the strip $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\}$ with connected components $C_1, \dots, C_{k(\mu)}$ there are $\prod_i (2^{c_i(\mu)} - 1)$ choices of exterior arches which contain at least one exterior arch in each connected component. Given such a choice, slice all choosen exterior arches open and connect them planarly in the obvious way (without introducing crossings with the real interval $[-1, 1]$) to the right boundary $\{z \in \mathbf{C} \mid \mathrm{Re}(z) = 1\}$ of $\{z \in \mathbf{C} \mid -1 < \mathrm{Re}(z) < 1\}$. This defines a unique $2n$-meanderslice in $\mathcal{S}$ and this construction is easily seen to be bijective between between suitable choosen subsets of exterior arches in tame n-meanders and a basis of of $\mathcal{S}_{2n}$.

The proof of assertion (ii) is similar but slightly more involved and uses the inclusion-exclusion principle (sieve formula). Since we do not use it in the sequel we leave the details to the reader. ∎

Proof of Theorem 1.10. — Proposition 3.8 implies that the sequences

$$(s_n)^{1/n} \quad \text{and} \quad (q_n)^{1/n}$$

have a common limit γ which satisfies $(s_n)^{1/n} \leq \gamma \leq (q_n)^{1/n}$.

The inequality $\gamma^2 \geq \lambda$ follows from Proposition 3.10. ∎

4. Forest-meanders

The aim of this section is to study forest-meanders using a graded subalgebra $\mathcal{S}^f$ and a graded quotient algebra $\mathcal{Q}^f$ of the even algebra $\mathcal{MS}^e$ (which is the free subalgebra spanned by all 16 meanderslices of degree 2). There is again an inequality between the growth rate λ^f of forest-meanders and the growth of dimensions of the two associated algebras.

Definition 4.1. — Let $\sigma \in \mathcal{MS}$ be a meanderslice of degree $2n$ intersecting the real interval $(-1, 1)$ in $2n$ points $-1 < p_1 < p_2 < \ldots < p_{2n} < 1$. A loop l of σ is *even* if the last intersection p_j of l with the interval $(-1, 1)$ has even index (ie. j is even, $p_j \in l$ and $p_i \notin l$ for $i > j$). Otherwise l is called *odd*.

There are for instance 16 meanderslices of degree 4 which contain an odd loop.

Let $\mathcal{L}^o \subset \mathcal{MS}^e$ denote the vector space spanned by all even meanderslices containing an odd loop. The space $\mathcal{L}^o \subset \mathcal{MS}^e$ is a homogenous ideal of $\mathcal{MS}^e$ and we get a graded subalgebra $\mathcal{S}^f = \mathcal{R}^e/(\mathcal{R}^e \cap \mathcal{L}^o)$ (where $\mathcal{R}^e = \mathcal{R} \cap \mathcal{MS}^e$) and a graded quotient

$\mathcal{Q}^f = \mathcal{MS}^e/\mathcal{L}^o$. As in section 3 we have natural inclusions $\mathcal{R}^f \subset \mathcal{MS}^e$ and $\mathcal{R}^f \subset \mathcal{Q}^f$ which preserve the grading. Set $s_n^f = \dim((\mathcal{S}^f)_{2n})$ and $q_n^f = \dim((\mathcal{Q}^f)_{2n})$.

PROPOSITION 4.2. — *We have the inequalities*

$$s_n^f \leq q_n^f \leq (2n+1)^2 s_n^f \quad \textit{and} \quad s_a^f s_b^f \leq s_{a+b}^f \ , \quad q_a^f q_b^f \geq q_{a+b}^f \, .$$

The proof is analogous to the proof of Proposition 3.8.

TABLE 4.3. — *The first terms of the sequence s_n^f are*

$$s_0^f = 1 \ , \ 4 \ , \ 32 \ , \ 320 \ , \ 3536 \ , \ 41344 \ , \ 501264 \ \ldots$$

TABLE 4.4. — *The first terms of the sequence q_n^f are*

$$q_0^f = 1 \ , \ 16 \ , \ 240 \ , \ 3552 \ , \ 52224 \ , \ 764672 \ , \ 11163936 \ \ldots$$

THEOREM 4.5. — *The sequences $(s_n^f)^{1/n}$ and $(q_n^f)^{1/n}$ have a common limit $\gamma^f \geq \lambda^f$. One has moreover $(s_n^f)^{1/n} \leq \gamma^f \leq (q_n^f)^{1/n}$ for all n.*

Given any meander μ let us denote by $c^+(\mu)$ its number of exterior upper arches and by $c^-(\mu)$ its number of exterior lower arches. We denote moreover by F the set of all forest-meanders.

PROPOSITION 4.6. — *One has*

$$s_n^f = \sum_{\mu \in F_n} 2^{c^+(\mu)+c^-(\mu)}$$

and

$$q_n^f = \sum_{\mu \in F_n} (1 + c^+(\mu))(1 + c^-(\mu)) 2^{c^+(\mu)+c^-(\mu)} \qquad .$$

The proof is completely analogous to the proof of Proposition 3.10.

Proof of Theorem 4.5. — Proposition 4.2 implies that the sequences $(s_n^f)^{1/n}$ and $(q_n^f)^{1/n}$ converge and have a common limit γ^f such that $(s_n^f)^{1/n} \leq \gamma^f \leq (q_n^f)^{1/n}$. Proposition 4.6. implies the inequality $\lambda^f \leq \gamma^f$. ∎

5. The algebra $\mathcal{M}$ of meanders

In this section we define an interesting graded subalgebra $\mathcal{M} \subset (\mathcal{MS})^e$. As an abstract algebra $\mathcal{M}$ is free with infinitely many generators of arbitrary large degrees. As a vector space, $\mathcal{M}$ is simply the free vector space generated by all meanders (with an arbitrary number of components).

The algebra $\mathcal{M}$ has of course also a quotient corresponding to the quotient $\mathcal{Q}$ of $\mathcal{MS}$.

The free vector space $\mathcal{M}$ on all meanders can be endowed with an algebra structure as follows:

Given two meanders μ and ν we denote by $E^+(\mu),\ E^+(\nu)$ the set of all exterior upper arches of $\mu,\ \nu$ and by $E^-(\mu),\ E^-(\nu)$ the set of all exterior lower arches. Let $X^+ \subset E^+(\mu),\ Y^+ \subset E^+(\nu),\ X^- \subset E^-(\mu),\ Y^- \subset E^-(\nu)$ be subsets of exterior arches such that $\sharp(X^+) = \sharp(Y^+)$ and $\sharp(X^-) = \sharp(Y^-)$.

Define $\mu \,{}^{X^+}_{X^-}\,{}^{Y^+}_{Y^-}\, \nu$ as the (equivalence class of the) following meander. Choose first representatifs (systems of menadercurves) of μ and ν in $\mathbf{C}$ which do not intersect and such that ν is "at the right" of μ (ie we have $x < y$ for any $x \in \mu \cap \mathbf{R}$ and $y \in \nu \cap \mathbf{R}$). Cut now all exterior arches of $\mu \cup \nu$ which belong to $X^+ \cup Y^+$ in their midpoints thus obtaining $2\sharp(X^+) + 2\sharp(Y^+) = 4\sharp(X^+)$ upper "half-arches". Glue these half-arches together in the unique way such that the result is planar and such that each half-arch of μ is glued to exactly one half-arch of ν. Do the same with all lower arches in $X^- \cup Y^-$. We denote the resulting meander by $\mu \,{}^{X^+}_{X^-}\,{}^{Y^+}_{Y^-}\, \nu$ (Figure 5.1 shows an example).

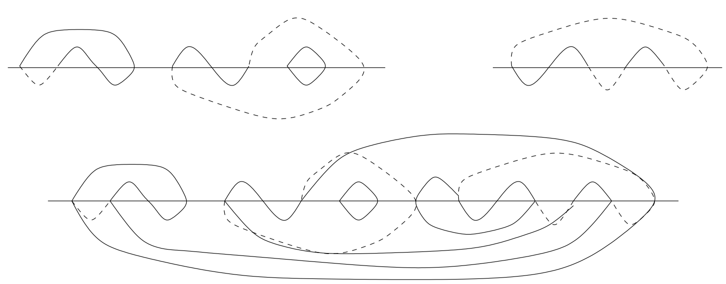

Figure 5.1. Two meanders μ and ν (with dotted subsets $X^\pm$ and $Y^\pm$) and the meander $\mu \,{}^{X^+}_{X^-}\,{}^{Y^+}_{Y^-}\, \nu$ (before erasing the dotted arches used in the construction).

Define now the product $\mu\nu \in \mathcal{M}$ as

$$\mu\nu = \sum_{\substack{X^+ \in E^+(\mu),\ Y^+ \in E^+(\nu),\ \sharp(X^+) = \sharp(Y^+) \\ X^- \in E^-(\mu),\ Y^- \in E^-(\nu),\ \sharp(X^-) = \sharp(Y^-)}} \mu^{X^+}_{X^-} \quad {}^{Y^+}_{Y^-}\nu \quad .$$

A little thought shows that this endows $\mathcal{M}$ with a structure of an associative graded algebra (the grading is given by $\deg(\mu) = n$ if μ is an n-meander). Figure 5.2 shows an example of a product.

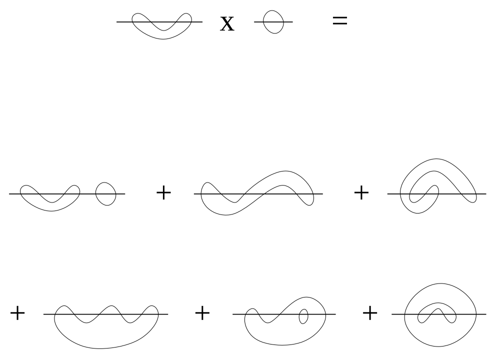

Figure 5.2. A product

Let $\mathcal{M}_n$ denote the vector space spanned by all n-meanders (with an arbitrary number of connected components). The dimension of $\mathcal{M}_n$ is then C_n^2 where $C_n = \binom{2n}{n}\frac{1}{n+1}$ is the n-th Catalan number (see for instance [**LZ**] for a proof of this).

Remarks 5.1

(i) Here is a slightly different (but equivalent) description of the product $\mu\nu$ of an m-meander μ with an n-meander ν. Consider an $(m+n)$-meander κ and cut it into two pieces along a vertical line through $x \in \mathbf{R}$ such that $\sharp([-\infty, x) \cap \kappa) = 2m$ and $\sharp((x, \infty] \cap \kappa) = 2n$ (one can always choose κ such that the above vertical line cuts every arch of κ at most once). The first part of κ contains then 2α upper half-arches and 2β lower half-arches. Glue the first of the upper half-arches to the second one, the third to the fourth and so on. Do the same with the lower half-arches and iterate this construction on the second piece. The meander κ appears with coefficient 1 in the product $\mu\nu$ if and only if the above construction yields μ on the first piece and ν on the second. Otherwise κ does not appear in $\mu\nu$.

(ii) Set $\alpha^+ = \sharp(E^+(\mu)), \alpha^- = \sharp(E^-(\mu))$ and $\beta^+ = \sharp(E^+(\nu)), \beta^- = \sharp(E^-(\nu))$. The product $\mu\nu$ is then a sum containing exactly

$$\sum_k \binom{\alpha^+}{k}\binom{\beta^+}{k} \sum_l \binom{\alpha^-}{l}\binom{\beta^-}{l}$$

distinct meanders.

(iii) The algebra structure on $\mathcal{M}$ can easily be deformed: Take two variables z_+ and z_- and set

$$\mu\nu = \sum_{\substack{X^+\in E^+(\mu),\ Y^+\in E^+(\nu),\ \sharp(X^+)=\sharp(Y^+)\\ X^-\in E^-(\mu),\ Y^-\in E^-(\nu),\ \sharp(X^-)=\sharp(Y^-)}} z_+^{\sharp(X^+)} z_-^{\sharp(X^-)}\ \mu_{X^-}^{X^+}\ {}_{Y^-}^{Y^+}\nu \quad .$$

Specializing z_+ and z_- to 1 yields of course the algebra structure described above.

(iv) The algebra $\mathcal{M}$ has of course also a quotient which simplifies computations invoving meanders: keep only track of exterior arches (and of their connected components of course). This quotient was used for instance for computing Table 1.6. We leave the details (which are quite straightforward) to the reader.

Let $\epsilon_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbf{Q}[q]$ be the linear map defined by $\epsilon_{\mathcal{M}}(\mu) = q^k$ for a meander μ having k connected components. We call ϵ the *augmentation map* (Caution: $\epsilon_{\mathcal{M}}$ is by no means a homomorphism of algebras).

Let O be the unique 1-meander (represented by the unit circle $\subset \mathbf{C}$).

THEOREM 5.2. — *We have*

$$\sum_k m_{n,k} q^k = \epsilon_{\mathcal{M}}(O^n)$$

where O^n is the n-th power in $\mathcal{M}$ of the unique 1-meander $O \in \mathcal{M}$ consisting of a circle centered at the origin.

Proof. — An easy induction on n. ∎

THEOREM 5.3. — *There exist an injective homomorphism of graded algebras $\iota : \mathcal{M} \longrightarrow \mathcal{MS}^{\mathrm{e}}$ of the meander algebra $\mathcal{M}$ into the even subalgebra $\mathcal{MS}^{\mathrm{e}}$ of the meander-slice algebra.*

Sketch of proof. — Let μ be a meander with τ exterior upper arches and β exterior lower arches. We send μ to a sum of

$$\sum_k (2k+1)\binom{\tau}{k}\sum_l (2l+1)\binom{\beta}{l} = (\tau+1)(\beta+1)2^{\tau+\beta}$$

meanderslices as follows. Choose a subset X of k exterior upper arches (there are $\binom{\tau}{k}$ such choices) and cut all these arches open thus getting $2k$ open strands. There are hence $2k+1$ possiblities to bend them to the left or to the right in a planar way. The same argument holds also for the exterior lower arches. Each such choice yields a unique slicemeander. Define now $\iota(\mu)$ as the sum over all possibilities. Figure 5.3 shows an example.

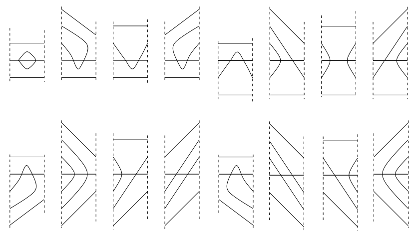

Figure 5.3. $\iota(O)$ for the unique 1-meander O

We leave it to the reader to convince himself that $\iota(\mu)$ defined as the sum of all these meanderslices yields indeed a homomorphism of graded algebras. ∎

6. Open problems and miscellaneous

The main open problem in the subject of meanders is of course the determination of the asymptotical behaviour (or even better a formula) of the numbers $m_{n,1}$ of connected n-meanders. Of course, understanding the asymptotics of any of the sequences $(m_{n,k})$, (m_n^t) or even (s_n) or (q_n) would also be helpfull.

Another interesting problem (less general than the above questions) is the question whether $\gamma^2 = \lambda$ or not.

Answers to the same questions involving forest-meanders would also be nice.

It would also be interesting to have a better understanding of all the algebraic structures (the different algebras introduced in this paper) associated to meanders. For instance the algebras $\mathcal{SM}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{S}^f$ and $\mathcal{M}$ are all free graded algebras. The first four of these algebras have in some sense canonical generators (given by suitable meanderslices). They have hence also a Hopf algebra structure given by the coproduct

$$\Delta(g_1 g_2 \cdots g_l) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k} g_{i_1} g_{i_2} \cdots g_{i_k} \otimes g_1 \cdots \hat{g_{i_1}} \cdots \hat{g_{i_2}} \cdots \hat{g_{i_k}} g_l$$

and counit $\delta(1) = 1$ if g is the unit and $\delta(g) = 0$ for any homogeneous element g of degre ≥ 1.

Does the algebra $\mathcal{M}$ have such a natural Hopf algebra structure (it clearly has a Hopf algebra structure since it is a free algebra but this structure depends on a choice of generators which is not canonical)?

Does the quotient algebra $\mathcal{Q}$ admit a Hopf algebra structure?

Let us also mention that the techniques of this papers allow easily the computation of lower bounds for the number of meanders of special types. Indeed, one can restrict oneself for instance only to meanders having at most say N exterior arches. Doing the computations in the suitable quotient (which keeps only track of exterior arches and their connected components) of $\mathcal{M}$ one gets a finite dimensional "transfer matrix" whose largest real eigenvalue is a lower bound for the corresponding exponential growth.

Call a meander $\mu \subset \mathbf{C}$ *symmetric* if $\mu = -\mu$ (where $-\mu = \{z \in \mathbf{C} \mid -z \in \mu\}$). Denote by $m^s_{n,k}$ the number of symmetric n-meanders with k connected components. One shows that the number of connected components of a symmetric n-meander has always the same parity as n. Connected symmetric meanders have hence always odd degrees and the first numbers $m^s_{2n-1,1}$ are as follows

$$m^s_{1,1} = 1\ ,\ m^s_{3,1} = 2\ ,\ m^s_{5,1} = 10\ ,\ m^s_{7,1} = 66\ ,\ m^s_{9,1} = 504\ ,\ m^s_{11,1} = 4210\ ,\ m^s_{13,1} = 37378\ .$$

One can show the inequalities

$$m_{n,1} \leq m^s_{2n-1} \leq s_{2n-1}$$

which imply

$$\sqrt{\lambda} \leq \lambda^s \leq \gamma$$

if the limit $\lambda^s = \lim\left(m_{2n-1,1}\right)^{1/(2n-1)}$ exists. (The techniques of this paper give no proof for the existence of this limit but it is very unlikely that $\liminf\left(m_{2n-1,1}\right)^{1/(2n-1)} < \limsup\left(m_{2n-1,1}\right)^{1/(2n-1)}$.)

Let $m^{t,s}_n$ be the number of tame symmetric n-meanders. The growth $\lim\left(m^{t,s}_n\right)^{1/n}$ (if it exists) of tame symmetric meanders equals also λ^s.

Finally, let us describe a subfamily of meanders with only very few connected meanders: Compose a n-arch system with the n-arch system $\cup\cup\cup\ldots\cup$ formed by n consecutive arches. We call meanders of this type *Narayana meanders* since the so-called Narayana numbers (see Exercice 36 of chapter 6 in [**S**]) enumerate them accordingly to their number of connected components. In particular, there exists only one connected Narayana n-meander.

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Bibliography

[A] V.I. Arnol'd. — *Sib. Math. J.,* **29** *(5) (1988)*, in Russian.

[FGG1] P. Di Francesco, O. Golinelli, E. Guitter. — *Meander, Folding and Arch Statistics*, Preprint.

[FGG2] P. Di Francesco, O.Golinelli, E.Guitter. — *Meanders and the Temperley-Lieb algebra*, Preprint.

[FGG3] P. Di Francesco, O. Golinelli, E. Guitter. — *Meanders: A Direct Enumeration Approach*, Preprint.

[LZ] S.K. Lando, A.K. Zvonkin. — *Meanders*, Selecta Math. Sov. **11** (1992), 117–144.

[S] R.P.Stanley. — *Enumerative Combinatorics*, vol. 2, Cambridge University Press.

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