HAL
archives-ouvertes.fr

Planetary System around the Pulsar PSR 1257+12

Laurent Nottale

▶ To cite this version:

Laurent Nottale. Planetary System around the Pulsar PSR 1257+12. Scale Relativity and Fractal Space-Time. A New Approach to Unifying Relativity and Quantum Mechanics, Imperial College Press, 2011, 978-1-84816-650-9. hal-03135303

HAL Id: hal-03135303

https://hal.archives-ouvertes.fr/hal-03135303

Submitted on 8 Feb 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Scale Relativity and Fractal Space-Time

A New Approach to Unifying Relativity and Quantum Mechanics*

Excerpt : Chapter 13.5.3, pp. 620-639

Planetary System around the Pulsar PSR 1257+12

Laurent NOTTALE
LUTH, CNRS and Paris-Meudon Observatory / PSL

February 10, 2021

Abstract

We apply the scale relativity framework, in which the formation of a planetary system is described in terms of a macroscopic Schrödinger equation, to the study of the three planets orbiting the pulsar PSR 1257+12. We describe in terms of wave packets the clouds from which the planets are formed, finally lying at their center of gravity. This leads us to predict that the semimajor axes of the planet orbits should be quantized as $n^2 + n/2$ (where $n = 5$, 7, 8 in this system, plus another possible planet at $n = 2$). We find prime integrals of the motion equation for the 3:2 near-resonant subsystem of two of the planets. This allows us to show that it has remained stable and that its evolution, since about one billion years, has not affected the structures present at the end of its formation. Our theoretical predictions can therefore be compared with its present state. We find that the observed ratios of planet semimajor axes differ from their theoretically expected values by only 4×10^{-4} and 6×10^{-5}, and that this remarkable agreement can be shown to be statistically highly significant.

Contents

*© Imperial College Press 2011

13.5.3. Planetary system around the pulsar PSR 1257+12

0.1 Precision structuring of PSR 1257+12 planetary system

The first extrasolar planetary system ever discovered has been the system of three planets found by Wolszczan around the pulsar PSR B1257+12 [16, 17]. Even though this star is not solar-like, this system deserves a special study [10, 11, 12], because (i) the pulse timing measurements allow a very precise determination of the orbital elements of the planets and (ii) their small masses (four Earth mass for two of them and the Moon mass for the third one) allow both the celestial mechanics models of evolution and the scale relativity model of formation to become also very precise. This system therefore stands out as a kind of ideal gravitational laboratory for studying the formation and evolution of planetary systems and putting models to the test.

The planets probably result from an accretion disk formed around the very compact star after the supernova explosion. Even though this is a secondary process, there is general agreement that the formation process should be similar to the standard picture [3].

We can therefore expect the purely gravitational formation process, described in the scale relativity approach by a macroscopic gravitational Schrödinger equation, to be still valid in this case. Moreover, the smallness of the planet masses implies very few perturbations between the subdisks from which the three planets have been formed and a negligible effect of self-gravitation, so that the theoretical predictions (based on conservation laws, in particular of the center of gravity of each subdisk) are expected to become very precise. The compacity of the star also suggests that planets be self-organized in terms of a smaller scale than the inner solar system (i.e., of a multiple of $w_0 = 144$ km/s).

On the one hand, the precision of pulse timing measurements has allowed to use the PSR B1257+12 system as a highly accurate probe of planetary dynamics. Indeed, it was soon pointed out that the near 3:2 resonance between the orbits of the two main planets should lead to precisely predictable and observable mutual gravitational perturbations [14, 8, 15, 9]. These non-Keplerian gravitational effects have been soon detected [17], and they have now been observed with high precision, yielding an irrefutable confirmation of the existence of planets around the pulsar and allowing a determination of the true masses and of the orbital inclinations of the planets [6].

On the other hand, the observed distances of the planets to the pulsar can be shown [10, 11] to be in very good agreement with the scale relativity theoretical expectations. To the first approximation, we expect the semimajor axes to show peaks of probability scaling as n^2, where n is integer. But, when perturbations to the simple one-object Kepler potential model are small and the trajectories are nearly circular (which is the case here, the present excentricities of the three planets being $e_A = 0$, $e_B = 0.0186 \pm 0.0002$ and $e_C = 0.0252 \pm 0.0002$ [6]), one may use the center of gravity conservation law to refine the theoretical expectation and state that, after the formation process the planets should lie at a distance given by the center of mass of the planetesimal wave packet, which scales as $n^2 + n/2$, i.e. $n^2(1 + 2/n)$ [4, 1]. This means a "correction" of 6 to 10 % with respect to the simple n^2 law for the pulsar planets.

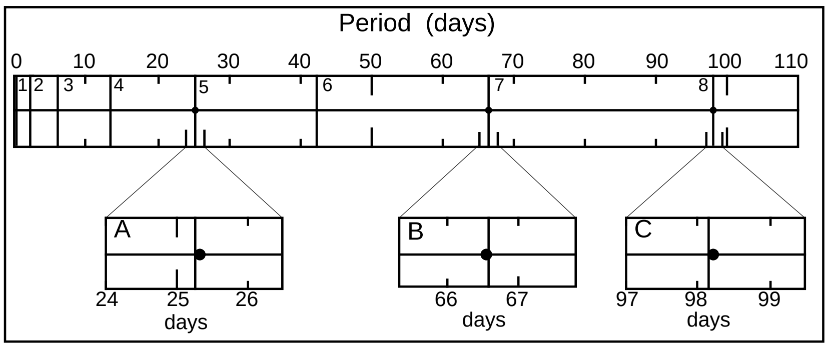

Figure 1: Comparison between the observed periods of the three planets observed around the pulsar PSR B1257+12 and the scale relativity expectation (1994 data). The agreement between the observed periods and the predicted ones is so precise that we have made three zooms by a factor of 10 in order to show the differences (less than 3 hours for periods of several months).

Already with the 1994 values of the orbital elements obtained by Wolszczan [17] from four years of observation, the agreement was excellent [10, 11]. The three observed planets A, B and C are indeed found to correspond to ranks $n = 5$, 7 and 8 with a high relative precision. With the n^2 law the agreement is already at the level of 1 % and 0.3 %. One finds:

$$(P_B/P_C)^{1/3} = 0.878 \text{ to be compared with } 7/8 = 0.875,$$

$$(P_A/P_C)^{1/3} = 0.636 \text{ to be compared with } 5/8 = 0.625.$$

With the $n^2 + n/2$ law, it becomes (see Fig. 1):

$$(P_B/P_C)^{1/3} = 0.8783 \text{ to be compared with } (52.5/68)^{1/2} = 0.8787,$$

$$(P_A/P_C)^{1/3} = 0.6366 \text{ to be compared with } (27.5/68)^{1/2} = 0.6359.$$

A factor of about 10 has been gained with this conservative law. The relative agreement between observation and theoretical expectation has become respectively 4×10^{-4} and 1.1×10^{-3} for the B/C and A/C ratios.

Today the comparison between the data and the theoretical predictions can be considered again, since the determination of the orbital elements of the three planets has been greatly improved thanks to the increase to 13 years of the observing time and to the account of gravitational mutual effects [6]. This improvement of the data and of their fitting model has led to a change by 5 σ of some of the derived orbital elements. This could have degraded the agreement with the scale relativity model. On the contrary, it has yielded a new improvement by a factor of about 20 on the A/C ratio. One now finds, for $P_A = (25.262 \pm 0.003)$ days, $P_B = (66.5419 \pm 0.0001)$ days and $P_C = (98.2114 \pm 0.0002)$ days [6]:

$$(P_B/P_C)^{1/3} = 0.8783 \;\text{ to be compared with }\; (52.5/68)^{1/2} = 0.8787,$$

$$(P_A/P_C)^{1/3} = 0.63597 \;\text{ to be compared with }\; (27.5/68)^{1/2} = 0.63593.$$

The relative agreement is now at the level of respectively 4×10^{-4} and 6×10^{-5} for the B/C and A/C ratios. The probability to find such a close agreement, accounting for all other possible fractional ratios with n up to 10, can be estimated to be less than 10^{-4} [11]. Moreover, using the standard pulsar mass $M = (1.4 \pm 0.1)$ $M_\odot$ in the relation

$$P_n = \frac{2\pi GM}{w^3}\,(n^2 + n/2)^{3/2}, \tag{1}$$

one obtains for the coupling constant of this system $w = (2.96 \pm 0.07) \times 144$ km/s, which is consistent with an expected integer multiple of the inner solar system constant w_0 by a factor of 3 and with the value of the Keplerian velocity $w_\odot = 435$ km/s at the Sun radius.

The precision of these results suggests to attempt improving the model, both concerning the formation as the evolution of the system. Indeed, the presently observed orbital elements result from the formation era followed by about one billion years of evolution, while the theoretical expectations correspond to the end of the formation. The comparison between the observational and theoretical orbital elements should therefore take the system evolution into account.

0.2 Heliumoid model of the PSR1257+12 planetary system formation

Let us address the problem of the formation of two planets from two subdisks in the scale relativity approach. The question to be solved is whether the self-gravitation and the mutual interaction of the subdisks leads to non negligible corrections to the previous one-body model.

Consider a star surrounded by a disk of planetesimals which can itself be separated in two parts, (B) and (C). The central star is of mass M and the individual planetesimals are assumed to have equal mass μ. The total Hamiltonian reads

$$\hat{H} = \sum_{B,C}\left(-\frac{\tilde{\hbar}^2}{2\mu}\Delta - \frac{M\mu}{r} - \sum_{B}\frac{\mu^2}{r_{12}} - \sum_{C}\frac{\mu^2}{r_{12}} - \sum_{BC}\frac{\mu^2}{r_{12}}\right), \tag{2}$$

where $\tilde{\hbar} = 2\mu\mathcal{D}$, with $\mathcal{D} = GM/2w_0$.

The three last terms are respectively the self potentials of the B and C rings and the potential of interaction between B and C, r_{12} being the interdistance between two planetesimals. Assume that the total number of planetesimals in B and C is respectively N_B and N_C. The masses of the B and C rings, and therefore of the planets B and C which will finally be formed from these rings, are

$$m_B = N_B\,\mu, \qquad m_C = N_C\,\mu. \tag{3}$$

The two first terms of the Hamiltonian then become

$$\sum_{B,C}\left(-\frac{\tilde{\hbar}^2}{2\mu}\Delta - \frac{M\mu}{r}\right) = -\frac{\tilde{\hbar}^2}{2\mu^2}\,(m_B\Delta_B + m_C\Delta_C) - \frac{Mm_B}{r_B} - \frac{Mm_C}{r_C} \tag{4}$$

The distribution of planetesimals in the rings B and C are given by solutions of the gravitational Schrödinger equation

$$\hat{H}\psi = E\psi, \tag{5}$$

where $\hat{H}$ is given by Eq. (2). In the first approximation, since $m_B/M \ll 1$ and $m_C/M \ll 1$ (the ratios are $\approx 10^{-5}$ for planets B and C of the PSR1257+12 system and $\approx 10^{-7}$ for planet A), the potential is strongly dominated by the central body, and ψ_B and ψ_C are solutions of the equation

$$2\mathcal{D}^2\Delta\psi + \left(\frac{E}{\mu} + \frac{M}{r}\right)\psi = 0. \tag{6}$$

In the second approximation, we can now use these solutions to derive the probability number density of planetesimals in the disks, $\rho_B = |\psi_B|^2$ and $\rho_C = |\psi_C|^2$. The self potential of ring B can now be written as a sum of the planetesimal interaction gravitational energy over all couples of planetesimals in this ring, namely,

$$-\sum_{i,j}\frac{\mu^2}{r_{12}} = -\sum_i N_B\,\mu^2\int_B \frac{|\psi_B|^2}{r_{12}}\,dV_B = -m_B^2\,I_{BB}. \tag{7}$$

This sum has therefore been reduced to the integral

$$I_{BB} = \int_B \frac{|\psi_{B1}|^2|\psi_{B2}|^2}{r_{12}}\,dV_1 dV_2. \tag{8}$$

Similar calculations can be made for the self-potential of C and for the BC interaction term. Namely, we define the integrals

$$I_{CC} = \int_C \frac{|\psi_{C1}|^2|\psi_{C2}|^2}{r_{12}}\,dV_1 dV_2, \tag{9}$$

$$I_{BC} = \int_{BC} \frac{|\psi_{B1}|^2|\psi_{C2}|^2}{r_{12}}\,dV_1 dV_2. \tag{10}$$

Then the Hamiltonian now takes the form

$$\hat{H} = -\frac{\tilde{\hbar}^2}{2\mu^2}\,(m_B\Delta_B + m_C\Delta_C) - \frac{Mm_B}{r_B} - \frac{Mm_C}{r_C} - m_B^2\,I_{BB} - m_C^2\,I_{CC} - m_Bm_C\,I_{BC}. \tag{11}$$

The three last terms therefore also give the correction to the total energy of the system due to self-gravitation and to interactions of the two rings, which reads

$$E_{\rm tot} = -\frac{1}{2}\,w_0^2\left(\frac{m_B}{n_B^2} + \frac{m_C}{n_C^2} + \frac{2m_B^2}{M}\,I_{BB} + \frac{2m_C^2}{M}\,I_{CC} + \frac{2m_Bm_C}{M}\,I_{BC}\right), \tag{12}$$

where n_B and n_C are the main quantum numbers of the B and C wave functions (namely, $n_B = 7$ and $n_C = 8$ for the PSR 1257+12 system), and where the I integrals are now dimensionless (namely, the lengths are expressed in gravitational "Bohr" units, $r_{\rm Bohr} = GM/w_0^2$).

To be complete, one adds the third planet A to the description, whose mass is yet ≈ 200 times smaller than that of planets B and C, so that the self-potential correction to its energy is negligible, while the interactions given by the integrals I_{AB} and I_{AC} may be relevant.

Finally the relative correction on the energy of the three planets is found to be

$$\frac{\Delta E_A}{E_A} = 2n_A^2 \left(\frac{n_B^2}{n_A^2 + n_B^2} \, \frac{m_B}{M} \, I_{AB} + \frac{n_C^2}{n_A^2 + n_C^2} \, \frac{m_C}{M} \, I_{AC} \right), \tag{13}$$

$$\frac{\Delta E_B}{E_B} = 2n_B^2 \left(\frac{m_B}{M} \, I_{BB} + \frac{n_C^2}{n_B^2 + n_C^2} \, \frac{m_C}{M} \, I_{BC} \right), \tag{14}$$

$$\frac{\Delta E_C}{E_C} = 2n_C^2 \left(\frac{m_C}{M} \, I_{CC} + \frac{n_B^2}{n_B^2 + n_C^2} \, \frac{m_B}{M} \, I_{BC} \right), \tag{15}$$

where we have attributed the interaction contribution to each planet according to its energy, and where the self-gravity and of planet A and its effects on planets B and C have been neglected.

The integrals I_{BB}, I_{CC} and I_{BC} are exactly those which are encountered in the helium and heliumoide quantum problem (see, e.g. [7]). Their numerical calculation for $n_A = 5$, $n_B = 7$ and $n_C = 8$, which are the values of the principal "graviquantum" number of the three planets of the PSR1257+12 system, and $l = n - 1$ (quasi circularity), yields:

$$I_{77} = 0.0173582, \; I_{88} = 0.0134383, \; I_{78} = 0.0221588, \; I_{57} = 0.0377702, \; I_{58} = 0.0307991.$$

Since the dimensionless energies are $E_n = -1/2n^2$, the perturbation can be expressed in terms of a correction on the expected main quantum numbers, $\delta n/n = -(1/2)\Delta E/E$. Using the measured values of the planet masses, $m_A = 0.020 \pm 0.002$, $m_B = 4.3 \pm 0.2$ and $m_C = 3.9 \pm 0.2$ Earth mass [6] and the standard pulsar mass of 1.4 $M_\odot$, one finds very small corrections,

$$\delta n_A = -0.000052, \;\; \delta n_B = -0.000090, \;\; \delta n_C = -0.000102.$$

However, in the absence of a precise knowledge of the mass of the pulsar, only ratios of quantum numbers (or equivalently, of periods, semimajor axes, or energy) can be compared. The above corrections can then be expressed in another way. We fix as reference $n_C = 8$, then the expected values of the two other planets become: $(n_A)_{\rm pred} = 5.000012$, while the observed value is $(n_A)_{\rm obs} = 5.00029 \pm 0.00020$, and $(n_B)_{\rm pred} = 7.000000$, while the observed value is $(n_A)_{\rm obs} = 6.996987 \pm 0.000008$.

As expected at the beginning of this study from the smallness of the planet masses, the corrections are negligible for this system, since they remain smaller than the observational errors despite the precision of the data. However, a new possible improvement of the determination of the orbital elements in the future may render them meaningful. Moreover, all this calculation is also relevant for multiple systems with larger planetary masses, in particular Jupiter-like planets, in which case the corrections are no longer negligible.

Problems and Exercises

Exercise 1 Apply the above heliumoid model of planetary system formation to the known multiple planetary systems with large planetary masses (several Jupiter masses), in order to estimate the self-gravity and interaction corrections to be applied to the values of the expected probability peaks.

∎

0.3 Evolution of a 3:2 near resonant two-planet system

0.3.1 Three-body perturbation theory

The general three-body perturbation equations (see Brouwer & Clemence [2]) in the Lagrange-Laplace theory have been applied by Malhotra [9] to the PSR 1257+12 planetary system. The gravitational interactions of low mass planets such as the PSR 1257+12 planets, which are $\approx 10^5$ times smaller than the pulsar mass, can be analysed in terms of osculating ellipses, i.e., of orbits that are instantaneously elliptical, but whose orbital parameters are time dependent. For a 3:2 near commensurability between two planets (1) and (2), identified respectively with planet B and C of the PSR 1257+12 planetary system, and assuming coplanar orbits, Malhotra found that the principal perturbation components in the interaction Hamiltonian are given by

$$\mathcal{H}' = -\frac{Gm_1m_2}{a_2}\left\{[\mathcal{P}(\psi,\alpha) - \alpha\cos\psi] + \frac{1}{2}A_1(\alpha)(e_1^2+e_2^2) - A_2(\alpha)e_1e_2\cos(\omega_1-\omega_2)\right.$$
$$\left. + C_1(\alpha)e_1\cos(\phi-\omega_1) + C_2(\alpha)e_2\cos(\phi-\omega_2)\right\}, \qquad (16)$$

where the planet masses are $m_1 \ll M_*$ and $m_2 \ll M_*$ (M_* being the central star mass), a_1 and a_2 are the osculating orbital semimajor axes, e_1 and e_2 the osculating eccentricities, $n_1 = 2\pi/P_1$ and $n_2 = 2\pi/P_2$ the mean orbital frequencies, and ω_1 and ω_2 the longitudes of periastron, and where (keeping Malhotra's notations)

$$\begin{aligned}
\alpha &= a_1/a_2, \\
\psi &= \lambda_1 - \lambda_2, \\
\phi &= 3\lambda_2 - 2\lambda_1, \\
\mathcal{P}(\psi,\alpha) &= (1 - 2\alpha\cos\psi + \alpha^2)^{-1/2}, \\
A_1(\alpha) &= +\frac{1}{4}\,\alpha\, b_{3/2}^{(1)}(\alpha) = +2.50, \\
A_2(\alpha) &= -\frac{1}{4}\,\alpha\, b_{3/2}^{(2)}(\alpha) = -2.19, \\
C_1(\alpha) &= -\frac{1}{2}\left(6 + \alpha\frac{d}{d\alpha}\right) b_{1/2}^{(3)}(\alpha) = -2.13, \\
C_2(\alpha) &= +\frac{1}{2}\left(5 + \alpha\frac{d}{d\alpha}\right) b_{1/2}^{(2)}(\alpha) = +2.59. \qquad (17)
\end{aligned}$$

The λ_j are the mean longitudes. The $b(\alpha)$ are Laplace coefficients. The numerical values of the A_j and C_j coefficients are those which corresponds to the value $\alpha = a_{01}/a_{02} = 0.771416(2)$ observed for the B and C planet ratio in the PSR 1257+12 system [6].

Three types of significant perturbations are apparent in this Hamiltonian:
(1) those related to conjonctions of the planets, described by the term $[\mathcal{P}(\psi,\alpha) - \alpha\cos\psi]$,
(2) secular effects responsible for the slow precession of the absides, described by the terms with coefficients A_1 and A_2, and
(3) the effects of the 3:2 near-commensurability of the mean motions, described by the remaining terms.

Malhotra wrote the first time derivative of the orbital elements as:

$$\frac{\dot{a_1}}{a_1} = +2\,\frac{m_2}{M_*}\,n_1\alpha\left[\frac{\partial}{\partial\psi}\mathcal{P}(\psi,\alpha) + \alpha\sin\psi + 2C_1(\alpha)e_1\sin(\phi-\omega_1) + 2C_2(\alpha)e_2\sin(\phi-\omega_2)\right] \tag{18}$$

$$\frac{\dot{a_2}}{a_2} = -2\,\frac{m_1}{M_*}\,n_2\alpha\left[\frac{\partial}{\partial\psi}\mathcal{P}(\psi,\alpha) + \alpha\sin\psi + 3C_1(\alpha)e_1\sin(\phi-\omega_1) + 3C_2(\alpha)e_2\sin(\phi-\omega_2)\right]. \tag{19}$$

Then, using the variables

$$h_j = e_j\sin\omega_j, \quad k_j = e_j\cos\omega_j, \tag{20}$$

for the two planets $j = 1, 2 = B, C$, the variations of the eccentricities and pericenters read

$$\begin{aligned}
\dot{h_1} &= A_{11}k_1 + A_{12}k_2 + B_1\cos\phi, \\
\dot{h_2} &= A_{21}k_1 + A_{22}k_2 + B_2\cos\phi, \\
\dot{k_1} &= -A_{11}h_1 - A_{12}h_2 - B_1\sin\phi, \\
\dot{k_2} &= -A_{21}h_1 - A_{22}h_2 - B_2\sin\phi.
\end{aligned} \tag{21}$$

Setting

$$\mu_1 = \frac{m_1}{M_*}, \quad \mu_2 = \frac{m_2}{M_*}, \tag{22}$$

the coefficients of the matrix A_{ij} and the vector coefficients B_j read [9]

$$\begin{aligned}
A_{11} = \mu_2\,n_1\alpha\,A_1(\alpha), &\quad A_{12} = \mu_2\,n_1\alpha\,A_2(\alpha), \\
A_{21} = \mu_1\,n_2A_2(\alpha), &\quad A_{22} = \mu_1\,n_2A_1(\alpha), \\
B_1 = \mu_2\,n_1\alpha\,C_1(\alpha), &\quad B_2 = \mu_1\,n_2C_2(\alpha).
\end{aligned} \tag{23}$$

0.3.2 Analytic solution

A partial analytical solution for this system of equations had been found by Rasio et al. [15] in a simplified case. They have treated the inner planet as a massless "test particle" perturbed by the outer planet, which was assumed to move on an unperturbed Keplerian orbit. To this approximation and neglecting the conjunctions (close encounters), they found that the system can be integrated in terms of a period-eccentricity relation that reads, for small eccentricities,

$$n_1 = n_*(1 + 3\,e_1^2), \tag{24}$$

where n_* is the value of the frequency for which $e_1 = 0$. However, this result holds only when $m_2 \gg m_1$, a condition which does not apply to the PSR 1257+12 system for which $m_1 \approx m_2$. In order to find a general analytical solution for the Malhotra system of equations, let us define a function that characterizes the conjunction effects on the semimajor axes variation,

$$\Gamma_c = \frac{\partial}{\partial\psi}\mathcal{P}(\psi,\alpha) + \alpha\sin\psi, \tag{25}$$

and a function that depends on eccentricities, namely,

$$\Gamma = C_1(\alpha)e_1\sin(\phi-\omega_1) + C_2(\alpha)e_2\sin(\phi-\omega_2). \tag{26}$$

In terms of these quantities, the semimajor axes equations now read

$$\dot{a_1} = +2\,\mu_2\, n_1 a_1 \alpha(\Gamma_c - 2\Gamma),$$
$$\dot{a_2} = -2\,\mu_1\, n_2 a_2(\Gamma_c - 3\Gamma). \tag{27}$$

Therefore the two equations can be combined into a relation which no longer depends on the eccentricities,

$$\frac{\dot{a_1}}{4\mu_2 n_1 a_1 \alpha} + \frac{\dot{a_2}}{6\mu_1 n_2 a_2} = \frac{1}{6}\,\Gamma_c. \tag{28}$$

We can now use Kepler's third law for the osculating orbits to write

$$n_1 a_1 \alpha = (GM_*)^{1/2} a_1^{1/2} a_2^{-1}, \quad n_2 a_2 = (GM_*)^{1/2} a_2^{-1/2}, \tag{29}$$

so that the above relation becomes:

$$M_*^{1/2}\left(\frac{1}{2m_2}\, a_1^{-1/2}\dot{a_1} + \frac{1}{3m_1}\, a_2^{-1/2}\dot{a_2}\right) = \frac{\Gamma_c}{3a_2}. \tag{30}$$

0.3.3 Prime integrals of the equations of motion

Under the approximation where the conjunction term is neglected, it leads to a prime integral of the motion,

$$3m_1\, a_1^{1/2} + 2m_2\, a_2^{1/2} = Q_a\,, \tag{31}$$

where $Q_a = 3m_1 a_{01}^{1/2} + 2m_2 a_{02}^{1/2} = \text{cst}$. This analytical result shows that the main oscillations of the two planet orbits are strongly coupled and are always in opposition. This effect agrees with the numerical integration of Wolszczan [17, Fig. 2]. If we now take the conjunction term (label c) into account, we can neglect the variation of a_2 in the right-hand side of Eq. (30), and we obtain the integral

$$3\mu_1 a_1^{1/2} + 2\mu_2 a_2^{1/2}\left(1 - \frac{1}{2}\,\mu_1\,\frac{2\pi}{P_2}\int \Gamma_c(t)dt\right) = Q_a. \tag{32}$$

For the system PSR1257+12, we have $\mu_1 \approx 10^{-5}$ in the correction term, $2\pi/P_2 = 23.37$ yr^{-1}, while the integral of Γ_c shows two components of amplitudes 0.05 and 0.25 yrs, so that the correction to the fluctuations of semimajor axes (which are themselves very small, of the order of 3×10^{-4}) remains smaller than 4×10^{-5} in proportion. Therefore the prime integral of the motion given by Eq. (31) is valid to a very good approximation.

Let us now consider the eccentricity equations. From the relations $e_1^2 = h_1^2 + k_1^2$ and $e_2^2 = h_2^2 + k_2^2$, we derive

$$e_1\dot{e_1} = h_1\dot{h_1} + k_1\dot{k_1}, \quad e_2\dot{e_2} = h_2\dot{h_2} + k_2\dot{k_2}. \tag{33}$$

From Eqs. (21) we obtain the expressions

$$e_1\dot{e_1} = A_{12}(h_1 k_2 - k_1 h_2) + B_1(h_1\cos\phi - k_1\sin\phi),$$
$$e_2\dot{e_2} = A_{21}(h_2 k_1 - k_2 h_1) + B_2(h_2\cos\phi - k_2\sin\phi). \tag{34}$$

From the very definition of h_j and k_j, we find

$$h_1 k_2 - h_2 k_1 = e_1 e_2 \sin(\omega_1 - \omega_2), \tag{35}$$

so that the derivative of the eccentricities finally read

$$\begin{aligned}
\dot{e_1} &= A_{12}\, e_2 \sin(\omega_1 - \omega_2) + B_1 \sin(\omega_1 - \phi), \\
\dot{e_2} &= -A_{21}\, e_1 \sin(\omega_1 - \omega_2) + B_2 \sin(\omega_2 - \phi).
\end{aligned} \tag{36}$$

We recognize in these expressions two contributions of respectively free and forced oscillations (see [9, 5]). For the PSR 1257+12 system, the free oscillations correspond to long-term motion (periods ~ 6200 yrs and ~ 92000 yrs [9, 5]), while the forced oscillation are on a shorter time scale (period $2\pi/(3n_2 - 2n_1) = 5.586$ yrs) and of smaller amplitude.

Let us consider only the long term motion characterized by the free oscillations. For them, the derivatives of the eccentricities read

$$\dot{e_1} = A_{12}\, e_2 \sin(\omega_1 - \omega_2), \quad \dot{e_2} = -A_{21}\, e_1 \sin(\omega_1 - \omega_2), \tag{37}$$

and we therefore find the relation

$$A_{21}\, e_1 \dot{e_1} + A_{12}\, e_2 \dot{e_2} = 0. \tag{38}$$

It can be integrated in terms of a new conservative quantity (valid for long-term motion),

$$A_{21}\, e_1^2 + A_{12}\, e_2^2 = \text{cst}. \tag{39}$$

With $A_{12} = \mu_2\, n_1 \alpha\, A_2$, $A_{21} = \mu_1\, n_2 A_2$, it yields a prime integral of long-term motion

$$\mu_1 n_2\, e_1^2 + \mu_2 n_1 \alpha\, e_2^2 = \text{cst}. \tag{40}$$

For the PSR 1257+12 planetary system, the relative variation of the semimajor axes is $\approx 2 \times 10^{-4}$, while the eccentricities vary by $\approx \pm 50\%$ on the long term. With $\alpha = a_1/a_2$ and $(n_2/n_1)^2 = (a_1/a_2)^3$ from Kepler's third law, the semimajor axes can therefore be considered as constant to the first approximation, as also shown by numerical integration [5], so that this conservative quantity can be written $m_1\, a_1^{1/2}\, e_1^2 + m_2\, a_2^{1/2}\, e_2^2 = \text{cst}$, or equivalently

$$Q_e = e_1^2 + \frac{m_2}{m_1}\, \alpha^{-1/2}\, e_2^2 = \text{cst}. \tag{41}$$

This result once again shows that the motion of planets B and C are tightly coupled and that their eccentricities oscillate on the long term in exact opposition (see Fig. 1c of Ref. [5]).

0.3.4 Expression for the eccentricity function Γ

Let us come back to the full motion, including the free and forced osciilations of the eccentricities. From equations (36), we derive expressions for $\sin(\omega_1 - \phi)$ and $\sin(\omega_2 - \phi)$, which we may now insert in the expression for the quantity Γ (Eq. 26). This quantity describes the eccentricity-dependent contribution to the variation of the semimajor axes. We find

$$\Gamma = \frac{C_1}{B_1} e_1 \dot{e_1} + \frac{C_2}{B_2} e_2 \dot{e_2} + \left(\frac{C_2 A_{21}}{B_2} - \frac{C_1 A_{12}}{B_1} \right) e_1 e_2 \sin(\omega_1 - \omega_2). \tag{42}$$

When replacing the various coefficients by their expressions, one finds that $C_2 A_{21}/B_2 = C_1 A_{12}/B_1 = A_2$, so that the last term vanishes and Γ is finally given by

$$\Gamma = \frac{e_1 \dot{e_1}}{\mu_2 n_1 \alpha} + \frac{e_2 \dot{e_2}}{\mu_1 n_2}. \tag{43}$$

To the approximation (very good in this context) where n_1, n_2 and α are constant, this function can now be easily integrated as

$$S\Gamma = \int \Gamma(t)dt = \frac{1}{2\mu_2 n_1 \alpha}(e_1^2 + Ke_2^2), \tag{44}$$

where we have set

$$K = \frac{m_2}{m_1}\,\alpha^{-1/2} = \frac{\mu_2}{\mu_1}\,\alpha^{-1/2}, \tag{45}$$

i.e., $K = (m_2/m_1)\sqrt{a_2/a_1}$. With the presently measured values of the masses of planets B and C, $m_B = (4.3 \pm 0.2)\ M_\oplus$ and $m_C = (3.9 \pm 0.2)\ M_\oplus$ [6], this parameter takes, for the PSR 1257+12 system, a value close to 1, namely,

$$K_{\rm PSR} = 1.03 \pm 0.10. \tag{46}$$

We recognize in the term $e_1^2 + Ke_2^2$ the hereabove long-term motion prime integral (Eq. 41).

0.3.5 Relation between semimajor axes and eccentricities

These results allow us to finally integrate analytically the semimajor axes equations in function of the eccentricities and to provide an exact expression for the incomplete Rasio et al. [15] formula. Let us first neglect the conjunction contribution. The equation for the semimajor axis of planet (1) writes

$$\frac{\dot{a_1}}{a_1} = -4\mu_2 n_1 \alpha\, \Gamma(t). \tag{47}$$

Using the expression obtained for Γ, it writes

$$\frac{\dot{a_1}}{a_1} = -4(e_1\dot{e_1} + Ke_2\dot{e_2}). \tag{48}$$

This equation is easily integrated as

$$\ln \frac{a_1}{a_{1*}} = -2(e_1^2 + Ke_2^2), \tag{49}$$

which may be approximated for small values of the eccentricities by

$$\frac{a_1}{a_{1*}} = 1 - 2(e_1^2 + Ke_2^2), \tag{50}$$

where a_{1*} is the (virtual) value of the semimajor axis for which $e_1 = e_2 = 0$. This relation applies to the PSR1257+12 system, in which the eccentricities of planets B and C run between about 0.01 and 0.03. We may also express this result in terms of variations of the semimajor axis and eccentricities with respect to some initial conditions, a_{10}, e_{10} and e_{20},

$$a_1 = a_{10}\{1 - 2[(e_1^2 - e_{10}^2) + K(e_2^2 - e_{20}^2)]\}. \tag{51}$$

This result allows to recover and to generalize the Rasio et al. [15] formula, $a_1 = a_{1*}[1 - 2e_1^2]$, in which e_2 was assumed to be constant (the term $K(e_2^2 - e_{20}^2)$ vanishes in this case). However, it also shows that the correct result cannot be obtained as a mere "test particle" limit $m_1/m_2 \to 0$, since this increases the factor K instead of decreasing it.

Let us finally derive the expression for the variation of the semimajor axis of the outer planet. Its equation reads

$$\frac{\dot{a_2}}{a_2} = 6\mu_1 n_2 \,\Gamma(t) = \frac{6}{K}(e_1\dot{e_1} + Ke_2\dot{e_2}). \tag{52}$$

It is integrated, for small eccentricities, as

$$\frac{a_2}{a_{2*}} = 1 + \frac{3}{K}\,(e_1^2 + Ke_2^2), \tag{53}$$

i.e., respectively to initial conditions,

$$a_2 = a_{20}\left\{1 + \frac{3}{K}\,[(e_1^2 - e_{10}^2) + K(e_2^2 - e_{20}^2)]\right\}. \tag{54}$$

We therefore find that the variation of the semimajor axes of the two planets (except for the small amplitude conjunction effects) depend only on the expression $e_1^2 + Ke_2^2$ which we have found to be a prime integral of the long-term motion. This allows us to conclude (within the approximations used in these analytical calculations) that there is no long-term variation of the semimajor axes of planets B and C. This result fully agrees with the Gozdziewski et al. [5] numerical integrations of their long-term motion, which also derived their stability.

0.4 Application to the PSR 1257+12 system evolution

0.4.1 Semimajor axes, analytic solutions

The evolution of the semimajor axis of the orbit of planet A of the PSR 1251+12 system is mainly determined by the conjunction effects by planets B and C. An analytic expression can be obtained for this evolution [13]: the corresponding period evolution, which is given from Kepler's third law by $a_A^{3/2}$, is plotted in Fig. 2.

The evolution of the semimajor axis of planet B (i.e., planet 1 in the previous two-planet analysis) can be analytically integrated under the form:

$$a_B(t) = a_{B0}\left\{1 - 4\,\mu_C\,\frac{2\pi}{P_{C0}}\,\alpha\left(S\Gamma(t) - \frac{1}{2}S\Gamma_c(t)\right)\right\}, \tag{55}$$

where

$$S\Gamma_c(t) = \int \Gamma_c(t)dt = \frac{\mathcal{P}(\psi(t)) - \mathcal{P}(\psi_0) - \alpha(\cos\psi(t) - \cos\psi_0)}{n_{B0} - n_{C0}}, \tag{56}$$

$$\mathcal{P}(\psi) = \left(1 + \alpha^2 - 2\alpha\cos\psi\right)^{-1/2}, \tag{57}$$

$$\psi(t) = (n_{B0} - n_{C0})t + \psi_0, \tag{58}$$

and

$$S\Gamma(t) = \int \Gamma(t)dt = \frac{1}{2\mu_C n_{B0}\alpha}[(e_B(t)^2 - e_{B0}^2) + K(e_C(t)^2 - e_{C0}^2)]. \tag{59}$$

The corresponding period evolution, which is given from Kepler's third law by $a_B^{3/2}$, is plotted in Fig. 3. It compares satisfactorily with the perturbation of orbital

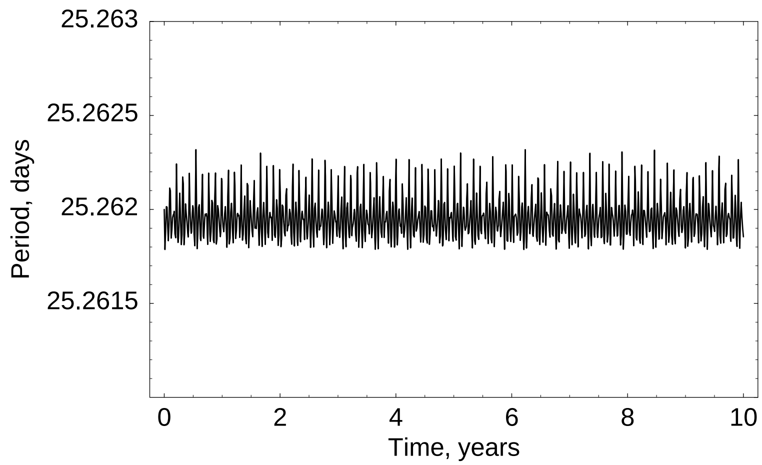

Figure 2: Evolution over 10 years of the period P_A of planet A in the PSR 1257+12 planetary system, plotted from its analytic expression. The fluctuations are mainly due to the conjunction effects by planets B and C. Except for this small oscillation, the period is stable on the long term (he motion of planet A is decoupled from the dynamics of planets B and C in the long-term scale [5]).

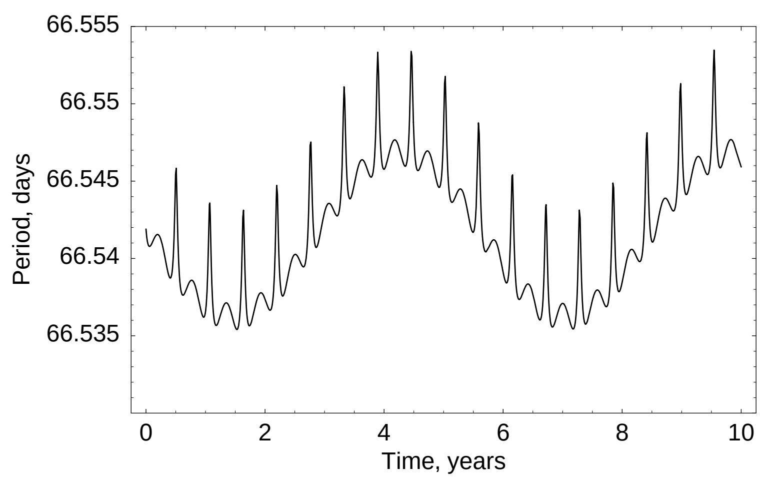

Figure 3: Evolution over 10 years of the period P_B of planet B in the PSR 1257+12 planetary system, plotted from its analytic expression (see text). The fluctuations are due to the mutual effects between planets B and C. Except for this small oscillation, the period is stable on the long term.

period derived from numerical integration of equations of motion by Wolszczan [17, Fig. 2].

Finally, the evolution of the semimajor axis of planet C (i.e., planet 2 in the previous two-planet analysis) can be analytically integrated under the form:

$$a_C(t) = a_{C0} \left\{ 1 + 6\, \mu_B \, \frac{2\pi}{P_{C0}} \left(S\Gamma(t) - \frac{1}{3} S\Gamma_c(t) \right) \right\} . \tag{60}$$

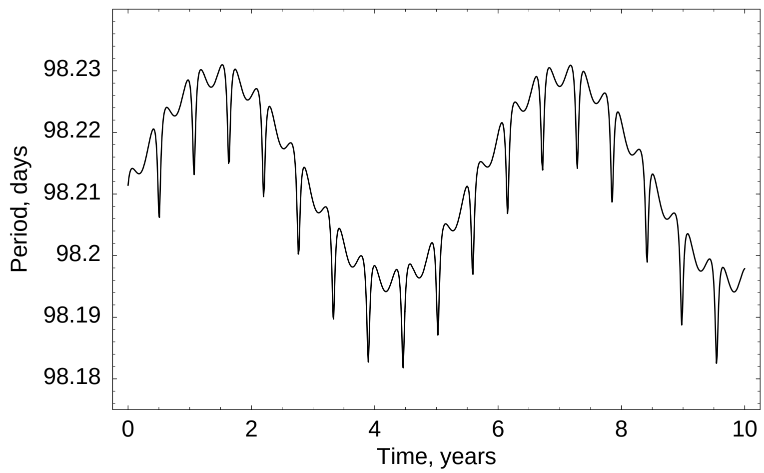

Figure 4: Evolution over 10 years of the period P_C of planet C in the PSR 1257+12 planetary system, plotted from its analytic expression (see text). The fluctuations are due to the mutual effects between planets B and C. Except for this small oscillation, the period is stable on the long term.

The corresponding period evolution, which is given from Kepler's third law by $a_C^{3/2}$, is plotted in Fig. 4. It also compares satisfactorily with the perturbation of orbital period derived from numerical integration of equations of motion by Wolszczan [17, Fig. 2]. The coupling of the two B and C orbits, which leads to motion in opposition of the two planets, is clear on Figs. 3 and 4.

0.5 Consequences for the formation model

0.5.1 Periods.

We can now compare to the expectation of the wave packet formation model ($n^2+n/2$), not only the mean semimajor axis ratio of planets B and C as previously done, but the full variation with time of this ratio. It can be expressed in terms of the ratio of effective quantum numbers n_B and n_C (which is expected to be equal to 7/8). The time evolution of the ratio $(n_B/n_C)/(7/8)$ is shown in Fig. 5.

An important point to notice is that this ratio, apart from very small fluctuations of about 3×10^{-4}, is stable on long time scales. This definitively justifies the validity of the comparison of today's periods with those expected from the formation model, despite the evolution of the system over billion years. Moreover, we can see in this figure that, when taking into account the mutual effects between planets B and C, the minimal difference between the value expected at the end of the formation era and the observed value has been decreased again by a factor of ~ 2, from 4.3×10^{-4} to about 2.5×10^{-4} (while it is 0.6×10^{-4} for the A/C ratio).

This high precision suggests to push further the model and to attempt to obtain a theoretical estimate of the width of the probability peaks. In the framework of the macroscopic quantum-type scale relativity approach, this width is given by a generalized Heisenberg relation $\Delta E \Delta t \approx 2m\mathcal{D}$, with $\mathcal{D} = GM/2w$. In this relation, the time fluctuation Δt may be estimated from the characteristic revival

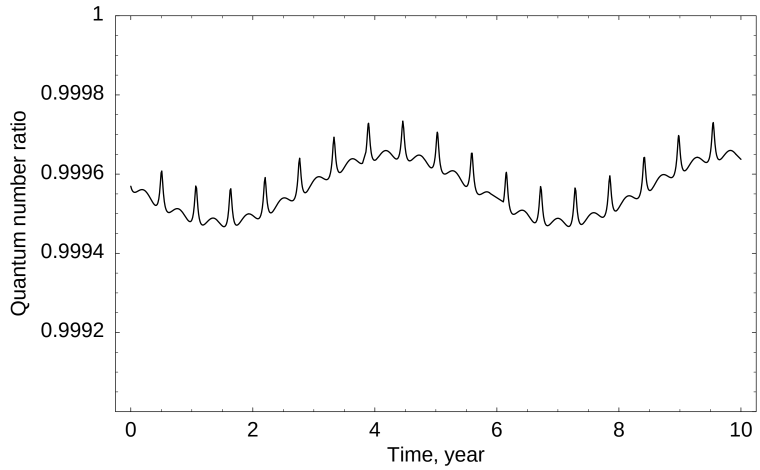

Figure 5: Evolution over 10 years of the ratio $[n_B(t)/n_C(t)]/[7/8]$ for the two planets B and C in the PSR 1257+12 planetary system. The minimal deviation from the expected value 1 reaches 2.5×10^{-4}.

time of localized wave packets. These wave packets are characterized by a revival time $t_{\rm rev} = 2nP/3$ where $P \propto n^3$ is the classical period and a superrevival time $t_{\rm sr} = 3n\, t_{\rm rev}/4$ [1]. Inserting this last value in the macroscopic Heisenberg relation, one obtains $\Delta E/E = 1/(\pi n^3)$, i.e. $\Delta n/n = 1/(2\pi n^3)$. For $n = 8$ this relation gives $\Delta n/n \approx 3 \times 10^{-4}$. Though this is a very preliminary estimate (one should take the self-gravitation of the wave packet into account), the fact that it is of the same order of size as the observed difference seems very encouraging.

0.5.2 Eccentricities

Concerning eccentricities, the precession and near-resonance effects between planets B and C imply an important relative variation of nearly a factor 3. One finds, both in the numerical integration [5] and in the analytic solution [13], that the two planet eccentricities vary in opposition in the range 0.0125 to 0.0285 on a period of about 6200 years (plus a smaller component of period about 92000 years). It would therefore have no meaning to attempt obtaining a theoretical expectation of the individual eccentricities from the formation model. However, we have seen that the two eccentricities combine in terms of a conservative quantity (Eq. 41), $Q_e = e_B^2 + K e_C^2$. With the observationnal values $e_B = 0.0186 \pm 0.0002$, $e_C = 0.0252 \pm 0.0002$, $m_B = (4.3 \pm 0.2)\ M_\oplus$ and $m_C = (3.9 \pm 0.2)\ M_\oplus$ [6] leading to $K = 1.03 \pm 0.10$, one finds $Q_e(\text{obs}) = 0.00100 \pm 0.00002(e) \pm 0.00004(K)$.

Since this quantity is an invariant prime integral of the long term evolution of the system, it must have been fixed at the end of the formation era. One may therefore consider the possibility of deriving it as one among the possible quantized values in the macroscopic quantum-type model. In this aim, let us carry the model farther.

The planetesimal wave packets, being not only quantum-like wave packets but also gravitational structures, are expected to concentrate to form protoplanets, then the planets by final accretion. But during this concentration phase, the conditions under which the geodesic equation (i.e., the fundamental equation of dynamics) may be integrated under the form of a Schrödinger equation still apply. This results in the appearance of a scale factor f on the gravitational coupling constant, and therefore also on the main quantum numbers. This is another manifestation of the combination of the scale invariance of gravitation with a macroquantum Schrödinger description, which differs profoundly from the behavior of the microscopic atomic quantum regime whose scales are fixed by the constancy of the Planck constant $\hbar$. But the quantization of the solutions at each stage and the conservation of energy finally implies that the scale factor be integer (this process is quite similar to that yielding hierarchically embedded levels of organization in planetary systems, as verified in our Solar System, see [10] and Secs. 13.3.6 and 13.4 of [13]).

Applied to the PSR 1257+12 planet formation, the initial ranks $n_B = 7$ and $n_C = 8$ are successively transformed into $n_B = 7f$ and $n_C = 8f$ with f increasing during the wave packet concentration. The coupling constant $w_\odot$ becomes correspondingly $fw_\odot$, then allowing the planet distances GMn^2/w^2 to remain the same. The width of the orbitals, on the contrary, being given by $\sigma_n \sim n^{3/2}$ [7], relatively decreases as $\sigma/a \sim f^{-1/2}$, as expected for such a concentration process.

Let us apply this process to the eccentricity quantization. We have seen that it is given by the amplitude of the Runge-Lenz vector [7], and is therefore expected to be quantized as $e_{kn} = k/n$, with $k = 0$ to $n-1$. The first quantized value is $e = 0$. It yields a satisfying first approximation for the B and C planet eccentricities, $< e > \approx 0.02$. To a better level of precision, the first excited value is obtained for $k = 1$, i.e., $e_B = 1/7f$ and $e_C = 1/8f$, with f integer. For $f = 6$, which corresponds to the beginning of the spatial separation of the orbitals, one obtains $Q_e = (1/42)^2 + K(1/48)^2 = 0.001014$ ($K = 1.03$ fixed), in fair agreement with the observed value 0.001000 ± 0.000018 (the agreement being preserved for all possible values of $K = 0.93$ to 1.13). Reversely, this result can be put to the test in the future since, if correct, it provides a value of K:

$$K = -\frac{e_B^2 - (1/42)^2}{e_C^2 - (1/48)^2} = 1.099 \pm 0.065, \tag{61}$$

from which an estimate of the mass ratio can be obtained,

$$\frac{m_C}{m_B} = 0.965 \pm 0.058, \tag{62}$$

more precise than the presently known value $m_C/m_B = 0.91 \pm 0.10$ [6]. A possible future improvement of the observational values of the eccentricities would still improve this estimate.

Problems and Exercises

Open Problem 1 Use the above model of formation and evolution to predict the distances and periods, in terms of probability density peaks, of other possible planets in the PSR1257+12 planetary system.

Hints: The $n^2 + n/2$ law yields in particular more probable short periods at 0.322 days ($n = 1$), 1.958 days ($n = 2$) and 5.96 days ($n = 3$). Wolszczan et al. [18] have obtained timing data for about 30 successive days. An analysis of the residuals of these data after substraction of the effects of the three known planets (the dispersion of these residual being still larger than the error bars) has yielded a marginal detection (at a significance level of 2.7 σ) of a planet with a period $P = 2.2$ days and a mass 0.035 $M_\oplus$, which is compatible with the $n = 2$ expectation [12]. The absence of planet for $n = 6$ may be well understood from the numerical integrations and simulations of Gozdziewski et al. [5], which show this particular zone to be unstable (although there is an extended stable zone between planets A and B). $\blacksquare$

References

[1] Bluhm R., Kostelecky V.A. & Porter B., 1996, *Am. J. Phys.* **64**, 944.

[2] Brouwer D. & Clemence G.M., 1961, *Methods of Celestial Mechanics* (New York: Academic).

[3] Currie T. & Hansen B., 2007, *Astrophys. J.* **666**, 1232.

[4] Gaeta Z.D. & Stroud C.R., 1990, *Phys. Rev.* **A 42**, 6308.

[5] Gozdziewski K., Konacki M. & Wolszczan A., 2005, *Astrophys. J.* **619**,1084.

[6] Konacki M. & Wolszczan A., 2003, *Ap. J. Lett.* **591**, L147.

[7] Landau L. & Lifchitz E., 1967, *Quantum Mechanics*, Mir, Moscow.

[8] Malhotra R., Black D., Eck A. & Jackson A., 1992, *Nature* **356**, 583.

[9] Malhotra R., 1993, *Ap. J.* **407**, 266.

[10] Nottale L., 1996, *Astron. Astrophys. Lett.* **315**, L9.

[11] Nottale L., 1998, *Chaos, Solitons & Fractals* **9**, 1043.

[12] Nottale L., 2006, in *Workshop Pulsars, Theory and Observations*, 16 and 17 January 2006, Institut d'Astrophysique de Paris, http://lpce.cnrs-orleans.fr/∼pulsar/PSRworkshop/Talks/L.Nottale.ppt.

[13] Nottale L., 2011, *Scale Relativity and Fractal Space-Time: a New Approach to Unifying Relativity and Quantum Mechanics*, Imperial College Press, London, 764 p.

[14] Rasio F.A., Nicholson P.D., Shapiro S.L. & Teukolsky S.A, 1992, *Nature* **355**, 325.

[15] Rasio F.A., Nicholson P.D., Shapiro S.L. & Teukolsky S.A, 1993, in "Planets around pulsars", *ASP Conference Series* **36**, 107.

[16] Wolszczan A. & Frail D.A., 1992, *Nature* **355**, 145.

[17] Wolszczan A., 1994, *Science* **264**, 538.

[18] Wolszczan A., 2000, preprint.