Excerpt from

FRACTAL SPACE-TIME AND MICROPHYSICS
Towards a Theory of Scale Relativity

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Chapter 7

PROSPECTS

7.1. Scale Relativity and Cosmology.

Introduction.

We have shown in the preceding chapter how the principle of scale relativity, even in its “special” version (i.e. *linear* in logarithm form), was able to shed new light on questions such as the charges and masses of elementary particles and the origin and values of some of the fundamental scales of microphysics. Scale invariance alone (as expressed in the standard renormalization group approach or in empirical models based on “scaling”) does not permit one to derive the existence of universal characteristic scales in Nature: on the contrary such scales break the naive scale invariance, and they have to be postulated on the ground of observations. We have demonstrated with scale relativity that going from scale invariance to *scale covariance* provides us with fundamental scales: the GUT scale emerges from the splitting of the Planck scale (since the Planck *length scale* and the Planck *mass scale* become different in the new theory) and the electroweak scale emerges from the postulated value $1/2\pi$ of the charge at infinite energy.

There is however an important question, which we already considered briefly in the previous section and which we want to address here more thoroughly: is it possible that every fundamental scale in Nature emerges from constraints which are set at the smallest scales ? We shall adopt a more general view: namely that the fundamental scales in Nature *are determined by constraints which are set at both the small and the large scales*. This will lead us to considering the largest scales in Nature, i.e., to study the relations of scale relativity with cosmology.

What are the arguments in favour of our conjecture ? One may first recall that, from a methodological point of view, the fractal approach allows one to do that naturally. As remarked above, the standard renormalization group is only a semi-group since one always integrates from the small (length) scale to the large scale. Conversely, fractals are built in the opposite way: a generator is defined at the large scale and used to define the structures at smaller scales. Thus fractals provide us with an opportunity to change the usual reductionist approach of science. We have already encountered some clues for the existence of constraints set at both the small and the large scales:

*Two semi-empirical formulae for the masses of the Z and W bosons have been given in Sec. 6.11 (Eqs. 6.11.7, 6.11.8): one equation relates $(2/3)m_Z+(1/3)m_W$ to the Planck scale, but the other relates $(4/9)m_Z+(5/9)m_W$ to the *electron scale*;[1]

*We have suggested an equation for the fine structure constant (6.11.15) where it is determined by a constraint written at the electron and Bohr scales, rather than integrated from the high energy bare charge.[2]

[1] Subsequent works have not supported the validity of these relations, which have therefore been given up.

[2] This approach has not been followed up in subsequent works.

*A fractal model has been given (Sec. 5.10) suggesting that the *muon* and *tau* masses may be connected with the *electron* mass.[1]

All these arguments lead to the same grand question: how is the scale of the electron, which owns the smallest mass for an elementary charged particle, determined? It has already been suggested that the mass scale of elementary particles is, in the end, determined by the global mass of the Universe: one of the strongest and most mysterious argument in favour of this thesis is the well-known large number coincidence:

$$(\hbar^2 H_0 / Gc)^{1/3} \approx m_e \alpha^{-1} ,$$

where H_0 is the present value of the Hubble constant ($\approx 3 \times 10^{-18}$ s^{-1}), which measures the expansion rate of the Universe. One could consider this coincidence as pure numerology, but, as remarked by Weinberg,[1] such a good agreement with this combination of constants is very improbable. The hope that this relation contains “a fundamental though as yet unexplained truth” led several physicists, in particular Dirac,[2] to proposing alternative cosmologies with varying constants (because H varies with time). We now know that such variations of constants are ruled out by observations (see Sec. 6.7), but the fundamental underlying questions remain to be asked. We shall see afterwards that scale relativity allows us to express these questions in a new way and to reactualize the Mach-Einstein principle.

That scale relativity must have things to say about cosmology is also apparent in the huge number of problems which remain open in today's cosmology and in the fact that most of these problems are related to scale and scaling laws. Let us list them (in a certainly non-exhaustive way):

*What is the value of the Hubble constant ? This is a fundamental scale problem, since the inverse of the Hubble constant gives the age of the Universe (to some multiplicative factor of order unity which is a function of the deceleration parameter q_0 and the cosmological constant Λ).

*What is the value of the cosmological constant ? There has been attempts at understanding the cosmological constant in terms of various physical phenomena, in particular as a vacuum energy density (see e.g. Weinberg[3]).

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[1] A possible justification of this relation, according to which the mu and tau lepton masses scale to lowest order as $m_\mu/m_e = 3 \times 4.1^3$ and $m_\tau/m_e = 3 \times 4.1^5$, has been given in the reference: Nottale L., 2004, *American Institute of Physics Conference Proceedings* **718**, 68-95.

There is however a trivial remark which is seldom made (except in the frame of the static Einstein model of the Universe) on this fundamental, universal and absolute constant: that its dimensional equation is the inverse of the square of a length:

$$\Lambda = \frac{1}{\mathbb{L}^2} \; .$$

The small value of the cosmological constant is connected with the fact that $\mathbb{L}$ is at the cosmological scale. A vanishing cosmological constant, as suggested by quantum gravity arguments,[4,5] corresponds to $\mathbb{L}$ infinite. But if the cosmological constant is finite, this means that there exists an *invariant* length of the order of the radius of the universe, while the universe and any length at the cosmological scale is subject to expansion (compare this paradox to the Planck scale paradox which we have presented in Sec. 6.7). It has already been remarked that there is a relation between scale invariance (and its breaking) and the value of the cosmological constant.[6]

*What determines the characteristic scales of galaxies, clusters and superclusters of galaxies, and large scale structures in the Universe such as the recently discovered Great Wall[7] and the possible 128 h^{-1} Mpc periodicity[8] ? (here h stands for the ratio of the Hubble constant over 100 km/s.Mpc and 1 Mpc= 3.08 x 10^{24} cm). The current (very active) attempts at understanding such questions are once again "reductive": the hope is to get the present structures from a theory of formation and evolution of structures which makes them start from the Big Bang (i.e. at small scale and remote time) and evolve to the present densities and radii.

*Scaling laws, apparently with a high level of universality, have been discovered in the hierarchical distribution of matter in the Universe. In particular the two-point correlation functions for galaxies, groups, clusters and superclusters of galaxies are all characterized by a power law $\xi(r) \propto (r_0/r)^{\gamma}$ with $\gamma \approx 1.8$ for every type of objects. Note that one of the most popular models for the distribution of galaxies is precisely the fractal and multifractal ones,[9-11] and this leads us back to the main theme of this book.

Among the reasons for applying scale relativity to cosmology, there is also the straigthforward argument that high energy particle physics describes the first instants of the Universe, and that the changes we have brought to this domain are expected to change the Big Bang theory. We shall briefly consider this case, but we shall see that scale relativity is also able to bring new insights into the domain of *observational cosmology*. The theory of scale relativity appears as a key, which, constructed to open the door of microphysics, proves capable also of opening another door, that of cosmology.

However we warn the reader that we shall only summarize the main lines of our arguments and results: this theme would need another full book, while the present essay is mainly devoted to the microphysical problem. In particular we shall assume that the reader is aware of the basics of theoretical and observational relativistic cosmology, as described, e.g., in the books of Weinberg[1] and Peebles.[9]

Moreover, this section should be considered as a preliminary model rather than a full theory at the present time. Indeed today's cosmology is described by an extremely coherent theory, Einstein's general (motion) relativity, while we are far from a self-consistent scale relativistic cosmology. We nevertheless hope that the construction we propose hereafter will be correct in its main lines thanks to the fact that scale relativity, as any theory of relativity, yields *universal* constraints (existence of a limiting velocity in motion relativity, of a limiting space-time scale in scale relativity), the consequences of which must apply to every domain of physics.

Reactualization of Mach's principle.

The principle known as Mach's principle, since Einstein's insistance on its importance for the understanding of inertia, actually contains several statements corresponding to different levels of relations between local properties (inertia) and global properties (the Universe).

The first level is the *definition of inertial systems*. Mach's main contribution was his insistence on *the relativity of any motion*. As a consequence, the motion of reference systems in which inertial forces are experienced (e.g., a mass in rotation, more generally accelerated systems) can be defined only *relatively to other masses*. General relativity and the principle of equivalence practically solve the problem: inertial systems are systems which are in free fall in the gravitational field determined, through Einstein's equations, by *the whole distribution of masses in the Universe*. Inertial systems are so defined only locally, because of the locality of the principle of equivalence. In general relativity, the definition of a global inertial system no longer has any physical meaning. There are two drawbacks in this solution.

The first is spin: a particle of vanishing radius may rotate around any axis passing through it, while it will always be considered in free fall according to general relativity.

The second is the apparent contradiction of the purely local nature of the solution brought by general relativity with what is actually observed. Observations seem to point towards the need for a global definition of inertial systems: this is the basis of Mach's proposed solution, that inertial forces must result from gravitational attraction of “distant stars” (Mach's principle), i.e., of distant matter in the Universe. Indeed a kind of “coherence” of inertial systems is observed on very different scales: the system in which we feel no centrifugal force on earth is nearly the same as that in which the sky is seen not to rotate;[1] the axis of the earth, up to precession, is always directed towards the same direction in spite of its motion around the Sun and of the displacement of the Sun in the galaxy.

This problem is solved by the *observed* hierarchical distribution of matter in the universe, but *not solved in principle*. In fact, as recalled in a previous section, inertia is experimentally found to be isotropic to a very high precision[1] ($\delta m/m = 0 \pm 10^{-20}$, but more recent experiments may reach 10^{-24}). Hence the contribution to inertial forces of masses and scales as large as our Galaxy or even the local supercluster must be dominated by contributions of far more distant masses. If Mach's principle is to be implemented in accordance with the principle of equivalence and of its experimental verification, the only acceptable solution is an effect of the Universe as a whole.

This leads us to the *second level* of “Mach's principle”. It was Einstein's initial hope that, if distant masses do determine the inertial systems, they must also determine the amplitude of inertial forces: more precisely, inertial forces being hopefully reduced to gravitational forces due to the Universe as a whole, the ratio of gravitational over inertial acceleration, i.e., the constant of gravitation G may be related to global parameters of the universe. It is well known that this hope was dashed: already in Schwarzschild's solution, there is a central gravitational (therefore inertial) mass in the absence of masses at infinity. In cosmology also, general relativity does not solve the problem in principle, since not all Friedmann-Lemaître (or more generally Robertson-Walker) models of the universe are “Machian”. Some models are devoid of mass; some others, even massive, do not verify the relation between some characteristic mass and length of the Universe which is needed to implement Mach's principle. Let us look at the form expected for such a relation.

Mach's principle may be achieved by requiring that the gravitational energy of interaction of a body with the universe (described to first order approximation as a total mass M situated at an average distance R) cancels its self-energy of inertial origin, $E = m\,c^2$:

$$\frac{GmM}{R} \approx m\,c^2 \;\Rightarrow\; \frac{G\,M}{c^2 R} \approx 1 \;. \tag{7.1.1}$$

The relation obtained is, except for a factor 2, the relation between a mass and its Schwarzschild (blackhole) radius. Hence Mach's (second level) principle is equivalent to the requirement that the Universe as a whole be a black hole.

One of the most detailed Machian model of the Universe was proposed by Sciama.[12,13] He adds to Einsteinian cosmology the requirement that “the gravitational field of the Universe as a whole cancels the gravitational field of local matter, so that bodies are free” and obtains Eq. (7.1.1). In his approach, it becomes very clear that Mach's principle could not be achieved in a scalar theory of gravitation like Newton's theory (indeed the force of the “left” part of the Universe cancels that of the “right” part at any point). Inertial forces may arise as an effect of the Universe only in a vectorial or tensorial theory: as shown by Sciama, inertia is an effect of *induction* of distant matter (in a sense similar to inductive force or current in electromagnetism). The inertial force which appears when we move in a non-inertial frame comes from the gravitational force which arises from the *acceleration* of the whole universe with respect to us.

Let us see what Eq. (7.1.1) does imply from the point of view of cosmological models. Only two solutions are possible: either M and R are constant, and one gets Einstein's model, the only possible static model among Robertson-Walkers' models (with $\Lambda \neq 0$), or the universe is non-static, as indicated by observations, so that R varies with time and Mach's principle can be achieved only in models where the characteristic mass M varies with time as the cosmological scale factor varies. This immediately excludes (in today's standard cosmology) spherical models, which are closed and in which the total mass of the Universe is constant.

In order to characterize ‘Machian’ models, let us define a characteristic mass $M = 4\pi\rho r^3/3$ in terms of the length $r = c/H$; this combined with Eq. (7.1.1) written as a Schwarzschild relation $2GM/c^2r = 1$ yields a density parameter:

$$\Omega = \frac{8\pi G\rho}{3H^2} = 1 \ . \tag{7.1.2}$$

Hence the Einstein-de Sitter model (with $k = 0$) is Machian. It is the only one in which Ω does not vary with time. This may also be seen in the expression for the mass which is observed inside the horizon ($z \rightarrow \infty$) of such a model

$$M_{\mathrm{H}} = \frac{4\,c^3}{G\,H_{\mathrm{o}}} \ .$$

Equation (7.1.2) explains why the problem set by Mach's principle is still with us. One might have been contented with general relativity and considered that the implementation of Mach's principle (second level) is not necessary. But *observations* tell us that Eq. (7.1.2) is true or nearly true: the measured values of Ω fall between 0.2 and 1, the value $\Omega = 1$ being preferred by its last large scale ($\gtrsim$ 100 Mpc) determination using IRAS galaxies and large scale motions.[14] So it seems legitimate to wonder why the observed universe is so close to achieving (or actually achieves) Mach's “second level” principle. Such an agreement reactualizes the impression that implementation of Mach's principle through *exact* equations in agreement with the principle of equivalence is needed indeed.

The third level of "Mach's principle" (the idea of which may be attributed to Einstein) is the conjecture that the mass of elementary particles is related to the whole mass of the Universe. Let us make more specific the meaning of this proposal. The units of mass are arbitrary. In the same way as Mach insisted on the relativity of all motions, he insisted also on the relativity of all masses: for him, a mass cannot be defined alone. One must consider two masses, which are then nothing but the inverse acceleration which they transmit to each other: $m_1\,\gamma_1 = m_2\,\gamma_2 \Rightarrow m_1/m_2 = \gamma_2/\gamma_1$. However special relativity obliges one to make such a viewpoint evolve. Mass is also energy, which may itself take a lot of forms (heat, radiation, kinetic energy...). The evolution must be still more radical when accounting for quantum mechanics. From the constants G, $\hbar$ and c, one may introduce the Planck mass $m_{\mathbb{P}}$ as a natural unit and write Newton's law (according to the relation $Gm_{\mathbb{P}}^2 = \hbar c$) as

$$F = \hbar c \ \frac{(m/m_{\mathbb{P}})\,(m'/m_{\mathbb{P}})}{r^2} \ . \tag{7.1.3}$$

Three situations may have occurred: (i) that no preferential scale of mass exists in Nature: the "third level Mach's principle" would have no meaning; (ii) that a preferential scale exists ("elementary particle"), but that this characteristic mass must precisely be the Planck mass: Eq. (7.1.3) would have accounted for such a situation; (iii) that preferential, universal and elementary masses exist in Nature, with a scale totally different from $m_{\mathbb{P}}$: this is the case "chosen" by Nature, since $m_{\mathbb{P}}/m_e = 2.38952(15) \times 10^{22}$. The origin of this ratio is one of the great mysteries of physics; its huge size suggests comparing it with the only *universal* mass ratio of an equivalent size, the ratio of the mass of the universe over the Planck mass, $M/m_{\mathbb{P}} \approx 10^{61}$.

Let us show how scale relativity allows one to set these problems in a completely renewed way.

Scale relativity and primeval Universe.

It is clear that the new structure of space-time implied by our reinterpretation of the physical meaning of the Planck scale radically changes our view of the primeval Universe. The first new physical law of cosmological importance is the disappearance of the zero instant from meaningful physical concepts. The evolution of the Universe does not begin any more at the instant "$t = 0$" (i.e. $log(t/t_0) = -\infty$), but at the Planck time "$t = \Lambda/c$". However this new structure should not be misinterpreted: in the new theory, the Planck scale owns all the properties of the previous zero instant. This means that temperature, redshift, energy, density and all the quantities Q which were previously diverging as t^{-k} are now diverging when t tends to Λ/c as

$$log\ \frac{Q}{Q_0} = \frac{k\ log\ (t_0/t)}{\sqrt{\left(1 - \frac{log^2(t_0/t)}{log^2(ct_0/\Lambda)}\right)}} \quad .$$

The scale factor of expansion of the Universe is also submitted to a new constraint: it can no longer become smaller than the Planck length Λ. This would be achieved if it initially evolves as $R = \Lambda^{1/2}(ct)^{1/2}$.

In the scale relativistic approach, Lorentz-like δ-factors are introduced, which are identified with variable anomalous dimensions. If one refers oneself to the present epoch $t_o \approx 5\ 10^{17}$ s, one gets $\mathbb{C}_0 = log(ct_0/\Lambda) \approx 61$. Then $z \approx 5$, the redshift of the most distant presently observed objects corresponds to $V = log(t_0/t) \approx 1$, so that $V/\mathbb{C}_0 \approx 1/60$: this corresponds to a negligible correction $\delta \approx 1{+}10^{-4}$. The redshift of the isotropic Microwave Background Radiation, $z \approx 1000$, corresponds to $log(t_0/t) \approx 12$, i.e. to $V/\mathbb{C}_0 \approx 1/5$ and $\delta \approx 1{+}1/50$. This is an interesting result that the “scale relativistic domain” (i.e. here meaning the domain where the consequences of the existence of a *lower limit* to all scales are not negligible) actually begins at about $z \approx 1000$, and then nearly coincides with the radiation dominated era of the Universe in standard cosmology.

The main result of scale relativity concerning the primeval Universe is its ability to solve the causality/horizon problem. Let us recall the nature of this problem. When looking at two directions separated by a large angle, e.g. two opposite directions, we observe regions of the Universe which, for a large enough redshift, may have never been connected in the past. The problem is particularly strong concerning the microwave background radiation, due to its high isotropy[16] ($\delta T/T \lesssim 2\ 10^{-5}$) and its early origin ($z \approx$

1000): at least twenty such independent regions would be observed in the framework of standard cosmology.

Such causally disconnected regions should behave as completely independent universes, and it becomes very strange that no large fluctuation of the microwave background temperature is observed. The solution to this problem is usually searched for in the framework of inflationary cosmology.[17-19] However one may remark that inflation is to some extent an *ad hoc* solution, in particular as concerns its cause (scalar field now unobservable, primordial black holes...), that must be postulated additionally to the presently known content of the Universe. Moreover it does not solve the problem *in principle*: in its framework the *presently observed* regions of the universe would have been causally connected in the past, but this does not remain true in the distant future.

Scale relativity naturally solves the problem because of the new behaviour it implies for light cones. Though there is no inflation in the usual sense, since the scale factor time dependence is unchanged with respect to standard cosmology, there is an inflation of the light cone as $t \to \mathbb{A}/c$.

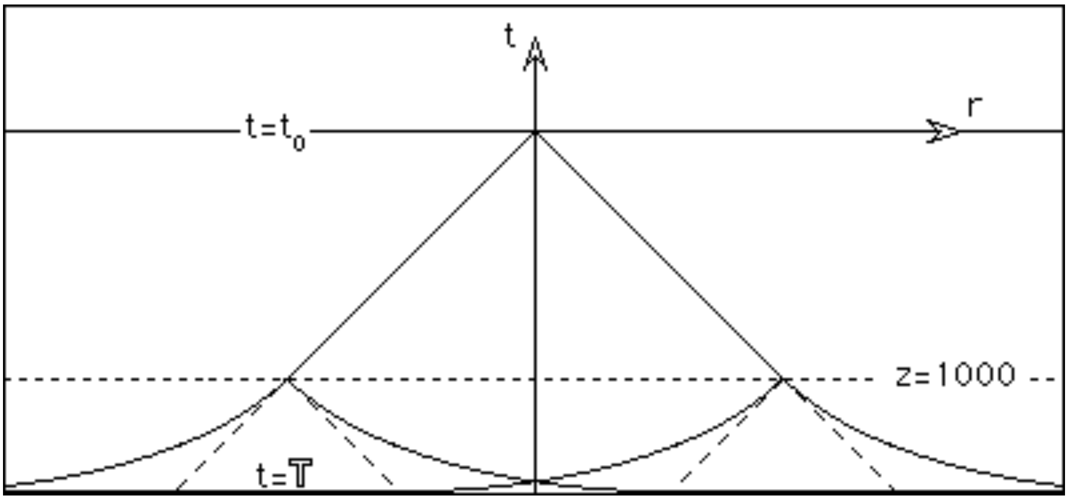

Figure 7.1. Schematic representation of the scale-relativistic flare of light cones in the primeval Universe. Two distant regions of the universe seen in opposite directions are causally disconnected in standard cosmology without inflation, since their past light cones (dotted lines) do not cross. In scale relativity, all points of the Universe become causally connected at the Planck time $\mathbb{T}$.

This may be understood by an analysis of the new properties of space-time at scale $\mathbb{\Lambda}$ implied by Eq. (6.8.1) . The fact that $\mathbb{\Lambda}$ is invariant under dilatations means that when observed at resolution $\mathbb{\Lambda}$, the distance between any two points reduces to $\mathbb{\Lambda}$ itself. Indeed, following our analysis of Chapter 2, the numerical result of a distance measurement is given by a dilatation ρ applied to the basic unit that cannot be taken smaller than the resolution. This means that there is a complete degeneration of space-time when looked at resolution $\mathbb{\Lambda}$ (compare with the partial degeneration of null geodesics on light cones). This property was already possessed by the previous primeval singularity (R=0) but this singularity was actually excluded from the evolution of the Universe in the previous theory: for example an open infinite Universe would have been infinite at any arbitrarily small time $t \neq 0$, while reduced to the singularity at $t = 0$. Here we have a continuous evolution from the particular scale $\mathbb{\Lambda}$ to larger ones. The new light cone evolution is illustrated in Fig. 7.1, where it may be seen how the various light cones flare when $t \to \mathbb{\Lambda}/c$ and cross themselves, allowing causal *connection between any two points of the Universe*. This definitely solves the causality problem.

Scale dependence in present cosmology.

As in microphysics, there is in cosmology a fundamental dependence of physical laws on scale. At scales for which the cosmological principle of homogeneity and isotropy is fulfilled, the Universe is found to be in expansion. Indeed all solutions (except one) of Einstein's equations based on the cosmological principle are non-static. The observation of the universal redshift of galaxies and the redshift-distance relation (Hubble law) indicates that this non-staticity is presently an expansion. This means that, at large scales, all physical variables (distances, time, density, temperature...) vary in terms of a universal scale factor R, which characterizes the Robertson-Walker metric:

$$ds^2 = c^2\, dt^2 - R^2(t)\ dl^2 \ ,$$

where we recall that dl^2 is a spatial element which may take only three forms corresponding to constant curvature spaces (hyperbolic, flat or spherical). For example, one has $T \propto R^{-1}$, $\rho \propto R^{-3}$ in dust universes,

$t \propto R^{3/2}$ in flat models, etc... (see e.g. Ref. 1). Fortunately, this scale factor (the so-called "Universe radius") is directly observable, since it is related to redshift z by the relation

$$\frac{R}{R_0} = (1 + z)^{-1} \quad ,$$

where R_0 is its value at the present instant. Thus this scale dependence is both observed and predicted by theory (general relativity).

The distribution of matter in the Universe is also found to be scale dependent. There is at present no satisfactory theoretical explanation of this "scaling". It is often described in terms of a two-point correlation function,[9,20] $\xi(r)$, which measures the deviation of the observed distribution of galaxies with respect to an uniform (Poissonian) one. Namely, one writes that the probability that one object lies between r and $r+dr$ from another object is given by

$$P(r)\, dr = 4\pi\, r^2 \, [1 + \xi(r)]\, dr \ .$$

It is observationally found that, for most classes of objects of cosmological importance (galaxies, groups and clusters of galaxies, superclusters of galaxies), $\xi(r)$ is well represented by a power law:

$$\xi(r) = \left(\frac{r_o}{r}\right)^{\gamma} \quad ,$$

where the correlation length r_o depends on the type of object considered (about 5 Mpc for giant galaxies, 20 Mpc for clusters), but where $\gamma \approx 1.8$ *whatever the type of object.*

A popular approach to the question of the distribution of galaxies is a fractal and multifractal one.[9-11] The correlation function is related to another measure of correlation[21], the *correlation integral* $C(r)$. It measures the probability of finding another point in a sphere of radius r centred on a point of the distribution, so that

$$\frac{dC(r)}{dr} = 4\pi\, r^2 \, [1 + \xi(r)] \quad .$$

Then the hereabove form observed for $\xi(r)$ means that, for small r, $C(r)$ varies with scale as

$$C(r) \propto \left(\frac{r}{r_o}\right)^D \qquad \text{with} \qquad D = 3 - \gamma \quad .$$

D is the *correlation dimension* and is equal to a fractal dimension in the simplest case. Such a form may be obtained from a renormalization group equation:

$$\frac{dC(r)}{d\ln r} = a + D\ C(r) \ ,$$

where the correlation and fractal dimension D is now interpreted as an anomalous dimension (here $D = \delta$ since this is the dimension of a set of points, i.e., a "dust" of topological dimension $D_T = 0$).

So the value $\gamma = 1.8$ is translated to the fractal and anomalous dimension $D = 1.2$ for the distribution of galaxies. Several models of the formation of a hierarchical distribution of matter, e.g., by fragmentation, naturally yield[10,11] $D = 1$. The unsolved question is why $D = 1.2$, rather than $D = 1$.

However, even the simple fractal model is problematic: it predicts that $1 + \xi(r)$ is a power law, rather than the $\xi(r)$ observed, and it has a constant fractal dimension, while at very large scales one expects to find $D = 3$ (uniformity).

The value $D = 1$ is also encountered for the local distribution of matter observed around the various objects. Hence the observation of flat rotation curves in the outer parts of spiral galaxies[22] leads to the conclusion that they are embedded in supermassive halos of dark matter having a density $\rho \propto r^{-2}$, i.e. a mass distribution $M(r) \propto r^D$ with $D = 1$. In the same way, the observed halos of clusters of galaxies show a similar distribution $M(r) \propto r$ in the mean.[23]

The static non-static relative transition.

There is a fundamental question concerning the expansion of the Universe which is seldom explicitly asked: *where does the expansion stop*?

It is clear from several arguments that a static non-static transition must exist.

First, if the cosmological scale factor R was to be applied to any length in Nature, then it would become a trivial scale factor that would disappear from the equations. It was already remarked by Laplace that Newton's theory of gravitation is scale invariant: “One of the remarkable properties [of Newtonian attraction] is that if the dimensions of all the bodies in the universe, their mutual distances, and their velocities were to increase or diminish proportionately, they would describe curves entirely similar to those which they at present describe; so that the universe reduced to the smallest imaginable space would always present the same appearance to observers. The laws of nature therefore only permit us to observe relative dimension”.[24,10] If Laplace had added the fact that the size of objects is determined either by fields different from the gravitational one, or by the local gravitational field rather than the global, he would have predicted a relation of proportionality between distance and velocity, i.e. the Hubble law.

The expansion of the Universe can indeed be interpreted as a variation with time of the cosmological units relatively to local (atomic) units. Note also that the Friedmann-Lemaître and Robertson-Walker solutions to Einstein's equation are based on a description of the material content of the Universe as a perfect fluid. When applied to the present Universe where the basic constituents are galaxies (provided that some smoothly distributed dark matter should not be the dominant component), this means that galaxies are identified with the basic particles of a gas, so that, as in thermodynamics, the theory is not expected to apply at scales of the order of the particle size.

It is indeed known that a typical giant galaxy like ours (of radius $\approx$10 kpc), even if it is entailed in differential rotation, is globally static. The velocity field of clusters of galaxies shows an external halo which links up to expansion, while there is an inner static region of size of about its core radius (100-200 kpc). This indicates that transition from staticity to non-staticity, i.e. from scale independence to scale dependence, is a *relative* transition.

The staticity of objects under their own gravitational field amounts to writing their equilibrium, i.e., to writing the virial theorem. This leads to a well-known general relation between mass, velocity dispersion and radius (see Ref. 1, p. 477):

$$l \approx \frac{G\, m}{<v^2>} \,. \tag{7.1.4}$$

We suggest that this relation plays the same role in cosmology as the de Broglie length $\lambda = \hbar/mv$ in microphysics. Note indeed that both relations give a length in terms of mass, velocity and a fundamental constant, $\hbar$ at small scale and G at large scale. Also remarkable is the fact that, if one looks for a situation where they would be equal, one gets $m = (\hbar v/G)^{1/2} = m_{\mathbb{P}}\sqrt{(v/c)}$, which is nothing but the Planck mass $m_{\mathbb{P}}$ when $v = c$. So for possible cosmological constituents of mass smaller than the Planck mass (in particular elementary particles), the two scale dependent microphysical and cosmological domains connect, without being separated by a classical scale independent domain (see Fig. 7.2).

The comparison with the fundamental transition lengths of microphysics goes on with the Compton length $\lambda = \hbar/mc$. Making $v = c$ in (7.1.4) yields, up to a factor of 2, another fundamental length of general relativity, namely the Schwarzschild radius corresponding to mass m. (For cosmological constituents as small as elementary particles, for example the isotropic microwave background, the hereabove formula may not apply: anyway in this case, one expects the transition length to be at the microphysical scale).

Up to now it has been assumed that the size of objects were of no direct cosmological importance. We propose rather that the largest static sizes of objects are an essential element for understanding cosmology, since they define a "phase" transition from staticity to non-staticity, or, in other words, a scale of symmetry breaking for scale covariance. We think that it is not by chance that the supermassive dark matter halos or galaxy clusters halos both correspond to fractal dimension D=1 (i.e., as the topological dimension is zero, to anomalous dimension $\delta = 1$) and to the transition region from scale independence to scale dependence. It is remarkable in this respect that the two-point correlation function, when calculated at "small" scale (10 kpc-100 kpc) for, e.g., dwarf galaxies close to giant ones yields $\gamma \approx 2$ rather than 1.8.[20,25]

Let us apply a scale transformation $\rho = r'/r$ to the correlation integral and write it in logarithm form:

$$log \frac{C'}{C_0} = log \frac{C}{C_0} + \delta \; log \; \rho \quad , \tag{7.1.5}$$

$$\delta' = \delta = 1 \qquad (r > l, \; r' > l) \quad .$$

It is certainly clear to the reader that we are now once more in exactly the same situation as in microphysics (but with an inversion between the smallest scales and largest scales). The hereabove scale transformation is a Galilean group transformation which holds for scales *larger* than the "virial length" l, while the anomalous dimension δ jumps from $\delta = 1$ to $\delta = 0$ below this length, which plays the role of a static / non-static transition.

The nature of the cosmological constant.

As in microphysics, we are tempted to conclude that the right structure imposed by the principle of scale relativity is the Lorentz group rather than the Galileo group. Then the whole mathematical development of Chapter 6 is applicable to the cosmological problem, and we arrive at the conclusion that there should exist in Nature an *upper* scale, unpassable, universal, invariant under dilatations (thus in particular invariant under the expansion of the Universe), which would hold all the previous properties of *infinity*. Let us name $\mathbb{L}$ this new length scale.

As in microphysics, we are led to asking ourselves whether a length which already exists in present physics could be identified with this new structure. Remark that the microphysical solution, the Planck length $\mathbb{\Lambda} = (\hbar G/c^3)^{1/2}$, is the only solution that can be constructed with the three basic fundamental constants. Then we can already say that $\mathbb{L}$ should be the product of $\mathbb{\Lambda}$ by a constant, absolute, and pure number $\mathbb{K}$:

$$\frac{\mathbb{L}}{\mathbb{\Lambda}} = \mathbb{K} \quad .$$

The present theory has already introduced an universal, absolute, and unvarying constant with is defined at the cosmological scale: this is the cosmological constant Λ, which is defined as the inverse of the square of a length. As recalled above, this length must be both at the scale of the Universe ($>10^{28}$ cm) and not subjected to its expansion, for general relativity to remain self-consistent. Recall also that the general equations satisfying general covariance are Einstein's equation including a cosmological constant term.

So we propose to reinterpret the cosmological constant as resulting from the existence of the new upper scale $\mathbb{L}$:

$$\mathbb{L} = \frac{1}{\sqrt{\Lambda}} .$$

With this interpretation for $\mathbb{L}$, we get a first estimate for the pure number $\mathbb{K} \approx 10^{61}$: we shall attempt at estimating more precisely its value afterwards. Note in this respect that this interpretation is consistent with the analysis of Ref. 6 and the recent results by Hawking[4] and Coleman[5], who obtained a vanishing cosmological constant from quantum gravity arguments. Indeed the Hawking-Coleman approach remains in the frame of the Galileo group of dilatations, while in our frame the Galileo group corresponds in cosmology to the limit $\mathbb{L} \to \infty$, i.e. $\Lambda \to 0$.

Let us briefly consider the possible implications of this proposal for our understanding of large scale structures and of Mach's principle. A more extensive account is outside the scope of the present book and will be given elsewhere.

Vacuum energy density.

There have been attempts to reinterpret the cosmological constant as vacuum energy density, $\rho_V \approx \Lambda c^2/G$ (see Ref. 3 and references therein), i.e., with our notations,

$$\rho_V = c^2/G\mathbb{L}^2 .$$

The problem encountered with this interpretation is that a calculation of the vacuum energy density in standard quantum theory gives a divergent result. Assuming a cutoff at the Planck scale, we get the “Planck energy density”

$$\rho_{\mathbb{P}} = \frac{c^5}{\hbar G^2} \ .$$

With the current lower limit on the possible values of the cosmological constant, $\Lambda < 3\ 10^{-56}$ cm^{-2}, the two estimations differ[3] by a ratio of $\approx 10^{120}$.

We suggest the following solution to this problem. We assume that the vacuum energy density is *an explicitly scale dependent* quantity for every possible scales in the Universe. The vacuum is clearly one of the cosmological constituents for which scale covariance is unbroken (there is no classical domain). Then it is solution of a renormalization group equation

$$\frac{d\rho_V}{dlnr} = k\ \rho_V \ ,$$

so that it varies between two scales r_1 and r_2 as $(r_1/r_2)^k$. We may now admit that both the above values of the vacuum energy density are correct, one defined at the Planck scale and the other at the cosmological scale. Their ratio is

$$\frac{\rho_{\mathbb{P}}}{\rho_V} = \left(\frac{\mathbb{L}}{\mathbb{\Lambda}}\right)^2 = \mathbb{K}^2 \ ,$$

and we get a self-consistent scheme by taking $k = -2$.[1] This approach is not fully scale-relativistic. Including the scale Lorentz-factors (Chapter 6) yields a vacuum energy density which becomes infinite at $r = \mathbb{\Lambda}$ and null at $r = \mathbb{L}$. In this case the hereabove values correspond to two fundamental length scales, one (microphysical) we have identified with the Grand Unification scale, and a new one (cosmological) we shall come back to hereafter.

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[1] This solution is, however, partially unphysical since it cannot be claimed to be valid between the extreme (Planck and cosmic) scales. Indeed, this would correspond to an energy varying as r (in order to obtain a vacuum energy density varying as r^{-2}), which is the Schwarzschild mass-radius relation. Now this relation is indeed valid at the Planck scale (since the Planck length is half the Schwarzschild radius of a Planck mass) and, as suggested here, at the cosmic scale (which would imply that the universe is in its own black hole horizon, see what follows). A more general solution including a breaking of strict scaling (i.e., the combination of a geometric cosmological constant and of varying vacumm energy density) has been later proposed (see what follows and L. Nottale, 1996, *Chaos, Solitons & Fractals* **7**, 877).

******The Universe at its own resolution*.

Before going on, the nature of $\mathbb{L}$ must be specified. Indeed we cannot be satisfied with the definition of $\mathbb{L}$ as a "length", since the concept of distance in general relativity, and in particular cosmology, depends on the method of measurement. Recall that one defines luminosity distance, angular-diameter distance, proper-motion distance, etc.., which all have different expressions in terms of redshift (see e.g. Ref. 1). So the new upper scale must be characterized, not only by a simple number, but also by a global description of the Universe when seen at that scale. The length $\mathbb{L}$ is actually defined as the "radius" of a Universe which, *when seen at its own resolution, becomes invariant under dilatations*. Only one cosmological solution of Einstein's equations is unaffected by expansion: the Einstein static spherical model. So we suggest that, at resolution $\mathbb{L}$, the Universe is described by the *Einstein spherical model*. But the interpretation is different from that of standard cosmology. Recall that in scale relativity we have changed the law of composition of dilatations, so that the structure of the Universe at the upper scale $\mathbb{L}$ does not impose anything on its structure at any other smaller scale. At resolution $\mathbb{L}$, there is a degeneration of space-time (as at resolution $\mathbb{A}$ and at velocity c). The whole set of various possible models (hyperbolic, flat, spherical) are retrieved at smaller scales, owing to the fact that their properties are defined in a purely local way. We claim that, while an integration of these local properties is approximately correct on 'small' scales, this may no longer be the case when pushing the integration to scales of the size of the Universe itself.

**Mach's principle and large numbers*.

The existence of the universal scale $\mathbb{L}$ allows one to consider Mach's principle in a new way. The main difficulty encountered in previous attempts of implementation of the various levels of Mach's principle was that a *time varying* scale, c/H_0, was used as the fundamental cosmological scale in the equations. We now have a horizon for the Universe which, although it owns all the properties of infinity, is given by a finite, constant and universal measure.

We have seen hereabove that the 'second level' of Mach's principle may be translated by the requirement that the Universe be a black hole.

Applying this requirement to the Universe at resolution $\mathbb{L}$, described by Einstein's model, we find that the maximal separation between any two points is $\pi\mathbb{L}$. Then we expect that the Universe be characterized by an effective mass $\mathbb{M}$ such that

$$\frac{2}{\pi}\,\frac{G}{c^2}\,\frac{\mathbb{M}}{\mathbb{L}} = 1 .$$

This result is self-consistent, since this is exactly the expression for the total mass of Einstein's model. It yields an interpretation for one of the large number coincidences

$$\frac{\mathbb{M}}{m_{\mathbb{P}}} = \frac{\pi}{2}\,\mathbb{K} .$$

The value $\mathbb{K} \approx 10^{61}$ would yield a characteristic mass $\approx 10^{56}$ g $\approx 10^{23}$ solar masses, in agreement with observations ($\approx 10^{11}$ galaxies of 10^{12} solar masses).

The ‘third level’ of Mach's principle has still more radical implications. Its achievement would imply a connection between the mass of elementary particles and the mass of the Universe. Let us suggest a (still very rough) solution to this problem. We start from the fact that the electron is the lightest elementary charged particle. The appearance of the upper scale $\mathbb{L}$ implies the existence of a characteristic minimal energy $E_{\min} = \hbar c/\mathbb{L}$. Now, (i) assume that the mass of the electron is of pure electromagnetic origin. This defines a scale r_0 such that

$$\frac{e^2}{r_0} = m_e\, c^2 .$$

That is, $r_0 = \alpha\,\lambda_c$ is Lorentz's classical radius of the electron. Then, (ii) assume that the gravitational self-energy of the electron at scale r_0 precisely equals the smallest possible energy $\hbar c/\mathbb{L}$. We obtain the equation

$$\frac{G\, m^2(r_0)}{r_0} = \frac{\hbar c}{\mathbb{L}} ,$$

where $m(r_0)$ is the effective mass at scale r_0, i.e., $\alpha^{-1}m_e$, except for the small scale dependence of α (<1%). This reasoning finally yields

$$\alpha \, \frac{m_{\mathbb{P}}}{m_e} \; = \; \mathbb{K}^{1/3} \; . \tag{7.1.6}$$

This is a possible road toward an explanation for Dirac's large number coincidence. The fact that $\mathbb{L}$, instead of c/H, appears there is the key point, since it allows us to have an absolute relation rather than a time-dependent one. The agreement is indeed remarkable: with $\mathbb{K} \approx 10^{61}$, one gets $\mathbb{K}^{1/3} \approx 2 \times 10^{20}$, while $\alpha \, m_{\mathbb{P}}/m_e = 1.7437(1) \times 10^{20}$. Conversely, if we admit this formula to be correct, we find a precise estimate for the fundamental scale factor $\mathbb{K}$:

$$\mathbb{K} = \; 5.3018(10) \times \; 10^{60} \; .[1] \tag{7.1.7}$$

By using, in (7.1.6), the value of the inverse fine structure constant at scale r_0 , $\alpha^{-1}(r_0) = 136.3$ (see Chapter 6), instead of its low energy value, one finds $\mathbb{K} = 5.388 \times 10^{60}$. If our scheme is globally correct, one important problem for physics will be the origin of this pure number. We shall not answer this question here, but only indicate a possible road towards its solution. Equation (7.1.6) may be written in terms of the scale relativistic constants $\mathbb{C}$. We have a universal constant $\mathbb{C}_U = ln\mathbb{K}$, which is related to the electron constant $\mathbb{C}_e = ln(m_{\mathbb{P}}/m_e)$ and to the constant $\mathbb{C}_0$ at scale r_0 by the relations

$$\mathbb{C}_U \; = \; 3 \; (\mathbb{C}_e + ln\alpha) \; = 3 \; \mathbb{C}_0 \; .$$

If we admit the above estimates for $\mathbb{K}$, we get $\mathbb{C}_U = 139.83 \pm 0.01$,[2] which is 2% off the low energy fine structure constant. This opens the hope that, in a way similar to our conjecture for the determination of the electroweak scale from the ‘bare’ charge, $\mathbb{C}_v = \alpha_1{}^{-1}(\mathbb{A}) = 4\pi^2$, the universal constant $\mathbb{C}_U$ can be ultimately determined by the value of the electric charge at scale $\mathbb{L}$.[3]

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[1] Using the 2006 recommended values of the constants, we find $\mathbb{K} = 5.3000(12) \times 10^{60}$.

[2] Using the 2006 recommended values of the constants, we find $\mathbb{C}_U = 139.82281(22)$.

[3] Suggestions of variation of the electric charge (i.e., equivalently, of the fine structure constant) at cosmological scales have been made these last years, but they have not been confirmed. On the other hand, subsequent works in the framework of the scale relativity theory have led to the proposal of a relation between the mass of the electron (considered as mainly of electromagnetic origin) and its charge, that reads to lowest order $\mathbb{C}_e = (3/8)\alpha^{-1}$(L. Nottale, 1994, in "Relativity in General", (1993 Spanish

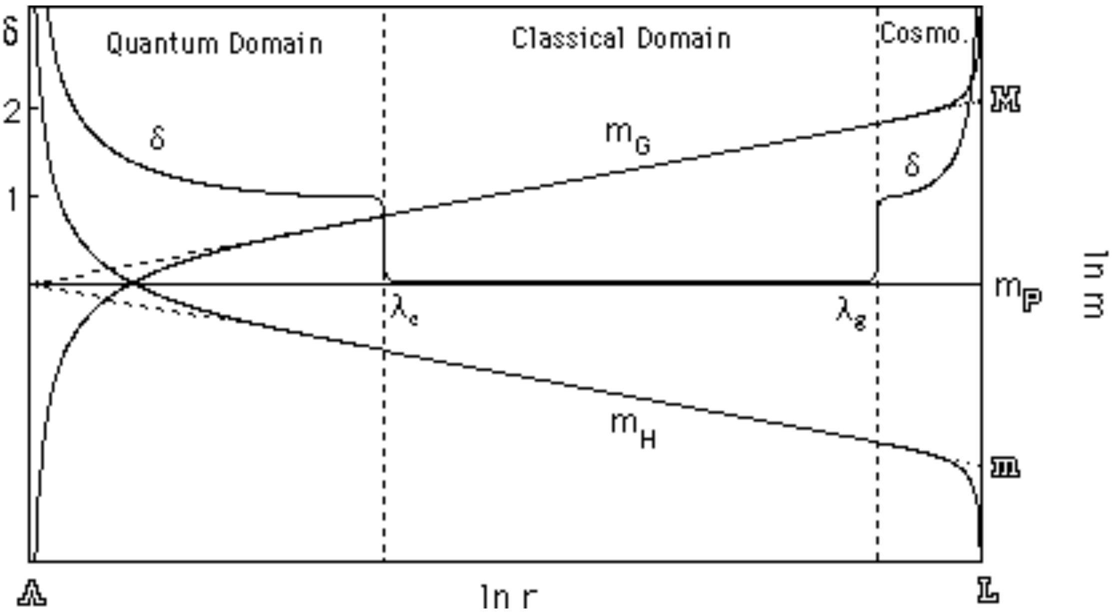

Figure 7.2. The three, quantum, classical and cosmological, domains, from the smallest scale $\mathbb{\Lambda}$ to the largest scale in Nature, $\mathbb{L}$ (according to scale relativity). The two transitions between these domains are not absolute, but *relative* to the system considered (its mass and velocity or velocity dispersion). We have plotted in this diagram the variation of the anomalous dimension δ: in the classical domain, its null value corresponds to scale independence. Also shown is the variation in terms of length scale of the fundamental mass scales given by the generalized Schwarzschild (m_G) and Compton (m_H) formulae. The Planck mass $m_{\mathbb{P}}$ plays the role of a zero point for mass scales.

From the above estimate, one can deduce the maximal length in the Universe, $\pi\mathbb{L}$ = 8.97 Gpc (i.e., 29.2 x 10^9 light years), and the value of the cosmological constant:

$$\Lambda = \frac{1}{\mathbb{L}^2} = 1.36\ 10^{-56}\ \text{cm}^{-2}\ .$$[1]

Relativity Meeting), Salas, Ed. J. Diaz Alonso and M. Lorente Paramo (Frontières), p.121). In this case $\mathbb{C}_U$ can be expressed in terms of the mere fine structure constant, namely, $\mathbb{C}_U = (9/8)\alpha^{-1}+3\ ln\alpha$ (+small corrections).

[1] Using the 2006 recommended values of the constants, we find $\Lambda = 1/\mathbb{L}^2 = 1.36281(41)\ 10^{-56}\ \text{cm}^{-2}$.

This corresponds to the reduced cosmological constant $\lambda_0 = \Lambda c^2/3H_0{}^2 = 0.36\, h^{-2}$.[1]

**Slope of correlation function.*

Let us come back to the question of the distribution of structures in the Universe. In scale relativity the fractal/anomalous dimension now varies with scale. What is the upper distance to be used for the scale δ-factors for this case? We deal with volumic effects, and the volume of the Einstein universe is $2\pi^2\mathbb{L}^3$, instead of the Euclidean $(4/3)\,\pi\ (\pi\mathbb{L})^3$. So the limiting volume-distance is expected to be $\mathbb{L}_V = (3/2\pi^2)^{1/3}\mathbb{L} \approx 0.534\ \mathbb{L}$. Choosing $C = C_0$ for the correlation integral, we get a new relation (see Fig. 7.2):

$$\delta\,(r) \;=\; \frac{1}{\sqrt{\left(\,1 - \dfrac{log^2(r\,/\,\ell)}{log^2(\mathbb{L}_V/\,\ell)}\right)}} \quad ,$$

where ℓ is the static-expansion transition, and r the distance between two galaxies (that indeed plays the role of a resolution). Then the exponent of the two-point correlation function is given by $\gamma = 3-\delta$.

Consider galaxies. Their typical radius is $\ell = 10$ kpc. Combined with the above determination of $\mathbb{L}_V$, this yields $\mathbb{C} = 5.18$ (in logarithm base 10). Then, while $\gamma = 2$ at a scale of 10 kpc, we predict $\gamma = 1.8$ at a scale of 10 Mpc, in good agreement with observations.[20] It is expected to subsequently fall farther ($\gamma = 1.65$ at 30 Mpc, $\gamma = 1.43$ at 100 Mpc).

Consider clusters of galaxies. Their core radius is $\ell \approx 100$ kpc. This yields $\mathbb{C} = 4.18$ (in logarithm base 10). We predict $\gamma = 1.86$ at 10 Mpc, 1.75 at 30 Mpc and 1.56 at 100 Mpc, also in good agreement with what are observed.[26] In such an interpretation, the apparently universal value $\gamma \approx 1.8$ would come from the fact that the distances at which the correlation function is well measured is not an absolute scale but depends on the objects themselves (radius and mean interdistance) in such a way that it roughly corresponds to a given relative scale ($V/\mathbb{C} \approx 0.55$).

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[1] Using the 2006 recommended values of the fundamental constants and recent precise determinations of the Hubble parameter, $H_0 = 73 \pm 3$ km/s.Mpc, we find $\Omega_\Lambda = \Lambda c^2/3H_0{}^2 = 0.38874(12)\,h^{-2}$, which is fairly well supported by the 2006 observational determinations including the WMAP 3 years results, $\Omega_\Lambda(\text{obs}) = (0.384 \pm 0.047)\,h^{-2}$ (Spergel et al., 2006).

**New fundamental scales.*

One of the problems with the simple fractal model is its inability to reconcile the locally fractal distribution with a globally uniform one. In the scale-relativistic approach, δ and γ vary with scale, so that uniformity is reached for $\delta = 3$, i.e., $V/\mathbb{C}=2\sqrt{(2)}/3$. This corresponds to a *scale of transition to uniformity* of about 750 Mpc.

In microphysics, we have seen that the separation between the scale of length-time and the scale of energy-momentum led to the emergence of a new scale which we identified as the Grand Unification scale. The same is true in cosmology: we expect the largest structured scale $\mathbb{l}$ to be given by $log(\mathbb{l} / \ell) = \mathbb{C}/\sqrt{(2)}$. Being concerned with linear structures, one must take $\mathbb{L}_l = \pi \mathbb{L}$ in the computation of $\mathbb{C}$. For giant galaxies, assuming $\ell = 10 \pm 2$ kpc, this yields $\mathbb{l} = 160 \pm 8$ Mpc. This result is remarkable, owing to the recent discovery of a periodicity at 128 h^{-1} Mpc in deep redshift narrow-cone surveys.[8] Identifying the observed and predicted wavelengths provides us with a new precise determination of the Hubble constant:

$$H_0 = 80 \pm 4 \text{ km/s.Mpc} ,$$

in excellent agreement with recent determinations[27,28] from precise indicators, $H_0 = 82 \pm 7$ km/s.Mpc and 72 ± 5 km/s.Mpc. Combined with our estimate of Λ, this would yield a reduced cosmological constant $\lambda_0 = 0.56$: such a high value may help solve the problem of the age of the Universe.[1]

For clusters of galaxies, $\ell \approx 100$ kpc yields $\mathbb{l} = 315$ Mpc. This result, twice the periodicity of galaxies, is consistent with the fact that galaxies are themselves members of clusters with a high rate. Conversely, one may use the constraint that the ratios of the periodicities of various levels of the observed hierarchy must be an integer to derive this hierarchy: we find that the length-scale ratio of one level to the following one must be $k^{2+\sqrt{2}}$, i.e., 10.7 for k=2, 42.5 for k=3, 114 for k=4, which compare well with the observed hierarchy.[29]

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1 This expectation has been totally confirmed in the following years, in particular from 1998 with the first precise observational measurements of the cosmological constant (using type I SNe, COBE and WMAP, gravitational lensing, etc.). With the 2006 value of H_0 (see preceding note), we predict a reduced cosmological constant $\Omega_\Lambda = 0.729 \pm 0.060$, to be compared with the experimental value 0.72 ± 0.03. A precise value of the Hubble parameter may therefore be derived, $h = 0.735 \pm 0.015$.

We conclude this section by once more stressing the fact that the above results should be considered as tentative. A coherent scheme appears to emerge, but it remains to be demonstrated that the application of our approach to cosmology can be made consistent with the firmly established constraints of general relativity.

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