Excerpt from

FRACTAL SPACE-TIME AND MICROPHYSICS
Towards a Theory of Scale Relativity

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Chapter 3
FROM FRACTAL OBJECTS TO FRACTAL SPACES

3.3. Fractal Curves in a Plane.

Let us now come to our first attempts to define fractals in an intrinsic way and to deal with infinities and with their non-differentiability. We first consider the case of fractal curves drawn in a plane. The von Koch construction may be generalized in the complex plane by first giving ourselves a base (or "generator"[21]) F_1 made of p segments of length $1/q$. The coordinates of the p points P_j of F_1 are given, either in Cartesian or in polar coordinates (see Figs. 3.5 and 3.6) by:

$$Z_j = x_j + i\, y_j = q^{-1}.\, \rho_j\, .\, e^{i\,\theta_j} \quad , \quad j = 1 \text{ to } p.$$

Let us number the segments from 0 to p-1. Then another equivalent representation would be to give ourselves either the polar angle of the segment j, say ω_j, or the relative angle between segments $j-1$ and j, say α_j. We further simplify the model by choosing a coordinate system such that F_o is identified with the segment [0,1]. The length of the individual segments is now $1/q$, and the fractal dimension will be given by

$$D = ln\ p/ln\ q.$$

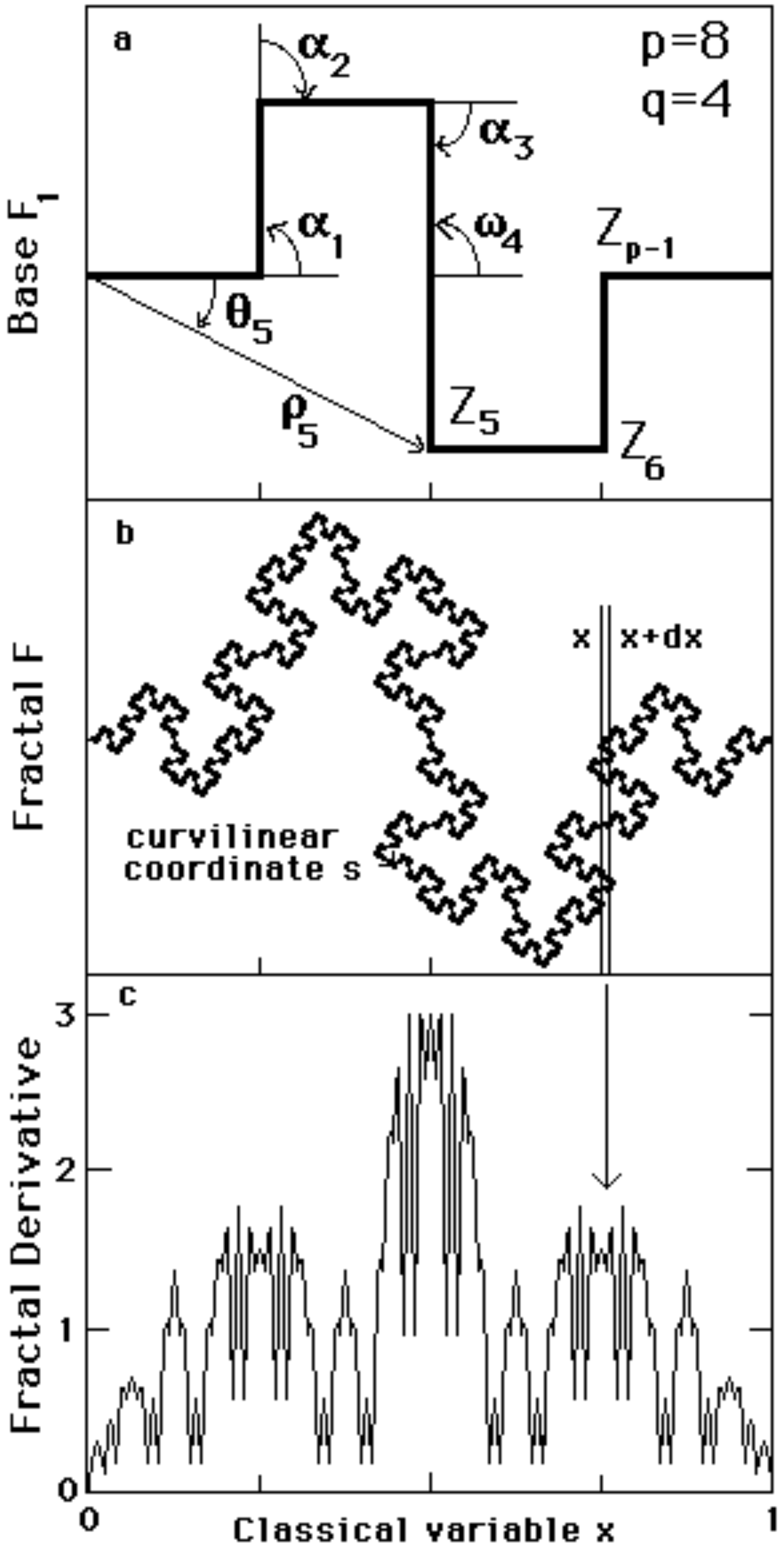


Figure 3.5. Construction of a fractal curve from its generator (or base) F_1. Figure **a** defines the structural constants used in the text. A curvilinear coordinate s is defined on the fractal curve. In (**c**) its fractal derivative is plotted at approximation ξ_4 (see Sec. 3.7).

The following additional relations hold between our “structural constants”:

$$\alpha_j = \omega_j - \omega_{j-1} \quad ; \quad \sum_{j=0}^{p-1} \alpha_j = 0$$

$$\omega_j = \omega_o + \sum_{k=0}^{j} \alpha_k \tag{3.3.1}$$

$$q = \sum_{j=0}^{p-1} e^{i\omega_j} \quad ; \quad Z_{j+1} - Z_j = q^{-1} \; e^{i\omega_j}$$

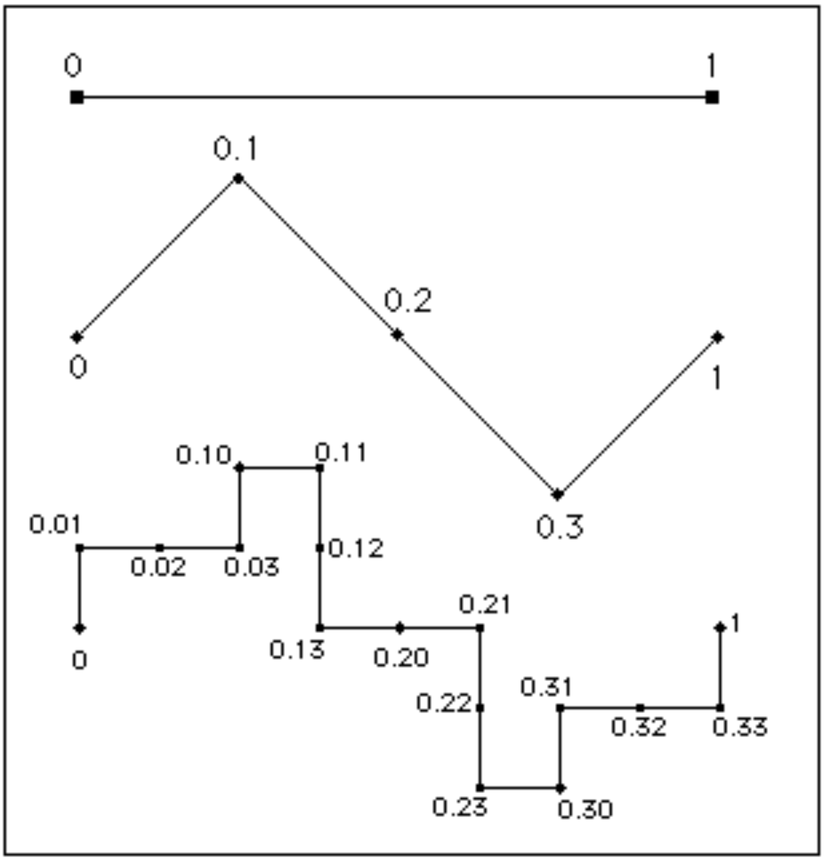


Figure 3.6. Parametrization of a fractal curve in the counting base p (in the case shown here, p=4, $q=2\sqrt[3]{2}$, so that D=4/3; the generator of this fractal has a nonzero slope at origin, so that the slope on the fractal is never defined in this case : see also Fig. 3.9).

These relations include the case when $\omega_o \neq 0$ (for j=0, the first equation writes $\alpha_o = \omega_o - \omega_{p-1}$). The ω_j 's and α_j 's are complete and independent sets of parameters, so that ω_o in the second relation must be expressable in terms of the α_j 's. This is indeed achieved by solving the equation $\Sigma\, sin\, \omega_j = 0$.

A parameter s may now be defined on the fractal and written in terms of its expansion in the counting base p (see Fig. 3.6):

$$s = 0.s_1 s_2 ... s_k ... = \Sigma_k\, s_k\, p^{-k} \quad , \tag{3.3.2}$$

with each s_k taking integer values from 0 to $p-1$. This parameter is a normalized curvilinear coordinate on the fractal curve. The hierarchy of its figures reproduces the hierarchical structure of the fractal. This allows us to write the fractal equation in the form[11]

$$Z(s) = Z_{s_1} + q^{-1}\ e^{i\omega_{s_1}} \Big[\, Z_{s_2} + q^{-1}\ e^{i\omega_{s_2}} [\, Z_{s_3} + ...\,]\Big] \ .$$

We now set

$$\varphi_{s_k} = \omega_{s_1} + \omega_{s_2} + ... + \omega_{s_{k-1}} + \theta_{s_k} \quad ,$$

and the parametric equation of the fractal becomes

$$Z(s) = \sum_1^{\infty} \rho_{s_k}\ e^{i\varphi_{s_k}}\ q^{-k} \quad . \tag{3.3.3}$$

This equation may still be generalized to the case where some additional transformation is applied to the generator (for example some fractals are constructed by alterning the orientation of the generator). This may be described by an operator $S_j = e^{\sigma_j}$, so that one obtains a generalized equation:

$$Z(s) = \sum_1^{\infty} \rho_{s_k}\ e^{\Sigma_j \sigma_{s_j} + i\varphi_{s_k}}\ q^{-k} \quad .$$

The above equations yield an “external” description of the fractal curve, in which, for each value of the curvilinear coordinate s, the two coordinates $x(s)$ and $y(s)$ in the plane may be calculated ($Z(s) = x(s) + i\, y(s)$). In terms of s, $x(s)$ and $y(s)$ are *fractal functions*, for which successive approximations $x_n(s)$ and $y_n(s)$ may be built. Though only one value of x and y corresponds to each value of s (while the reverse is false), their fractal character is revealed by the divergence of their slope when $n \to \infty$. Their fractal dimension is the same as that of the original fractal curve. This is illustrated in Fig. 3.7 for the fractal curve of Fig. 3.5.

The structure of Eq. (3.3.3) is remarkable, since it evidences the part played by p on the fractal and q in the plane: $(s = \sum s_k p^{-k}) \leftrightarrow [Z(s) = \sum C_k(s) q^{-k}]$. An "intrinsic" construction of the fractal curve may also be made.[11] Placing ourselves on F_n, we only need to know the change of direction from each elementary segment of length q^{-n} to the following one. On the fractal generator F_1, these angles have been named α_i. The problem is now to find $\alpha(s)$.

The points of F_n which are common with F (those relating the segments) are characterized by rational parameters s written with n figures in the counting base p, $s=0.s_1 s_2 ... s_n$. Let us denote by s_h the last non-null digit of s, i.e.

$$s = s_1 / p + s_2 / p^2 + ... + s_h / p^h \quad .$$

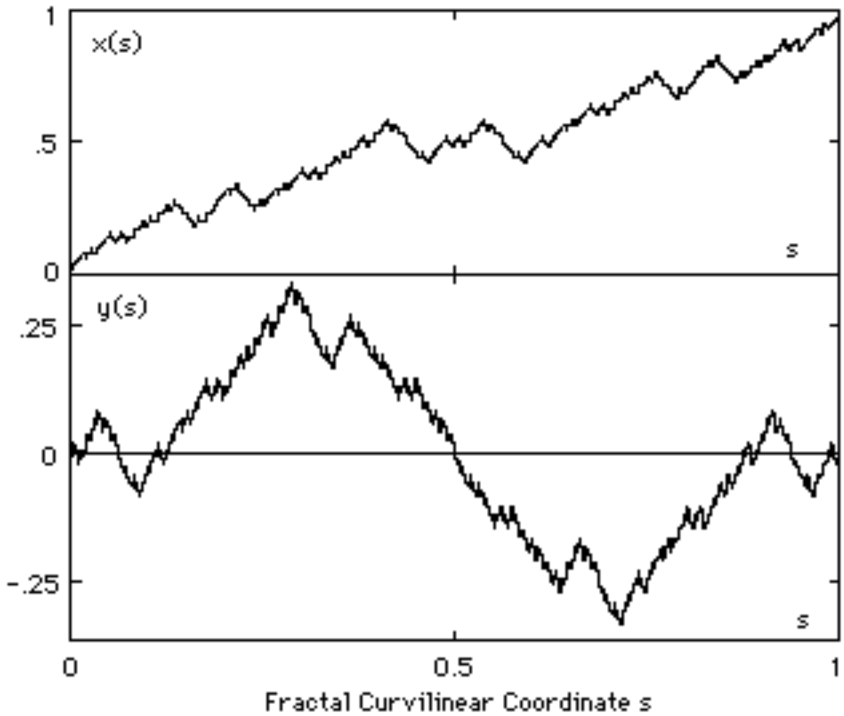


Figure 3.7. The coordinates *x* and *y* of the fractal curve of Figs. 3.5 and C1 in terms of the normalized curvilinear coordinate intrinsic to the fractal, *s*. The function *x(s)* and *y(s)* are fractal functions which themselves vary with scale.

It is easy to verify, provided that $\alpha_0 = \omega_0 - \omega_{p-1} = 0$ (which is a necessary condition for self-avoidance), that the relative angle between segment number $(s.p^n - 1)$ and segment number $(s.p^n)$ on F_n is given by[11]

$$\boxed{\alpha(s) = \alpha_{s_h}} \quad . \tag{3.3.4}$$

This formula completely defines the fractal in a very simple way, uniquely from the (p-1) structural angles α_i, and independently of any particular coordinate system in the plane (x,y). Drawing of fractal curves based on (3.3.4) is at least 5 times faster than from (3.3.3).

Let us end this section by considering another generalization of the fractal construction. The von Koch-like fractals are discontinuous in scale, because of the discreteness of the method that consists in applying generators. Continuity of the construction may be recovered by introducing intermediate steps between F_n and F_{n+1}. We define a sublevel of fractalization, k, such that $0 \leq k < 1$, and we generalize (3.3.4) in the following way: placing ourselves on F_n,

$$\text{if } s_n = 0 \quad \text{then} \quad \alpha(s) = \alpha_{s_h} \quad ,$$
$$\text{if } s_n \neq 0 \quad \text{then} \quad \alpha(s) = k\, \alpha_{s_n} \quad .$$

The result is illustrated in Figs. 3.8 and C2, in which a fractal curve is plotted in terms of the space variable x and of the scale variable $ln\delta x$.

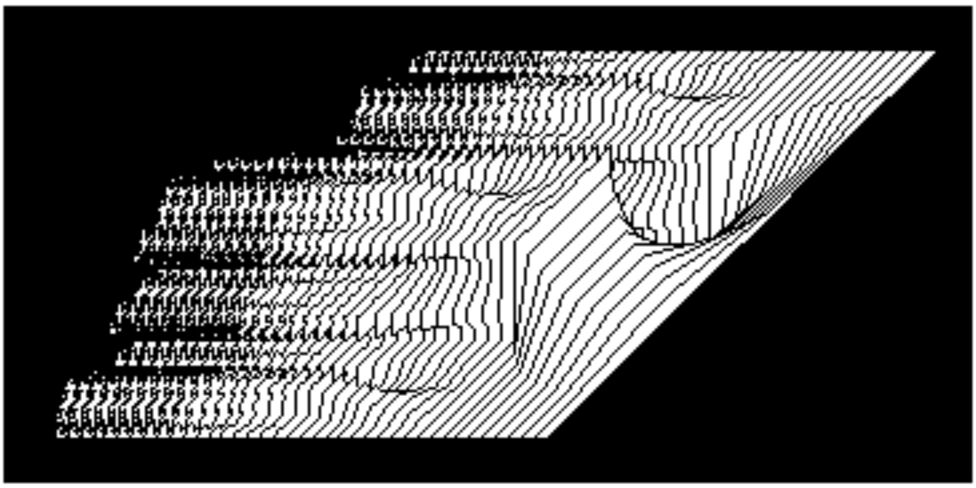


Figure 3.8. The fractal curve of Fig.3.5 in terms of resolution (see also Fig. C2).

We may now use this method to illustrate the periodic self-similarity of fractals: we show in Fig. 3.9 the result of a zoom on two different fractals, one with zero slope at the origin of its generator, the other with a nonzero slope. In the first case, zooming amounts to a translation, while in the second case there is a never-ending *local* rotation at the origin, while the *global* shape of the fractal is conserved.

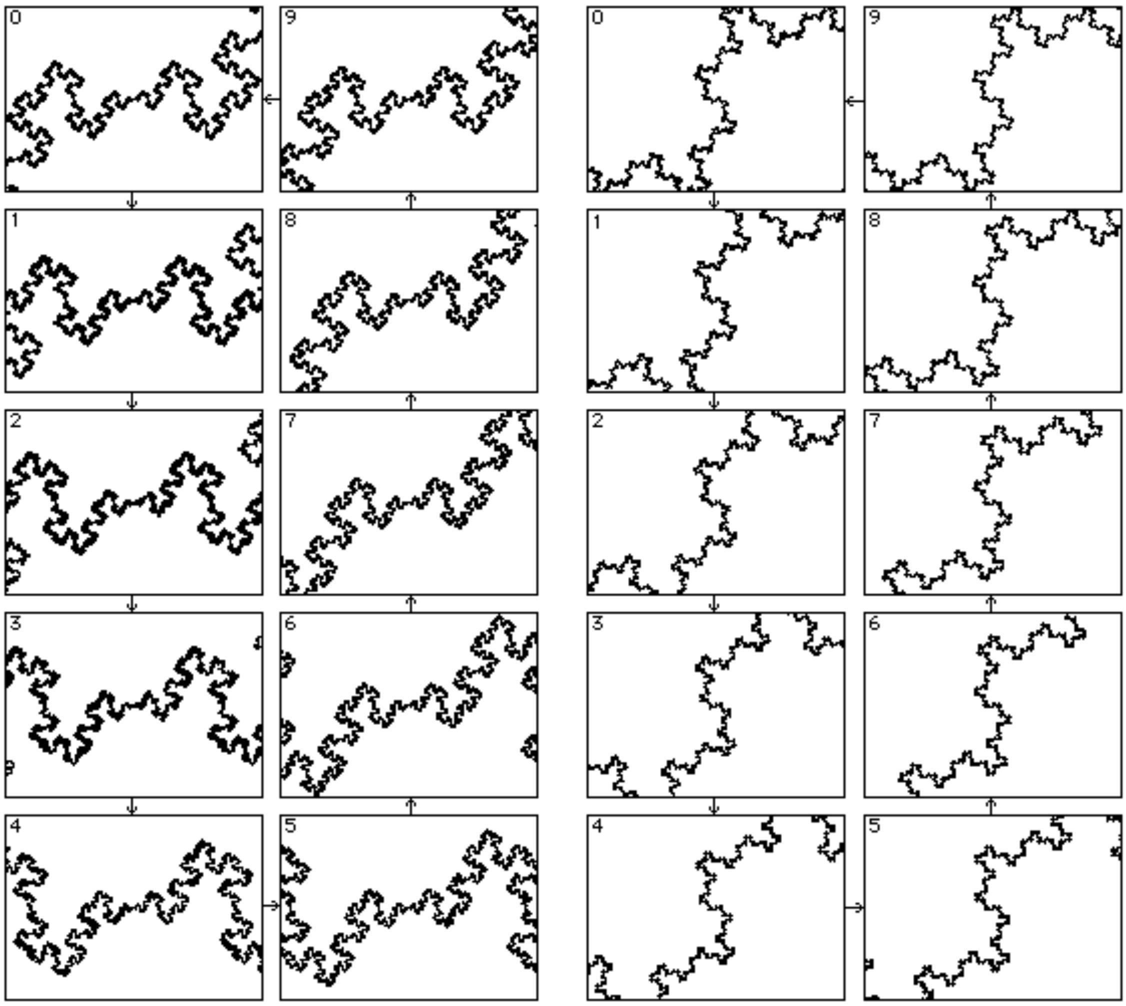


Figure 3.9. Zoom on two fractal curves. Figures 0 to 9 constitute a movie; the subsequent Fig. 10 is exactly identical to Fig. 0.

3.4. Non-Standard Analysis and Fractals.

We have proposed[11] to deal with the infinities appearing on fractals, and then to work effectively on the actual fractal F instead as on its approximations F_n by using Non-Standard Analysis (NSA).

It has been shown by Robinson[27] that proper extensions $^*\mathbb{R}$ of the field of real numbers $\mathbb{R}$ could be built, which contain infinitely small and infinitely large numbers. The theory, first evolved by using free ultrafilters and equivalence classes of sequences of reals[28], was later formalized by Nelson[29] as an axiomatic extension of the Zermelo set theory. We do not intend to give here a detailed account of this field which is now developed as a genuine new branch of mathematics; we shall just recall the results which we think to be most relevant for application to fractals.

Let us briefly recall the ultrapower construction of Robinson. Though less direct than the axiomatic approach (which actually has compacted into additional axioms all the essential new properties of Robinson's construction), it allows one to get a more intuitive contact with the origin of the new structure. Indeed the new infinite and infinitesimal numbers are built as equivalence classes of sequences of real numbers, in a way quite similar to the construction of $\mathbb{R}$ from rationals. So, in the end, some of the ideal character of the new numbers is found to be already present in real numbers.

Let $\mathbb{N}$ be the set of natural numbers. A free ultrafilter U on $\mathbb{N}$ is defined as follows.

U is a non empty set of subsets of $\mathbb{N}$ [$P(\mathbb{N}) \supset U \supset \varnothing$], such that:

(1) $\varnothing \notin U$
(2) $A \in U$ and $B \in U \Rightarrow A \cap B \in U$.
(3) $A \in U$ and $B \in P(\mathbb{N})$ and $B \supset A \Rightarrow B \in U$.
(4) $B \in P(\mathbb{N}) \Rightarrow$ either $B \in U$ or $\{j \in \mathbb{N} : j \notin B\} \in U$, but not both.
(5) $B \in P(\mathbb{N})$ and B finite $\Rightarrow B \notin U$.

Then the set $^*\mathbb{R}$ is defined as the set of the equivalence classes of all sequences of real numbers modulo the equivalence relation:

$$a \equiv b \quad , \quad \text{provided} \quad \{j : a_j = b_j\} \in U \quad ,$$

a and b being the two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of $^*\mathbb{R}$ if it holds termwise for a set of indices which belongs to the ultrafilter. For example:

$$a < b \quad \Leftrightarrow \quad \{j \;:\; a_j < b_j\} \in U \;.$$

$\mathbb{R}$ is isomorphic to a subset of $^*\mathbb{R}$, since one can identify any real $r \in \mathbb{R}$ with the class of sequences $Cl(r,r,...)$. It is the axiom of *maximality* (4) which ensures $^*\mathbb{R}$ to be an ordered field. In particular, thanks to this axiom, a sequence which takes its values in a finite set of numbers is equivalent to one of these numbers, depending on the particular ultrafilter U. This allows one to solve the problem of zero divisors: indeed the fact that (0,1,0,1,...).(1,0,1,0,...) = (0,0,0,...) does not imply that there are zero divisors, since axiom (4) ensures that one of the sequences is equal to 0 and the other to 1.

That $^*\mathbb{R}$ contains new elements with respect to $\mathbb{R}$ becomes evident when one considers the sequence $\{\omega_j = j\} = \{1,2,3,..\ n,\ ...\}$. The equivalence class of this sequence, ω, is larger than any real. Indeed, for any $r \in \mathbb{R}$, $\{j : \omega_j > r\} \in U$, so that whatever $r \in \mathbb{R}$, $\omega > r$. It is straightforward that the inverse of ω is infinitesimal.

Hence the set $^*\mathbb{R}$ of hyper-real numbers is a totally ordered and non-Archimedean field, of which the set $\mathbb{R}$ of standard numbers is a subset. $^*\mathbb{R}$ contains infinite elements, i.e. numbers A such that $\forall n \in \mathbb{N},\ |A| > n$ (where $\mathbb{N}$ refers to the set of integers). It also contains infinitesimal elements, i.e. numbers ε such that $\forall n{\neq}0 \in \mathbb{N},\ |\varepsilon| < 1/n$. A finite element C is defined: $\exists n \in \mathbb{N},\ |C| < n$. Now all hyper-real numbers may be added, substracted, multiplied, divided; subsets like hyper-integers $^*\mathbb{N}$ (of which $\mathbb{N}$ and the set of infinite hyper-integers $^*\mathbb{N}_\infty$ are subsets), hyper-rationals $^*\mathbb{Q}$, positive or negative numbers, odd or even hyper-integers, etc...may be defined, and more generally most standard methods and definitions may be applied in the same way as for the standard set $\mathbb{R}$. But the different sets or properties are classified as being either internal or external.[28,29]

An important result is that any finite number a can be split up in a single way as the sum of a standard real number $r \in \mathbb{R}$ and an infinitesimal number $\varepsilon \in \mathfrak{I}$: $a=r+\varepsilon$. In other words the set of finite hyper-reals contains the ordinary reals plus new numbers (a) clustered infinitesimally closely around each ordinary real r. The set of these additional numbers $\{a\}$ is called the monad of r. More generally, one may demonstrate that any hyper-real number A may be decomposed in a single way as $A=N+r+\varepsilon$, where $N \in {}^*\mathbb{N}$, $r \in \mathbb{R} \cap [0,1[$ and $\varepsilon \in \mathfrak{I}$.

The real r is said to be the "standard part" of the finite hyper-real a, this function being denoted by $r=\boldsymbol{st}(a)$. This new operation, "*take the standard part of*", plays a crucial role in the theory, since it allows one to solve the contradictions which prevented previous attempts, such as Leibniz's, to be developed. Indeed, apart from the usual strict equality "=", one introduces an equivalence relation, "≈", meaning "infinitely close to", defined by $a \approx b \Leftrightarrow \boldsymbol{st}(a-b)=0$. Hence two numbers of the same monad are infinitely close to one another, but not strictly equal.

A practical consequence is that a very large domain of mathematics may be reformulated in terms of NSA, in particular, concerning physics, the integro-differential calculus.[28] The method consists of replacing the Cauchy-Weierstrass limit formulation by effective sums, products and ratios involving infinitesimals and infinite numbers, and then taking the standard part of the result. Hence the derivative of a function will be defined as the ratio

$$df/dx = \boldsymbol{st}\Big\{[f(x+\varepsilon) - f(x)] / \varepsilon\Big\} \quad ,$$

with $\varepsilon \in \mathfrak{I}$, provided this expression is finite and independent of ε. The integral of a function is defined from an infinitesimal partition of the interval $[a,b]$ in an infinite number $\boldsymbol{\omega}$ of bins, as a sum

$$\int_a^b f(x)\, dx = st\Big(\sum_{i=1}^{\omega} f(x_i)\, \delta x_i \Big) \quad ,$$

provided it is finite and independent of the partition. The number of bins, infinite from the standard point of view, is assumed to be a given integer belonging to ${}^*\mathbb{N}_\infty$. It is said to be *-finite. Such a method allows, more generally, a new treatment of the problem of infinite sums. A summation from 0 to ∞ may be replaced by a summation over an *-finite number of terms ranging from 0 to $\boldsymbol{\omega} \in {}^*\mathbb{N}_\infty$. The sum will be said to converge if for different $\boldsymbol{\omega}$'s, its standard part remains equal to the same finite number.

Thanks to its ability to deal properly with infinite and infinitesimals, Non-Standard Analysis is particularly well adapted to the description of fractals. To this purpose we have proposed[11] to continue the fractalization process (F_o, F_1, ..., F_n, ...F) up to an *-finite number of stages $\boldsymbol{\omega}$. This yields a curve $F_{\boldsymbol{\omega}}$, from which the fractal F is now defined as

$$F = \boldsymbol{st}\,(F_{\boldsymbol{\omega}}) \quad . \tag{3.4.1}$$

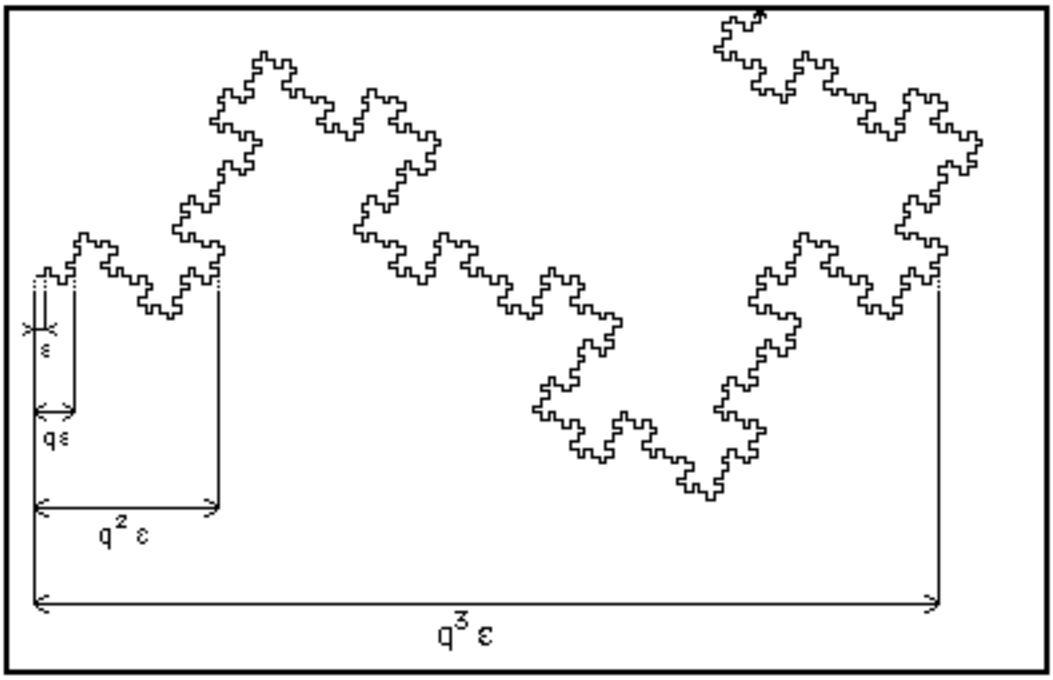


Figure 3.10. Infinite magnification of a non-standard fractal curve F_ω. The elementary segment has a length $\varepsilon = q^{-\omega}$. The whole figure is an internal structure of the zero point.

This means that we define an *-curvilinear coordinate from its expansion in the base p:

$$^*s = 0.s_1...s_k...s_{\boldsymbol{\omega}} = \sum_{k=1}^{\omega} s_k\, p^{-k}$$

and that the equation of $F_{\boldsymbol{\omega}}$ is given by Eq. (3.3.3) now summed from 1 to $\boldsymbol{\omega}$, $Z(s)$ being its standard part. One of the main interests of the introduction of the curve $F_{\boldsymbol{\omega}}$ is that it contains and sums up all the properties of each of the approximations F_n, and also of their limit F. Moreover a meaning may now be given to the length of the fractal curve. The length of F_ω is a number of $^*\mathbb{R}_\infty$:

$$L_\omega = L_o\,(p/q)^\omega = L_o\, q^{\omega(D-1)},$$

and the (non-renormalized) curvilinear coordinate on F_ω is $\xi = {}^*s.L_\omega$. Between 0 and s, it is made of $s.p^\omega$ segments of length $q^{-\omega}$. The "surface" of F_ω may now also be defined as

$$S_\omega = L_o\,(p/q^2)^\omega = L_o\, q^{\omega(D-2)} \quad ,$$

which is an infinitesimal number when D<2. When taking the standard part of all these quantities, one finds again that the length is undefined (infinite) and the surface is null for D<2 and finite for D=2 (i.e. a curve in $\mathbb{R}^2$ becomes plane-filling).

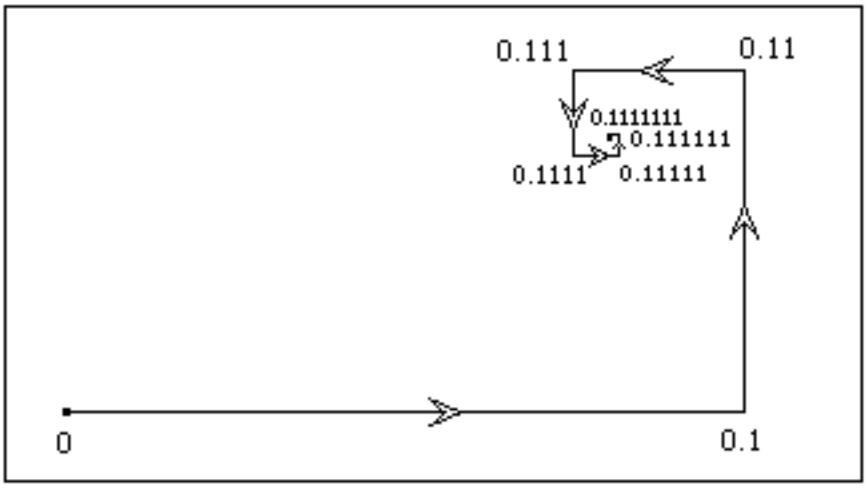


Figure 3.11. Schematic representation of the way to the point of intrinsic coordinate 0.11111...=1/(p−1) on a fractal curve: it is reached by an infinite spiral, so that the slope of the fractal curve cannot be defined for this point.

The non-differentiability of the fractal is now visualizable in a new way, by the fact that any standard point of the fractal may be considered as structured: when viewed with an infinite magnifier, it is found to contain all the values of the slopes owned by the complete fractal[11] (see Fig. 3.10). However we have made the remark that a kind of differentiability can be defined for fractals, which we have called ε-differentiability.[11] It consists in imposing that any part of the fractal magnified by q^ω be differentiable:

indeed there is an infinite number of curves F_{ω}, the standard part of which is the same fractal F; an ε-differentiable one may be *a priori* chosen. This is in fact equivalent, from a practical viewpoint, to imposing that each approximation F_n be differentiable, which is always a possible choice.

This concept of ε-differentiability has been recently clarified and generalized by Herrmann.[30] The consequence for applications to physics is not negligible: this means that the non-differentiable fractal may be built as the limit of a family of differentiable curves, for which the usual integro-differential formalism may then be recovered (see Sec. 3.8).

Let us close this section by an additional illustration of the nature of the non-differentiability of fractals. Consider the point of curvilinear coordinate $s = 1/(p-1) = 0.1111..$ on a fractal curve such that $\omega_0=0$ and $\omega_1=\pi/2$. As shown in Fig. 3.11, this point is reached as the limit of an infinite spiral; this is also the case of most points of a fractal, with most of the time far more complicated patterns.

3.5. Fractal Curves in Space.

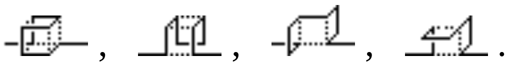
The above results concerning the equations of fractal curves in the plane are easily generalizable to curves drawn in higher dimensional space (see Figs. 3.12 and C3). In $\mathbb{R}^3$ the rotation complex operators $e^{i\omega_k}$ are simply replaced by 3-dimensional rotation matrices R_k. The generator F_o is defined by the coordinates of its p points, $U_k = (X_k, Y_k, Z_k)$ and the point of parameter s will be defined by the vector[11,31]

$$U(s) = \sum_k R_{s_k} R_{s_{k-1}} \ldots R_{s_1} U_{s_{k+1}} q^{-k} \quad . \qquad (3.5.1)$$

In the same way, if one gives oneself the relative rotation matrices A_k on the generator, the intrinsic equation of the fractal is given by the last non-null figure (of rank h) of the s -expansion :

$$A(s) = A_{s_h} \quad . \qquad (3.5.2)$$

A particularly interesting subclass of such fractal curves, concerning the physical aims of the present book, is the class of curves of fractal dimension 2, since, as will be seen in the following sections, this is the universal fractal dimension of particle paths in quantum mechanics. They are built from $p=q^2$ segments of length $1/q$ (in the case of perfect self-similarity). The case of orthogonal generators in $\mathbb{R}^3$ is particularly simple. For example, with q=3 (and thus p=9), one can construct, among others, the following generators:



Thanks to the relation $2(3^2-1) = 4^2$, one can combine them to obtain a large class of (q=4, p=16) generators, for example:

and several other combinations. We give in Fig. 3.12 some examples of such fractal curves drawn to higher order approximations.

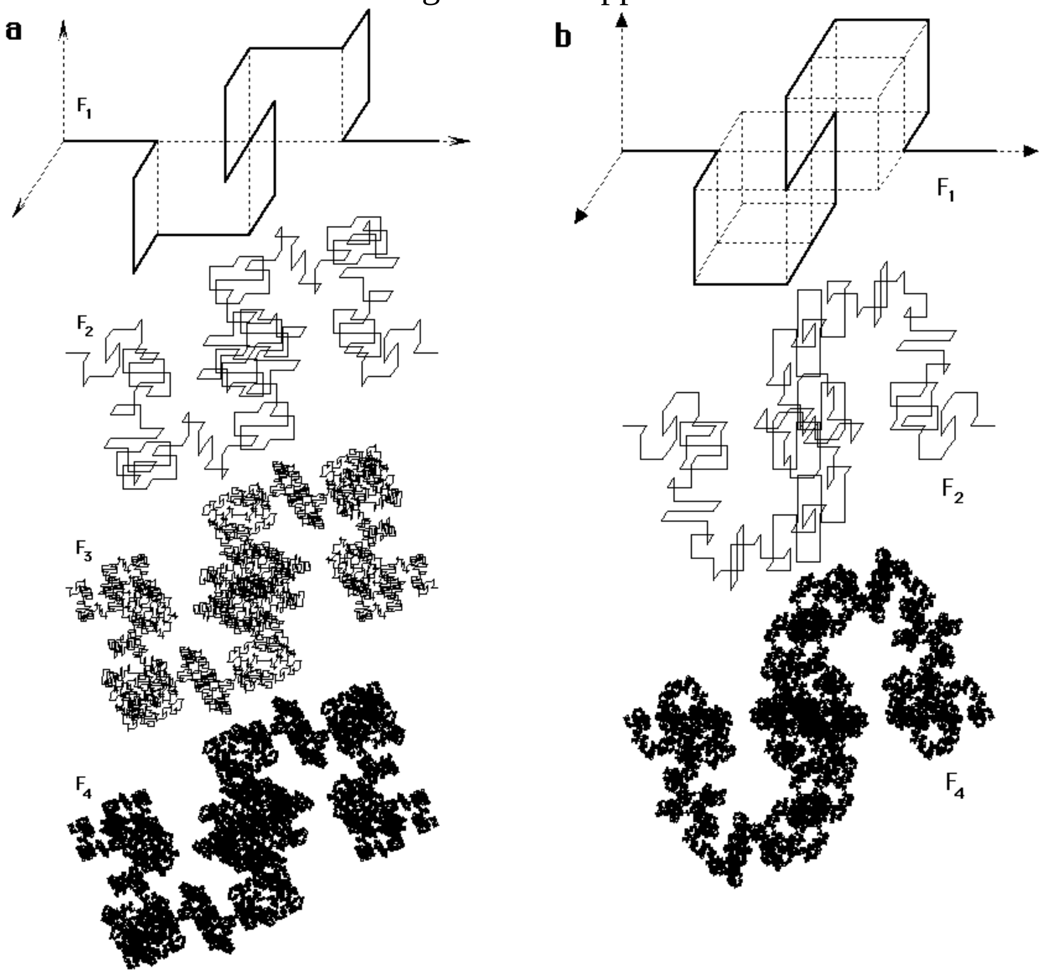


Figure 3.12 a and b

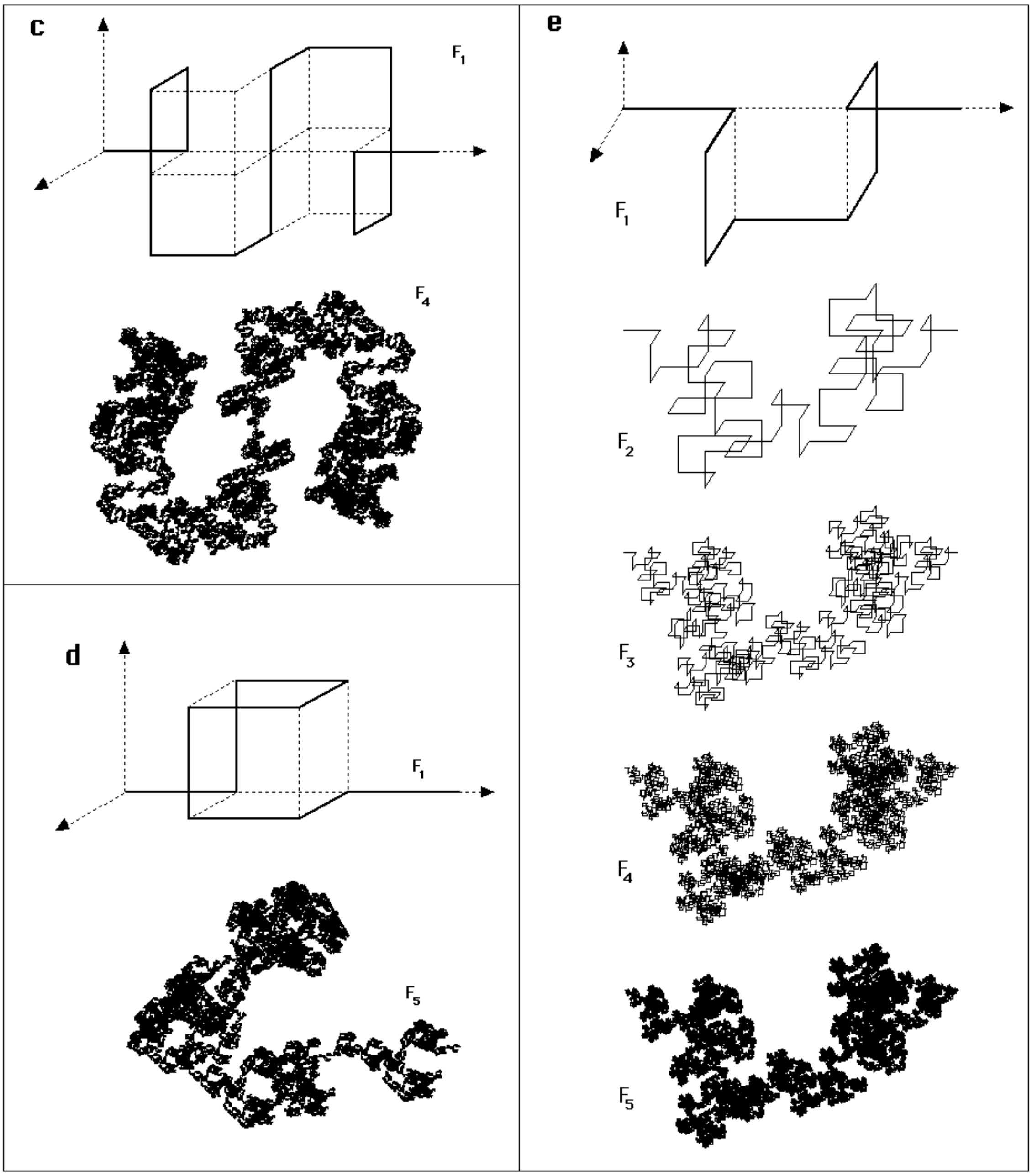


Figure 3.12 (cont.). Drawing of successive approximations of fractal curves of fractal dimension 2 in space (R^3). The curves (**a**), (**b**) and (**c**) are based on q=4 and p=16 (i.e. their generators are made of 16 segments of length 1/4), while (**d**) and (**e**) are based on q=3 and p=9. Note that the generator of (**a**) is built from a symmetrisation of (**e**), with its last segment excluded (see text).

3.6. Fractal Surfaces.

Although the study of fractal curves is already instructive for the understanding of the properties of fractal spaces, since the geodesical lines of fractal spaces or space-times are particular fractal curves, problems more specific for spaces (as compared to functions or applications) begin to be encountered when studying surfaces (i.e., fractals of topological dimension 2). We illustrate this point hereafter by a preliminary study of a particular class of fractal surfaces made of orthogonal sides.

Let us attempt to obtain a first member of this class by a two-dimensional generalization of the well-known fractal curve shown in Fig. 3.5. F_o being a square of side 1, one gets the generator F_1 of Fig. 3.13, made of 48 new squares of length 1/4. If one now wants to build F_2 by the usual "fractalization" method, i.e. replacing each one of the 48 squares by a scaled version of F_1, a new and specific difficulty appears: matching of the structures at their boundary. While F_1 may be matched to itself when there is no rotation from one face to the adjacent one, this is no longer the case when the relative rotation is $\pm\pi/2$. Hence a complete description of such a fractal implies not only the giving of one generator, but also of the various matching conditions from one side to the other. More generally a generator of that type may be built by giving ourselves, in a Cartesian system of coordinates (x,y,z), the altitudes Z_{ij} (i=0 to $q-1$, j=0 to $q-1$) of the horizontal faces in F_1. Then the total number of faces is

$$p \;=\; q^2 \;+\; \Sigma_{ij} \,|\, Z_{i+1,j} - Z_{ij} \,|\; + \;\Sigma_{ij} \,|\, Z_{i,j+1} - Z_{ij} \,| \quad .$$

If this number is conserved whatever the connections, the fractal dimension is $D=ln(p)/ln(q)$. Note that there would be an additional term if F_1 contained faces turning backwards, in which case the number of horizontal faces would be larger than q^2.

As shown in the previous example, even when one wants to ensure the highest level of self-similarity, the generator cannot be unique, except in the special case where the external faces are all of zero altitude, i.e. $Z_{0j} = Z_{q-1,\,j} = Z_{i0} = Z_{i\,,q-1} = 0$ (see Figs. 3.14 and C4).

A solution which allows one to keep at maximum the same structure whatever the position and resolution consists in having only the outer distribution of faces changed (thus defining a "connecting box"), while the internal structure is kept invariant (see Fig. 3.13d:

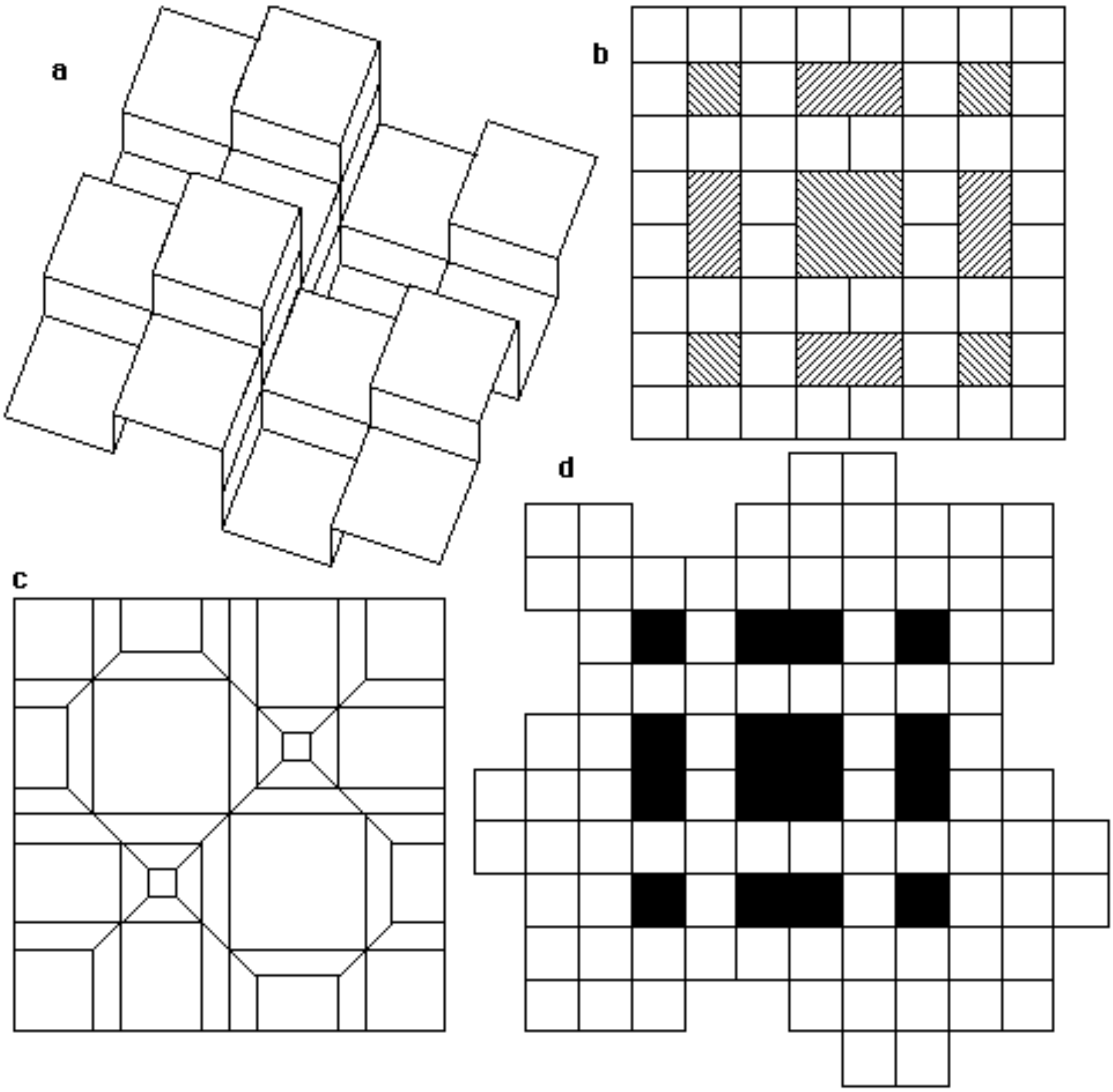


Figure 3.13. Various representations of the generator of an orthogonal fractal surface, which generalizes the fractal curve of Figs. 3.5 and 3.7. (**a**) The generator in 3-space $\mathbb{R}^3$ (see also Fig. **d**). Its is made of 48 squares of side 1/4. (**b**) An unfolded plane version of the same fractal surface. The surface of Fig.**a** may be obtained by making holes in the hachured squares of Fig.**b**, then by folding it into the shape of the generator of Fig. 3.5a, in both x and y directions. The points related by the hachures of the degenerated faces come in contact. (**c**) Lattice representation of the same fractal surface generator. The metrics on this lattice is such that each segment has the same length 1/4. (**d**) An improved version of the same generator in the representation (**b**), (but now with q=6: note that such a fractal would still have multiple points). It allows connection of orthogonal faces to the next order of the fractal construction, F_2. The right side of (**d**) may be matched to its upper side if there is no rotation between the faces. It may be matched to the same side rotated of π, i.e., to the left side of (**d**) if the two faces are orthogonal.

this is a particularly interesting case since the same border is used whatever the relative angle of the faces, but a rotation of 180° must be imparted to one face in order to connect it to the other one with a relative angle of $\pi/2$).

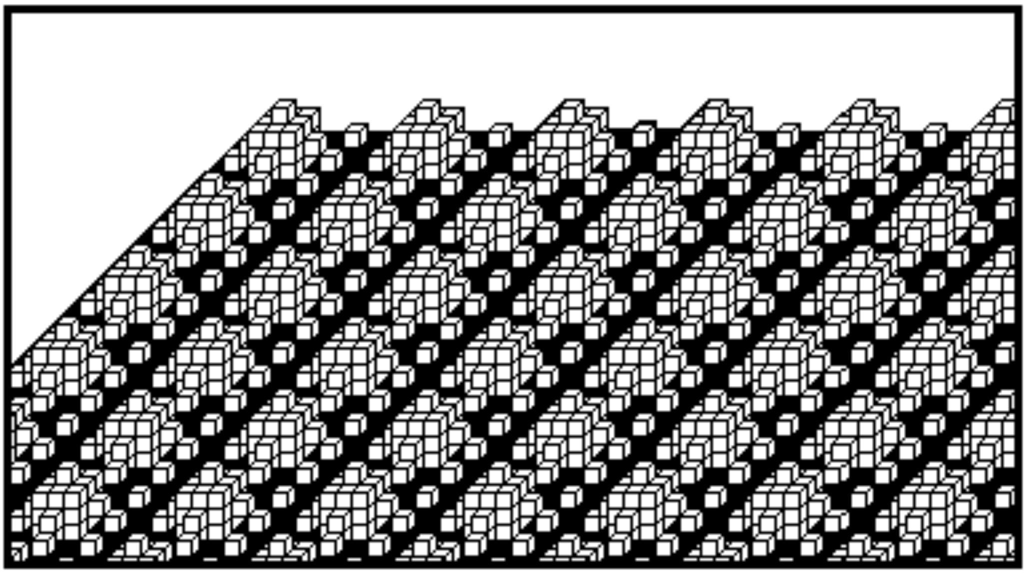


Figure 3.14. Periodic fractal surface (approximation F_2).

The effective construction of such a fractal is also instructive. It may indeed be obtained from the folding of a lacunar fractal drawn in the plane. For example the structure of Fig. 3.13a is obtained from a 8x8 square in which the regular pattern of holes of Fig. 3.13b has been made. This result is generalizable to any fractal surface of this type. This planar diagram may be seen as the representation of the generator in a particular curvilinear coordinate system, (x, y), in terms of which the metric may be degenerated. While the metric inside the uncut faces is Euclidian, $ds^2 = dx^2 + dy^2$, the metric inside the holes may be $ds = 0$, dx, dy, $dx + dy$, $dx - dy$, depending on the way the faces are connected after folding. Finally, pursuing to infinity the fractalization process will yield, in terms of such coordinates, a metric on the fractal surface alternatively Euclidian and degenerated, this in a fractal way.

An alternative and maybe more powerful representation is the drawing of a lattice in the plane (Fig. 3.13c). The metrical properties of this structure are set by the requirement that all segments are of equal length, but the definition of a 2-dimensional coordinate system is more difficult. The various points may be connected to 3, 4, 5 or 6 adjacent points. This remark leads us to a first and quick look at the question of curvature on fractal surfaces. For such orthogonal fractals (always assuming continuity and self-avoidance), there are clearly 4 possibilities concerning the connections of faces around a given point, as shown in Fig. 3.15. The standard one is 4 faces and corresponds to flatness. A point belongs to only 3 faces at the vertex of a cube, and this corresponds to infinite positive curvature: lets us call it {+1}. Finally there are two cases of infinite negative curvature, with 5 and 6 faces, which we call {–1} and {–2}, i.e. $\{C = 4 - S\}$, where S is the number of segments starting from the point considered. (It may be remarked that $\Sigma C_i = 0$ on the vertical lines in the absence of backward turning faces). These results, straightforward when considering the generator F_1, apply as well to any point of the fractal F, since F may be considered as made of all the vertex points of its various approximations F_n. Hence the curvature of a fractal surface is *a fractal alternance of infinite positive and negative curvatures*.

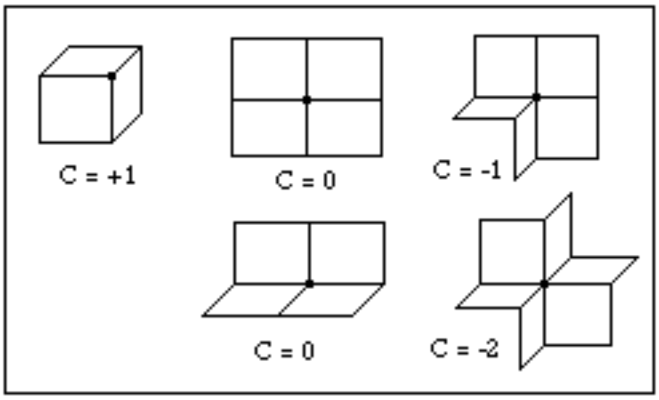


Figure 3.15. The 3 possible cases of infinite curvature on orthogonal fractal surfaces, C = –2, –1, and +1 (+ two flat cases C = 0).

Additional illustrations of fractal surfaces, some having differentiable generators, are given in Figs. 3.16, C5 and C6. For a more general mathematical description of fractal surfaces based on two variable fractal functions, see Massopust.[32] We shall also consider in Sec. 3.10 the general case of fractal spaces, of which fractal surfaces are the particular 2-dimensional achievement.

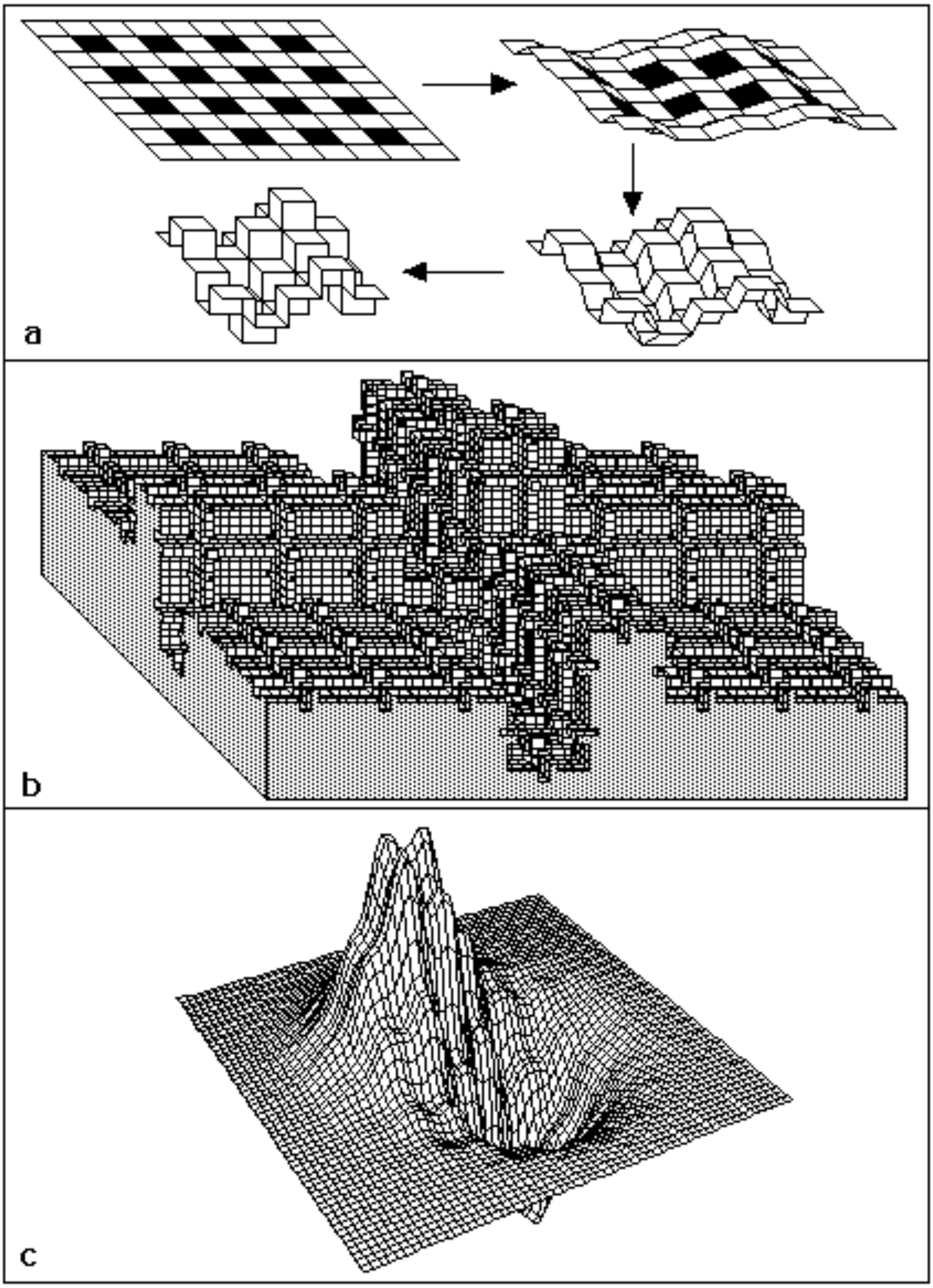


Figure 3.16. Fractal surfaces. Figure **a**: steps in the folding of a lacunar generator yielding the generator of a continuous fractal surface. Figure **b**: construction to second order of an orthogonal fractal surface. Figure **c**: second order approximation of a surface built up from a differentiable generator. (See also Figs. C4-6).

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