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THE SIGNIFICANCE OF THE MARKET PORTFOLIO

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The Significance of the Market Portfolio

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Arguments for creating a market to allow trading the portfolio of all endowments in the entire world, the “market portfolio,” are considered. This world share market would represent a radical innovation, since at the present time only a small fraction of world endowments are traded. Using a stochastic endowment economy where preferences are mean variance, it is shown that creating such a market may be justified in terms of its contribution to social welfare. It is also argued that creating a market for world shares is attractive for certain reasons of robustness and simplicity.

The portfolio of all endowments in the world, the “world portfolio” or, more commonly, the “market portfolio,” has great significance in the capital asset pricing model (CAPM) in finance.¹ The Sharpe–Lintner CAPM characterization of optimal risk sharing implies that in equilibrium no one will be subject to a random shock that is not shared by everyone else. Thus the CAPM gives us the “mutual fund theorem,” which asserts that only one risky portfolio need be available to individual investors, the mutual fund that holds shares in the world portfolio. In this article we seek further clarification of the significance of the world portfolio beyond the bounds of the restrictive assumptions of the CAPM.

The original version of the CAPM was designed to describe how agents should invest in existing financial assets.² Thus all agents have some stock of wealth and they must choose how much of their wealth to invest in each asset. There is some zero-cost intermediary that allows the agents to purchase the assets. One of the key insights of the CAPM is that each agent needs only the shares in the world portfolio and the risk-free bond to be available to them to trade in so that they obtain their optimal allocation of risk. It is in this sense that the world portfolio is so important in the CAPM.

In our analysis we will drop the (highly unrealistic) assumptions of the CAPM that all risks are tradable and that all agents have some nonstochastic

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¹ We will rely in what follows on the terms “world portfolio” or “world shares” to avoid any possible confusion with commonly used empirical proxies that are also often called the “market portfolio.” The term “market portfolio” is an appropriate term to refer to the portfolio of all endowments in the world when each endowment is marketed. The terms “world portfolio” or “world shares” are more appropriate generic terms for the portfolio of all endowments in the world, whether each endowment is marketed or not.

² CAPM will refer to the Sharpe–Lintner version unless specified otherwise.

stock of wealth which they invest; we include in our model nonfinancial endowments such as labor income. Thus in general no one will be able to hold the aggregate endowment unless unprecedented new institutional arrangements are made to permit it to be traded. We instead develop a CAPM-type model in which each individual has an exogenous random endowment that is initially not tradable, and we will consider adding one, two, or more contracts that make it possible to buy or sell portfolios of claims on the endowments. We assume that these contracts are to be traded in markets open to everyone, and a market price will be generated such that total excess demand by all agents is zero. Thus, by creating these contracts, we are creating new markets for portfolios of endowments, making a risk tradable that had not been so before.

It has been stressed by Roll (1977) and many others that the world portfolio is untradable, and so the return on this portfolio is unobserved. Not only are there currently no markets to trade claims on human capital, but also there are no international liquid markets for claims on housing, social security wealth, and other assets that together account for a large part of individuals' wealth. A market for shares in the world portfolio itself, as well as for other aggregates of endowments, could, however, easily be established. While Roll is right that the returns on the world portfolio are unobserved, the *dividends* paid on the world portfolio *are* observed. Every country of the world has published national income and product accounts, and the aggregated national incomes are the total dividend on the world portfolio. While these measures of national income are imperfect, they are the outcome of years of work by economists and the reworked versions, as in the Penn World Table or the World Bank World Tables, are widely regarded as reasonably good indicators of true national income. To create a market for shares in the world portfolio, we need only create cash-settled futures markets for long-term claims on the aggregated national incomes of the world. Longs in these unusual futures contracts would receive a dividend, from shorts, proportional to world income. Creating futures markets for long-term claims on national incomes is discussed in Shiller (1993a,b), and these same methods could be extended to allow creation of a market for the entire world. Creating this market for claims on world income, or for claims on other income aggregates, would allow price discovery for the market value of claims on these aggregates, revealing prices that have never been observed before and allowing hedging that would be impossible without these markets.

We confine our attention to designing N contracts, where N is small, in order to prescribe in simple terms the most important risk management actions that should be taken by groups of people and to ask if a market for the world portfolio is the most important market, or is even in the span of the most important markets. Most people take no more than simple prescriptions

from existing models. Practitioners usually do not use the CAPM to arrive at precise definitions of optimal portfolios, but merely refer to the prescription of the CAPM that investors should hold the world portfolio of investable assets. The indexed funds that are now commonplace were designed as a step toward making this possible. But this common prescription disregards the correlation of portfolio returns with other endowments, traded risks with nontraded risks.

It is very important, at the time financial innovation takes place, to consider what are conceptually the most important markets. We cannot have liquid markets for everything, and history shows that markets that are not sufficiently valuable to participants will not succeed, and markets will sometimes disappear when better markets are created.

A market for shares in the world portfolio sounds at first like it ought to be the most important possible market to create; it is after all the market for everything. We shall argue in this article that there are general conditions under which it is indeed the most important. These conditions are different from the conditions that are responsible for the importance of the world portfolio in the CAPM.

We begin, however, by showing what appears to be the opposite, that, curiously, in a simple general equilibrium exchange model of all possible markets to create, a market for shares in the world portfolio would be, by a social welfare criterion, a least important market to create, not a most important market (Theorem 2). Within the simple assumptions of this model, we see that if we are in the business of creating markets for endowments that are not now tradable, then there is a natural order to creating such markets. There is a most important market to create, and then, after this, a market that would be the next best market to create, and so on. The market for shares in the world portfolio turns out to be a completely unimportant market in this ordering, still not spanned by all the other markets when we get to the end of the ordering, and then the welfare gain to creating it is zero. This is not to say that the world shares would not be useful to people if they were created first, or if they were created second or third, only that there would always be something better to do instead. This result may be regarded as, in a sense, the *very antithesis of the mutual fund theorem*.

Neither will we ever want to create markets for individual endowments or for portfolios weighting all endowments with the same sign. Optimal contracts will always involve portfolios of risky endowments with both positive and negative quantities and their weighted sum is zero. The optimal contracts are thus always essentially swaps, that is, every optimal contract is a linear combination of the endowments with both positive and negative weights. This result may be regarded as in a sense the *apotheosis of swaps*.

These results from our simple model may seem, at some level, rather obvious. Consider, to simplify the analysis, a world with only two *identical* agents with independent identically distributed endowment risks. Obviously if we create a market for an endowment swap between them, then they can achieve perfect risk sharing by swapping half of their endowments; after the swap each will be receiving half of the world endowment, and perfect risk sharing will have been achieved. But if they are instead given an ability to buy or sell claims on the world endowment, the world portfolio, this does them no good: it does not allow them to sell off their own endowment risk and buy some of the other. Since they are identical, if they wanted to make any trades on the market for the world portfolio they would obviously want to make the same trades, both buying or both selling, and so there is no one to take the other side of the trade. It follows that there would be no trade in the market for the world portfolio. Since there is no trade, there is no welfare gain to creating it.

Our results in this article with this general equilibrium model are not quite as obvious as this simple story would suggest. Our model allows the agents to differ either in risk aversion or in endowment risks, and so they may have some use for the market for the world portfolio if it is created. The more risk-averse agent or the agent with the more risk may sell some shares in the world portfolio to the other for a fixed, riskless, sum, and both are made better off by the creation of the market for the world portfolio. One agent, the more concerned about risk, obtains risk reduction for a price; the other agent, the less concerned about risk, accepts more risk for the benefit of receiving the price. Our results suggesting that we would never want to create a market for shares in the world portfolio are not that the world shares are always useless, but instead that the world shares are always dominated by swaps, even in the case where there are millions of agents with different risk preferences and for a general nonsingular covariance matrix of endowments.

These results would appear to make it difficult to make a case for establishing a market for shares in the world portfolio. Nonetheless, we are able to make the case that the world shares may be extremely important, even most important. The conclusion that there is no need for a market for the world shares rested on the assumption that the market designers who are creating the new markets know everything about utilities and about the variance matrix of endowments. We show a representation of lack of knowledge of risk aversions on the part of the market designers that brings the world shares back to some potential significance (Theorem 3). Moreover, if lack of knowledge about preferences is high, and if we also deal with our lack of knowledge about variance matrices by using the exchangeability principle of Bayesian statistics and endowments are positively correlated, then we are led to the market for shares in the world portfolio as the first best market to create (Theorem 5). With this theorem we have come full circle, after proving the antithesis of the mutual fund theorem, making the market for world shares

least important, we then are able to resurrect it, under certain assumptions, to the place of the most important market, the optimal first market to create.

Of course, abstract theoretical exercises like that in this article will never prove that such a market will succeed in practice. Designers of new risk markets have a poor track record in predicting which markets will actually succeed, and this proposed world market is much more unusual than are most. People would have to change their way of thinking about risk management before they would want to start trading in a new market for the world portfolio. We think that it is plausible, however, that, after a transition period, many people eventually would take positions in this market. People have varying concerns and opinions about risks. These concerns and opinions might well translate into real buy or sell orders on a market for shares in the world portfolio. While some individuals may regard such a market as little more than marginally helpful for their risk management, everyone in the world is potentially interested in it. Open interest for this market is potentially very large.

One aspect of this story we do not model in this article is moral hazard. Take the two-agent economy considered above and suppose the two agents agree to share their future endowments. Suppose their future endowments are partly controllable, as for example by varying effort to produce labor income. Then, once the two individuals decide to share their future endowments, there is an incentive to work less since they are assured some endowment from the agent who took the other side of the contract. This problem is irrelevant if the two agents represent two large groups of agents, each agent within the group having the same endowment. In this instance, the contract would be written on aggregates of the groups and this aggregate is not affected by any one individual. We believe the moral hazard problem is very small for the situations we intend to apply these methods to. However, it must be recognized that there are other kinds of moral hazard risks. People could lobby their governments to abrogate contracts retroactively. People could individually take actions to make it difficult for others to collect. Margin requirements might deal with this problem, but only for people who have sufficient assets to pledge as margin. We will disregard these kinds of moral hazard problems.

Theorem 1 below is part of a framework developed in Athanasoulis (1995) and Shiller and Athanasoulis (1995); it was developed independently by Demange and Laroque (1995b). A related analysis is found in Duffie and Jackson (1989), Cuny (1993), and Willen (1997). See Geanakoplos (1990) for an introduction to general equilibrium with incomplete markets. Cass, Chichilnisky, and Wu (1996) show how the number of assets needed to obtain a complete markets solution can be greatly reduced by constructing a set of mutual insurance contracts and a smaller set of Arrow securities when compared to an Arrow–Debreu world. This is related to our results as we only need assets far less than the number of states of the world to obtain a first

best solution. We, however, consider which assets are best to construct if we do not complete assets markets. Demange and Laroque show (1995a) that in an economy with general utilities, when all residual risk is hedged, then the only important assets remaining to construct in the economy are nonlinear assets, such as options, whose realizations depend exclusively on the realization of the world portfolio.³ Our results are complementary to this Demange and Laroque result rather than being a competing result. Our analysis here starts from no markets at all, and studies a sequence of markets to allow linear spanning of the original endowments; Demange and Laroque (1995a) are considering moving yet beyond the linear spanning, and the subsequent nonlinear markets add to the importance of the world portfolio.

The article is organized as follows. We first lay out the assumptions of the general equilibrium model and then solve the agent's problem, assuming preferences are mean variance, for a given set of available contracts. In the body of the article, we will go beyond the two-by-two examples shown above and prove all our theorems using matrix theory for J agents since we intend to apply this theory to situations where J is very large. We seek optimal aggregates of endowments to be traded on one or a few new markets. The resulting expressions for equilibrium prices and quantities will be used in all subsequent parts of this article. We then go through several variations on the maximization problem faced by the market designers, differing in assumptions about the information available to the designers, considering both uncertainty about preferences and uncertainty about the variance matrix of endowments. When we apply the exchangeability principle to modeling uncertainty about the variance matrix, we find that with sufficient uncertainty about preferences and positively correlated endowments, the market for shares in the world portfolio is the first-best market to create. We then conclude with some practical advice for market designers. An appendix discusses the implications for the model of preexisting markets.

1. The Model

There are J agents in this economy indexed by $j = 1, \dots, J$, each representing an individual. The theory may also be used where each j represents a grouping of individuals, such as a country, but that is not our favored interpretation here, since our theory is designed to define groupings of individuals for market definition, not accept them exogenously. All random variables are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the set of states of the world and $\omega \in \Omega$ is the state of the world. \mathcal{F} is a σ -algebra of

³ Hakansson (1978), building on results of Ross (1976), proposed creating an array of supershares each of which pays one unit of value if the world portfolio obtains a certain value. He assumed the world portfolio was already traded. When Leland, O'Brien, and Rubinstein created their version, SuperShares, on the Chicago Board Options Exchange in 1992, they were obliged to substitute something tradable (the Standard and Poor's 500 Index) for the untradable world portfolio [see Rubinstein (1995)].

subsets of Ω known as events and $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ satisfying $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}(\Omega) = 1$ is a probability measure on (Ω, \mathcal{F}) held commonly by all agents in the economy.

There is a single good in the economy which is consumed. Each agent j has an endowment $x_j \in L^2(\Omega, \mathcal{F}, \mathcal{P})$, where $L^2(\Omega, \mathcal{F}, \mathcal{P})$ is the set of random variables which are square integrable, that is, have finite mean and variance. We will denote the demeaned stochastic endowment as $\tilde{x}_j = x_j - E(x_j)$. Define x to be the $1 \times J$ vector of random endowments in the economy and similarly let \tilde{x} be the $1 \times J$ vector of demeaned stochastic endowments. Then $E(\tilde{x}'\tilde{x}) = \Sigma$ is the $J \times J$ covariance matrix of the endowments in the economy. We will assume that Σ is nonsingular.⁴ Define $E(\tilde{x}'\tilde{x}_j) = \Sigma_j$ and $E(\tilde{x}_j\tilde{x}_j) = \Sigma_{jj}$.

The $N \leq J$ contracts indexed by $n = 1, \dots, N$ designed in this article are futures contracts. Let $f_n \in L^2(\Omega, \mathcal{F}, \mathcal{P})$ be the risky transfer made in the n th futures contract resulting in $f_n(\omega)$ units of consumption contingent on state $\omega \in \Omega$. To purchase contract n , the agent must promise today to pay a riskless price $p_n \in \mathbb{R}$ in the period where the state of the world is resolved. Thus if the state $\omega \in \Omega$ is realized, agents who take a long position in contract n receive $f_n(\omega) - p_n$, those who take a short position pay this amount. Define f to be the $N \times 1$ vector whose n th element is f_n and P to be the $N \times 1$ vector whose n th element is p_n . Without loss of generality we construct the futures contracts such that $E(f) = 0$ and $E(ff') = I_N$, where I_N is the $N \times N$ identity matrix.

We restrict our attention to contracts that specify payments that are linear in endowments. Demange and Laroque (1995b) showed that the optimally chosen risky transfers f must be in the space spanned by the initial endowment risks \tilde{x} . Consequently we define $f = A'\tilde{x}'$, where A is a $J \times N$ matrix and $A'_n\tilde{x}' \in L^2(\Omega, \mathcal{F}, \mathcal{P})$, $n = 1, \dots, N$ and A_n is the n th column of A . Therefore, according to our notation, $E(ff') = A'\Sigma A = I_N$.

2. Agents

Each agent has a utility function $U_j: L^2(\Omega, \mathcal{F}, \mathcal{P}) \rightarrow \mathbb{R}$. We make the simplifying assumption that each agent has mean variance utility as follows,

$$U_j = E(c_j) - \frac{\gamma_j}{2} \text{var}(c_j), \tag{1}$$

where c_j is the consumption of agent j , the same as the endowment plus proceeds from hedging. We allow the parameter γ_j to differ across agents, to reflect differing attitudes toward risk and therefore to produce differing

⁴ We are hard pressed to find reason to think that there is any individual who has no uncertainty in her endowment stream, or that there is perfect correlation between any individuals' incomes. The largest component of wealth for most individuals is the present value of their labor income (or human capital); labor income vicissitudes are unique to each person (household). Another very large component of wealth for individuals is their equity in housing, also unique to each person (household).

motivations to trade. Variation across agents in γ_j might also be interpreted as reflecting different perceptions of risk, or different opinions about risks. However, in the context of our model, these differences across agents in perceived risks will have to take the restrictive form only of different agents applying different scalar multiples to the same variance matrix, keeping the structure of the perceived variance matrix the same across agents.

Agents take the risky transfers f which is a vector $L^2(\Omega, \mathcal{F}, \mathcal{P})$ process and the futures prices $P \in \mathbb{R}^N$ as given and solves for their optimal futures positions q_j as

$$q_j = \arg \max_{q_j \in \mathbb{R}^N} \{U_j \mid c_j = x_j + q_j'(f - P)\}. \tag{2}$$

We can rewrite this in a simpler form as

$$q_j = \arg \max_{q_j \in \mathbb{R}^N} \left\{ E(x_j) - q_j'P - \frac{\gamma_j}{2} (\Sigma_{jj} + q_j' A' \Sigma A q_j + 2q_j' A' \Sigma_j) \right\}. \tag{3}$$

Remembering that $A' \Sigma A = I_N$, the optimal demand for this agent is

$$q_j = -\frac{1}{\gamma_j} P - \text{cov}(f, \tilde{x}_j) = -\frac{1}{\gamma_j} P - A' \Sigma_j. \tag{4}$$

This demand curve tells us that agent j will purchase more of a security as its price declines. The agent will purchase less of the security the more it covaries with the endowment since it provides less hedging service. To help the exposition of this article it is convenient to form the $N \times J$ matrix Q whose j th column is q_j and rewrite Equation (4) as

$$Q = -P \iota' \Gamma^{-1} - \text{cov}(f, \tilde{x}) = -P \iota' \Gamma^{-1} - A' \Sigma, \tag{5}$$

where Γ is the $J \times J$ diagonal matrix with the j th diagonal element equal to γ_j and ι the $J \times 1$ unit vector, all elements equal to one.

3. Equilibrium

The equilibrium condition in this economy is simply that the futures contracts are in zero net supply. We can represent equilibrium in this economy as

$$Q \iota = 0 = -P \iota' \Gamma^{-1} \iota - A' \Sigma \iota. \tag{6}$$

From the equilibrium condition in Equation (6) we can derive the equilibrium pricing equation,

$$P = -A' \Sigma \iota (\iota' \Gamma^{-1} \iota)^{-1}. \tag{7}$$

Definition. The share in the world portfolio (or world share) is defined by its dividend

$$f_m \equiv \tilde{x} m, \tag{8}$$

where m is a scaled unit vector

$$m \equiv \frac{\iota}{(\iota' \Sigma \iota)^{0.5}}. \tag{9}$$

If we multiply and divide the right-hand side of Equation (7) by $(t'\Sigma t)^{0.5}$, then the price of contract n depends on $A'_n \Sigma t (t'\Sigma t)^{-0.5} \equiv A'_n \Sigma m$, the covariance of contract n with the world share. Thus we can derive the CAPM pricing equation from Equation (7). If the covariance of a contract with the world share is zero, as for example with a risk-free contract, then the price of this contract is $p_f = 0$. [The prices in this framework are (minus) the risk premia in the capital asset pricing equation.] The price of the world share is $p_m = -[(t'\Sigma t)^{0.5}/t'\Gamma^{-1}t]$. It follows that

$$p_n - p_f = \frac{\text{cov}(f_n, f_m)}{\text{var}(f_m)} (p_m - p_f), \tag{10}$$

which is the familiar CAPM pricing equation and $\text{cov}(f_n, f_m)/\text{var}(f_m)$ is the familiar beta of the CAPM. Similar results are obtained by Magill and Quinzii (1996), Duffie and Jackson (1989), Oh (1996) and Mayers (1972). Substituting Equation (7) into Equation (5) we also obtain

$$Q = -A'\Sigma M \tag{11}$$

and we define $M \equiv I_J - u'\Gamma^{-1}(t'\Gamma^{-1}t)^{-1}$ and $A \equiv [A_1 : A_2 : \dots : A_N]$, where A_n is the n th column of A .

4. Market Design

The market designers' problem is to maximize welfare, total utility, in the economy given they are constrained to choose $N \leq J$ markets. The market designers will choose the $J \times N$ matrix A to maximize the sum of utilities in the economy.⁵ From Equation (3) we know that the j th agent's utility is given by

$$E(x_j) - q'_j p - \frac{\gamma_j}{2} (\Sigma_{jj} + q'_j A'\Sigma A q_j + 2q'_j A'\Sigma_j). \tag{12}$$

If we sum over all J agents, drop $E(x_j)$, and put this in matrix form we obtain

$$\text{tr} \left(-Q' P t' - \frac{1}{2} \Gamma (\Sigma + Q' Q + 2Q' A' \Sigma) \right), \tag{13}$$

where tr denotes the trace. If we substitute Equations (11) and (7) into Equation (13) we obtain

$$\text{tr} \left(\frac{1}{2} \Gamma M' \Sigma A A' \Sigma M - \frac{1}{2} \Gamma \Sigma \right), \tag{14}$$

where the term $\frac{1}{2} \Gamma \Sigma$ has no effect on the market designer's decision. Thus using $\text{tr}(AB) = \text{tr}(BA)$, the market designer's problem simplifies to

$$A \in \arg \max_{A_n \in \mathbb{R}^J, n=1, \dots, N} \{ \text{tr} (A' \Sigma M \Gamma M' \Sigma A) \mid A' \Sigma A = I_N \}. \tag{15}$$

This leads to a fundamental theorem shown separately by Demange and Laroque (1995b) and by Shiller and Athanasoulis (1995):

⁵ The market designer can choose to maximize a weighted sum of utilities rather than a simple summation. This does not have a substantial effect on the theory so we leave it out.

Theorem 1. *The A matrix that solves Equation (15) has columns corresponding to the N eigenvectors with highest eigenvalues of*

$$M\Gamma M'\Sigma. \tag{16}$$

Proof. See the appendix. ■

To obtain the results thus far we assumed that utilities were mean variance as in Equation (1). This same specification was used in Duffie and Jackson (1989). This is not an overly restrictive assumption. It is shown in Demange and Laroque (1995b) that if the endowments and the new contracts are normally distributed, and utilities are more general, increasing, concave, and twice continuously differentiable, then one will obtain similar results. If in addition the utility function is exponential (CARA), then one is able to obtain the identical market design results as in Theorem 1, see Demange and Laroque (1995b).⁶ Thus all of the results still hold when endowments and the new contracts are assumed to be normally distributed.

For the problem solved by the market designers, the problem is altered somewhat when assuming that utilities are more general. The result we obtain in this article, Theorem 1, is the definition of optimal markets that are constructed are independent of the number of markets constructed. Thus under the assumptions in this article, the first best market is always first best regardless of whether the designer of markets constructs one, two, or more markets. This is also true of the second-best market as well as the third best and so on. However, when the coefficient of absolute risk aversion is affected by market construction, that is, is a function of the A matrix, then the number of markets constructed do affect optimal market definition. This occurs with the more general utility specification. Thus in general the first-best market when one market is created is different from the first-best market when two markets are created, and so on.

5. The World Share Is in One Sense Least Important

It is now very easy to prove our result, noted above, that the share in the world portfolio is in a sense a least important market to allow trading in:

Theorem 2. *The A matrix that solves Equation (15) is orthogonal to $g \equiv t'\Gamma^{-1}(t'\Gamma^{-1}\Sigma\Gamma^{-1}t)^{-0.5}$, and all $N \leq J - 1$ markets together do not span the world portfolio.*

Proof. See the appendix. ■

⁶ CARA utility and normally distributed endowments is also used in Cuny (1993).

The intuition behind this results can be seen as follows. For each j construct the asset j with risky transfer $f_j = c_j - x_j$, where c_j is agent j 's consumption with complete markets. These assets enable all agents to achieve their complete market allocation. Since total consumption equals total endowments, market clearing implies that $\sum_{j=1}^J f_j = 0$ and thus we can drop any one of these assets and achieve the complete market allocation using only $J - 1$ assets. To show that the world portfolio is not an important asset, note that when preferences can be aggregated, as with mean variance preferences or more generally for linear risk tolerance preferences (with the same exponent for all investors, except when utility is exponential (CARA), so that monetary separation obtains), c_j is a linear combination of the aggregate endowment and the risk-free prices paid by agent j for the assets purchased. The optimal securities have risky transfers of the form $f_j = \alpha_j \sum_{j=1}^J x_j - x_j$, where $0 < \alpha_j < 1$ for all j . Thus it is clear that the risky transfers f_j , $j = 1, \dots, J - 1$ do not span the world portfolio. Furthermore since the α_j 's are less than one and strictly positive, it is clear that the contracts will all be swaps, with some positive elements and some negative elements. This gives us the "apotheosis of swaps."

Lemma 1. *In the case where agents have the same risk aversion, γ_j , the share in the world portfolio is orthogonal to all optimal contracts.*

Proof. See the appendix. ■

This lemma is particularly important since symmetry of risk aversions is likely to be assumed when designing new contracts.

To understand these results better let us consider a two-agent example: A two-agent example ignores some of the complexity that the optimal market solution method is supposed to handle, but it will make some basic concepts more transparent. We can then illustrate the solution to the market designer's problem on a simple two-dimensional graph (Figures 1 and 2), with the first element of A_1 , a_1 on the horizontal axis and the second element of A_1 , a_2 on the vertical axis. On these figures the constraint $A_1' \Sigma A_1 = 1$ is that the A_1 vector must end somewhere on the ellipse shown. The ellipse shown illustrates a case of positive correlation between the two endowments, where both endowments have the same variance and a correlation coefficient of one-half. On each figure, iso-welfare curves are parallel straight lines (one pair of which is shown); the further from the origin the higher the welfare.

The optimal vector A_1 must be orthogonal to g , which means that the vector is in the upper left quadrant (or lower right), and is not in the same quadrant where the world share vector m is. In Figure 1, the case is shown where all the γ 's are one, and so g equals m . The agent will use the optimal contract to swap half of her endowment risk for half of the other's, and both agents will end up holding a share of the world. In this case, the optimal contract is orthogonal to the world share, and the world share contract would

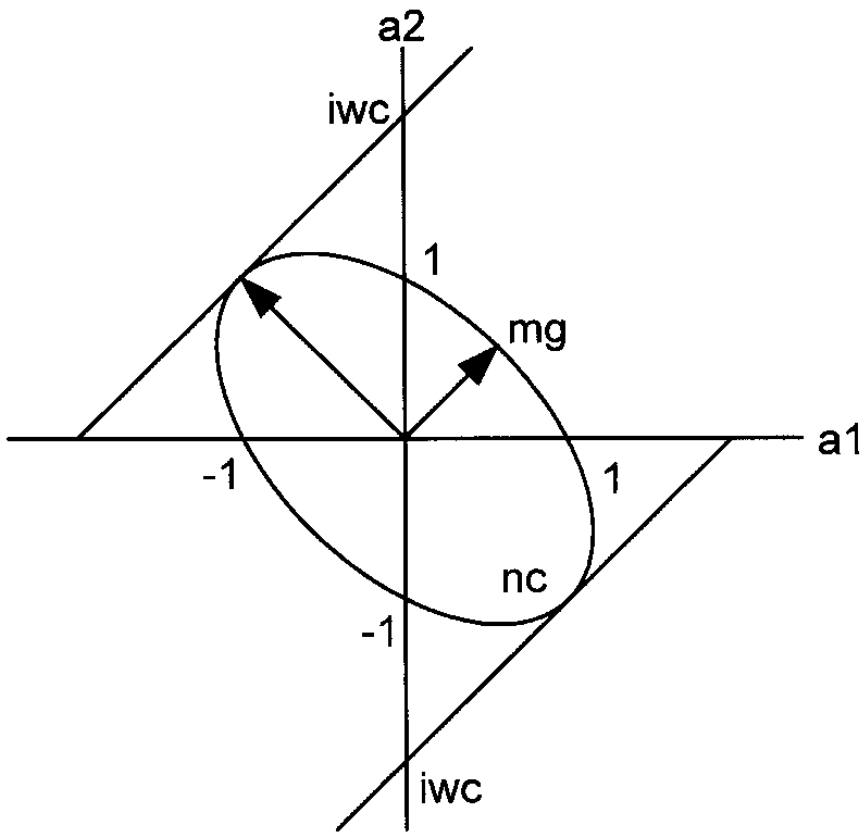


Figure 1

be utterly useless to the agents if it were created instead of the optimal contract. The optimal contract is found on the graph by finding the highest iso-welfare curve, *iwc*, that satisfies the constraint, tangent to the ellipse. Clearly in this symmetric situation there is no value to being able to trade the world share for these agents, as they would both like to take the same position.

In Figure 2, the case is shown where $\gamma_1 = 3$ and $\gamma_2 = 1$. Now, the *g* vector no longer coincides with the world share vector, *m*, and the optimal A_1 vector results in an unequal swap. In the swap, the more risk-averse agent gives up three times as much of the risky component of her endowment to the other agent, and pays a price to the other agent for doing so. After the swap, the more risk-averse agent is bearing only one-quarter of world endowment risk, the less risk-averse agent is bearing three-quarters. This is the Pareto optimal outcome: there are no more risk-sharing opportunities, and each agent is bearing world endowment risk in accordance with her own risk preferences. Note that in this case had we instead created the world share first, it would have been of some use though it would touch an iso-welfare curve that is closer to the origin. In both figures, the isoquants for the objective function

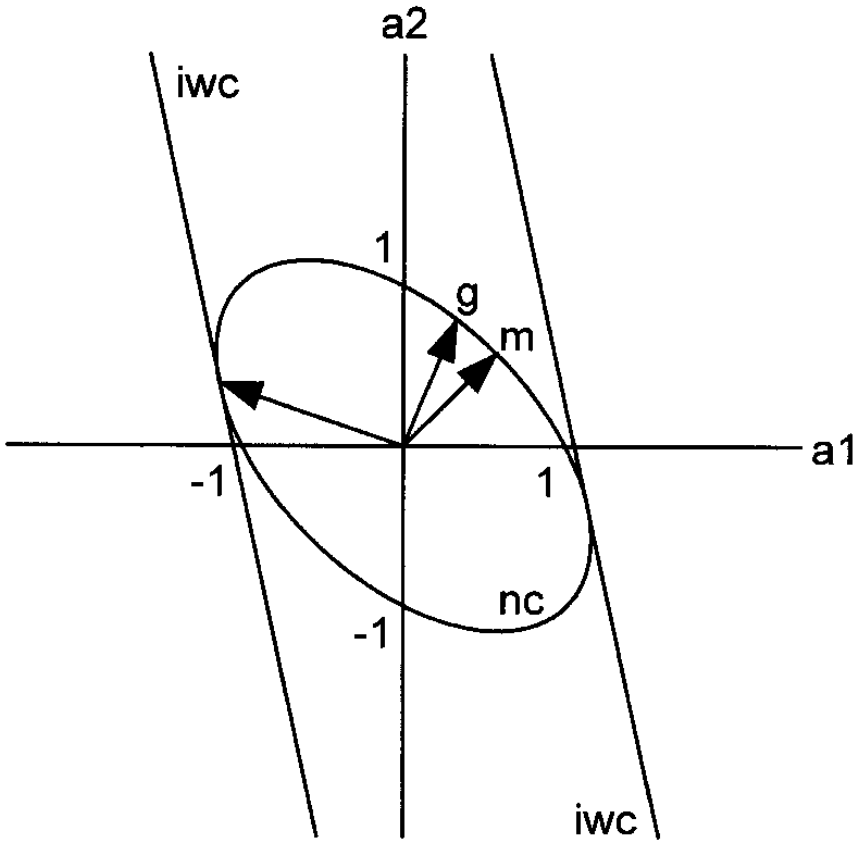


Figure 2

in Equation (15) are parallel straight lines with just such a slope that the tangency between them and the ellipse $A_1' \Sigma A_1 - 1 = 0$ occurs at a point defining a vector perpendicular to g .

6. Uncertainty About Preferences

The preceding analysis assumed great knowledge on the part of the market designers: the designers were assumed to know perfectly all utility functions. The unrealism of this assumption would appear to be an issue if we try to apply this analysis to the design of actual markets. We show that the relaxation of this assumption may restore the importance of a market for shares in the world portfolio.

Uncertainty about preferences poses a real problem to the market designers since we cannot assume that agents have the same uncertainty about their own preference parameters that the market designers do. Agents have perfect knowledge about their own preference parameters and maximize their expected utility knowing their γ_j . The above analysis of market equilibrium, Equations (6)–(11), must be evaluated for the agents' true risk preferences.

When we arrive at the market designers' problem, Equation (15), we face the problem that the market designers do not know the true M and Γ matrices. The market designers are assumed to know the Σ matrix, thus the only reason the market designers do not know agent's demands is that they do not know the coefficients of risk aversion. Supposing now that the true elements of Γ are unknown to the market designers, we will suppose that the market designers choose $N \leq J$ contracts to solve a maximization problem which is the same as Equation (15), but replacing the unknown value to be maximized in Equation (15) with its expected value:⁷

$$A \in \arg \max_{A_n \in \mathbb{R}^J, n=1, \dots, N} \{tr [E (A' \Sigma M \Gamma M' \Sigma A)] | A' \Sigma A = I_N\}. \quad (17)$$

Note that since M is a function of Γ , the expression involves expectations of a nonlinear function of Γ . In order to deal with Equation (17), we rewrite the matrix $A' \Sigma M \Gamma M' \Sigma A$ as

$$A' \Sigma M \Gamma M' \Sigma A \equiv A' \Sigma \Gamma \Sigma A - A' \Sigma u' \Sigma A (t' \Gamma^{-1} t)^{-1}. \quad (18)$$

One obtains Equation (18) by substituting in for M .

Theorem 3. *The A matrix that solves Equation (17) has columns corresponding to the N eigenvectors with highest eigenvalues of*

$$E(\Gamma) \Sigma - u' \Sigma E(t' \Gamma^{-1} t)^{-1} \quad (19)$$

Proof. Substitute Equation (18) into Equation (17) and proceed as in Theorem 1. ■

Note that, unless $E[\Gamma - u'(t' \Gamma^{-1} t)^{-1}]$ is singular, the matrix of Equation (19) is generally nonsingular, and so our conclusion above that only $J - 1$ markets are needed no longer holds. If there is no constraint on the number of markets constructed, the market designers will create all J contracts, and then the contracts will span the world portfolio. Let us assume the γ_j 's for all $j = 1, \dots, J$ are i.i.d. This assumption represents a symmetric state of knowledge of all individuals' risk aversion parameters. With this assumption we can rescale Equation (19) as

$$\Sigma - cu' \Sigma, \quad (20)$$

where $c = E(t' \Gamma^{-1} t)^{-1} / E(\gamma)$.

⁷ We are assuming that risk aversion parameters are independent of endowments. The risk aversion parameters are supposed to be characterizations of tastes, and not of the constraints imposed by the endowments, and so there is no reason to expect any dependency between them. More generally the market designer needs to know the joint distribution of the utility functions (risk aversion parameters) and endowments.

With Equation (20) we can easily take account of specific distributional assumptions about Γ . We need only derive the expected value and expected value of the harmonic mean of the elements of Γ to define the scalar c .⁸

The limiting case of this problem, when the variance of γ increases to infinity, is particularly interesting. This is the case where the market designers' information is becoming more diffuse. By this we mean the designers of markets do not know the coefficients of risk aversion of the agents and there is a higher probability of extreme differences in agents' risk aversion.

Theorem 4. *If γ_j , $j = 1, \dots, J$ are i.i.d. lognormal variates then as the variance, σ^2 , of $\ln(\gamma_j)$ goes to infinity, the A matrix that solves Equation (17) approaches a matrix whose columns are N eigenvectors of Σ with the corresponding highest eigenvalues.*

Proof. See the appendix. ■

If we are going to construct some contract given we do not know what every agent's utility function is and differences in risk aversions are likely to be large, what should the contract be? One wants to somehow maximize the probability that the contract will have the highest welfare improvement in the economy. As such the market designer should construct the contract that markets the largest component of risk in the economy. This is exactly the result of Theorem 4. Given we do not know each agent's risk aversion and differences in risk aversions are likely to be large, we have the best chance of large welfare improvement in the economy by allowing agents to hedge (trade) the most risk possible. The first principal component of Σ is unrestricted by our theory. It *could* have all positive elements and *could* approximate the share in the world portfolio.

If the first principal component of Σ is approximately the share in the world portfolio and its eigenvalue is large, then people have a substantial covariance with this market. Among those agents with similar market exposures, those who are more risk averse can sell a share of the world portfolio to less risk-averse agents, thereby reducing their risk. We do not need to know who is more risk averse in setting up markets to make this possible.

Let us return to the two-agent examples that were plotted in Figures 1 and 2. If we do not know which agent is the more risk averse, then this maximization problem facing the market designers is not as simple as it appeared from that figure. We do not know the position of the vector g , that

⁸ In the working paper version of our article (1997), Theorem 8 shows that if there are JK individuals, K belonging to each of J groups, and if incomes are perfectly correlated within each group, then the bigger problem of designing optimal markets for all JK people collapses to the simpler problem of Theorem 3 here for the J groupings of people. The presence of K individuals per "agent" does not introduce the need for any more than J markets. The heterogeneity of risk aversions across individuals, which we have termed diversity, in each group has the same effect, for the purpose of optimal market design, as does the uncertainty about risk aversions modeled explicitly here.

is whether Figure 1, Figure 2, or some other figure is relevant. Thus the position of the optimal A_1 vector cannot be determined.

We plot instead in Figures 3 and 4 the expected iso-welfare curve (eiwc) to the maximization problem [Equation (17)]. These are not parallel straight lines but ellipses. If we have only a little uncertainty about risk aversion, see for example Figure 3, where $c = 0.49$, the expected iso-welfare curves are elongated and near the origin resemble the parallel straight lines of Figure 1. But if our uncertainty about risk aversion is large, see Figure 4, where $c = 0$, the expected iso-welfare curves are elongated in the perpendicular direction. In the extreme case, where the uncertainty about agents' risk aversion makes it very probable that one is much more risk averse than the other, then, not knowing which is the more risk averse, the best contract we can design in this example is simply a market for shares in the world portfolio.

With very little uncertainty in these terms about the γ 's, the optimal A_1 for our two-agent example with i.i.d. γ 's will still be a vector perpendicular to the world share, a vector with a slope of minus one. Note that Figure 3, where $c = 0.49$, resembles Figure 1 in the vicinity of the origin. Figure 1 corresponds to $c = 0.5$. However, even a small amount of uncertainty means that there will still be a reason to create a second market, and A_2 will be

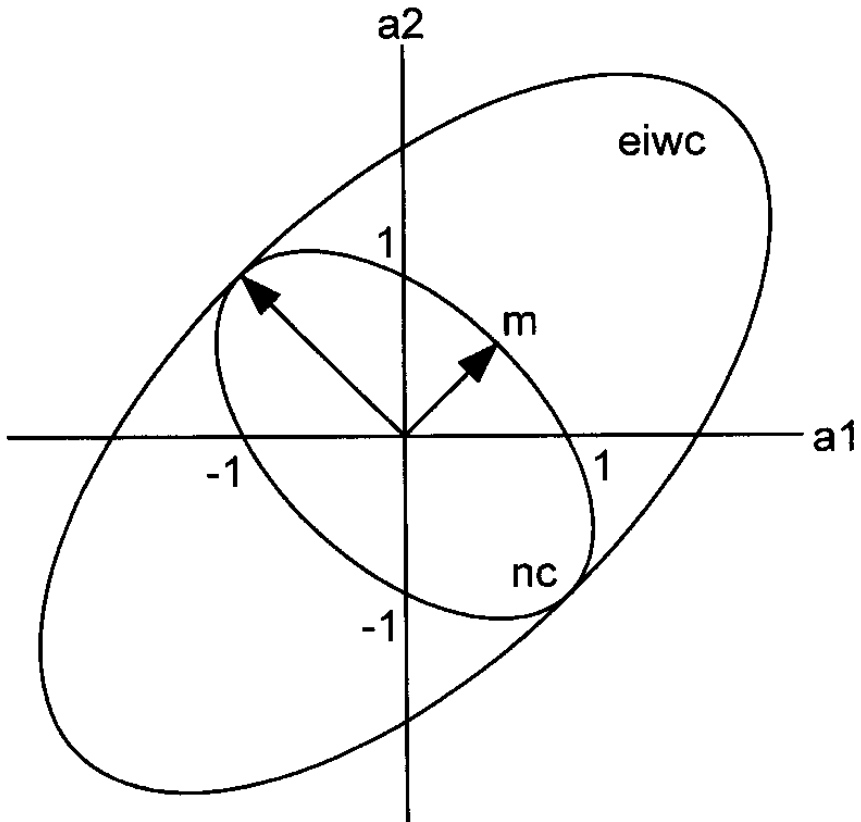


Figure 3

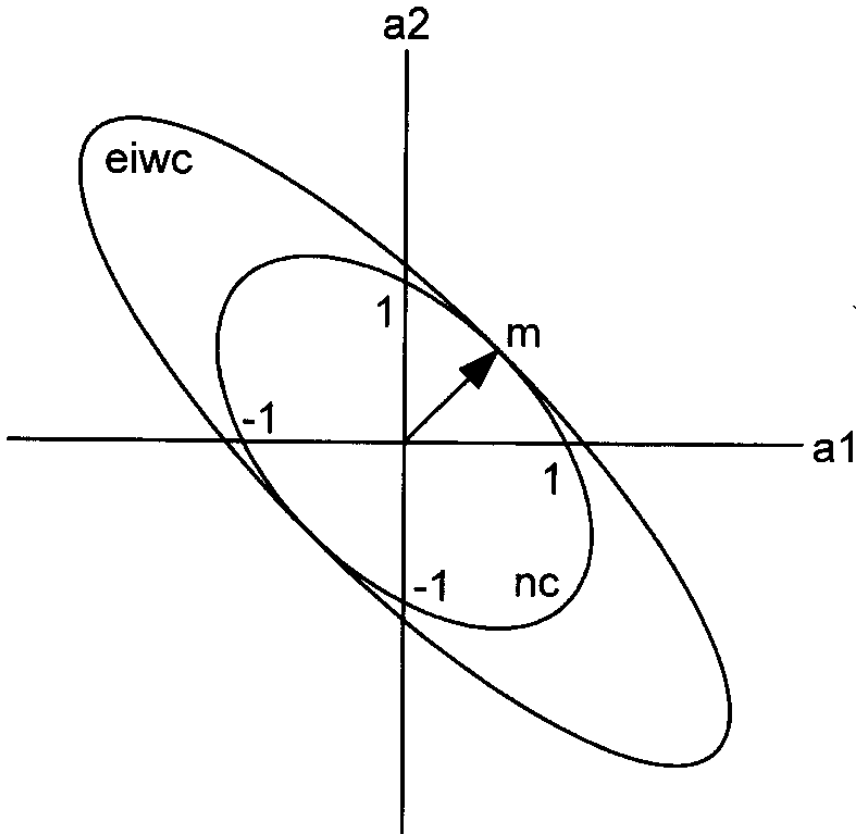


Figure 4

the world share vector, in the first quadrant, with slope of +1. As the uncertainty about the γ 's increases, the eigenvalue corresponding to A_1 shrinks relative to the eigenvalue corresponding to A_2 , and at some point becomes the lower; at this point we must switch the order of the columns of A , and the world portfolio becomes the best portfolio to create. What has happened finally is that uncertainty about the γ 's has become so great that we can no longer predict what kinds of swaps will be useful to agents. The world portfolio may still be useful if either agent is more risk averse than the other; that agent can sell part of the world component of the endowment to the other.

Note that this conclusion using the lognormal assumption might be generalized to other distributions but it is not true of all distributions of $\gamma_j > 0$ with finite means. The important point of the theorem is that the market designers' information about agents' utilities becomes more diffuse. If for some reason, as the variance approaches infinity, the market designers' information becomes less diffuse, then market designers can better construct

contracts since they have more information which results in more welfare improvement.

Consider, for example, a case where γ_j can only take on two values, γ_{j1} and γ_{j2} . γ_{j1} is fixed, the mean $\bar{\gamma}$ is fixed and we vary γ_{j2} . The probability we observe γ_{j1} or γ_{j2} are pr_1 and pr_2 , respectively. Thus we have

$$pr_1\gamma_{j1} + pr_2\gamma_{j2} = \bar{\gamma} \quad (21)$$

and

$$\text{var}(\gamma) = pr_1 (\gamma_{j1} - \bar{\gamma})^2 + pr_2 (\gamma_{j2} - \bar{\gamma})^2. \quad (22)$$

We increase the variance of γ_j by letting the higher value γ_{j2} approach infinity. As we do this we reduce the probability pr_2 that risk aversion for person j equals γ_{j2} . It is easy to show that in the limit, as the variance is increased to infinity, that is, as $\gamma_{j2} \rightarrow \infty$ the expected value of the harmonic mean of J values approaches γ_{j1} . In the limit, the probability approaches one that all J values are the same so that the probability approaches one that the expected value equals the harmonic mean of the J values. This example shows that all peoples risk aversion approaches γ_{j1} in the limit and thus as the variance goes to infinity, the market designers become more informed. Note here as well that the probability of extreme differences in risk aversions, as the variance goes to infinity, approaches zero.

7. Uncertainty about the Variance Matrix

The fact that there is fundamental uncertainty about the variance matrix Σ of endowments offers another reason to conjecture that the world share is important. If the market designers do not have data that will allow them to estimate Σ accurately, then, unless they have priors about Σ , they do not know who has a high variance of income, and who is more correlated with whom. If we also do not have information that certain agents have higher risk aversions than others, then we do not have any reason to stress any one agent over any other agent, or any one grouping of agents over any other. Symmetry suggests that the world portfolio, which weights all agents equally, would be important. In this section we will show a sense in which this suggestion is right.

Fundamental uncertainty about the variance matrix Σ is likely to be high. We attempted in an earlier article [Shiller and Athanasoulis (1995)] to estimate the variance matrix of world endowments by country using historical data on present values of national incomes, but find that our estimates are not highly reliable. There are not many decades of data on national incomes, and much of the differences across countries in variances and covariances appears to be due to a few major historical events that are not likely to be repeated in the same configuration. Japan was found to have a high variance, but we know that this is due to their post-World War II economic miracle.

Does this high estimated variance for Japan imply that Japan will have a high variance in the future? That conclusion would seem unwarranted; one might as well reach the opposite conclusion from the data, that Japan has achieved its transformation to an advanced economy and that henceforth its variance will be low. We found that World War II itself had a great influence on covariances across countries. Do we want to suppose that these covariances are an indication of future covariances? We do not expect World War II to happen again, and do not expect a war involving the same countries in the same way.

While there is probably some information in past data about future variances and covariances, it is worth exploring an alternative route to specifying the parameters we will use to define new markets. One may wish to use a simple prior for the expectation of the variance matrix Σ rather than an estimated Σ . If we assume that there are no historical data that are relevant to judging future variances, then the Σ in the formulas is entirely determined by our priors.

We refer to the principle of exchangeability for definition of priors, advocated by statisticians, de Finetti (1937) and Lindley and Smith (1972), and more recently by McCall (1991). The principle of exchangeability asserts that when our information about random variables is symmetric, when we cannot distinguish one from the other, then our subjective distribution for these variables should have the property that the marginal distribution of each variable is the same, that the marginal distributions of all pairs are the same, of all triplets are the same, etc. We apply the principle of exchangeability to individual people in the world, asserting that we, as market designers, do not have information to tell one from the other. It turns out that this application of the principle of exchangeability may allow us to reach the result that the world portfolio is most important.

Exchangeability thus requires that the variance matrix Σ has all the variances equal to each other (constancy along the diagonal) and all the covariances equal to each other (equality of all off-diagonal elements). Formally, this means that Σ has a simple two-parameter form: $\Sigma = aI + bu'$. We are interested in the cases where $a > 0$ and $b > 0$ and thus Σ is positive definite. Strictly positive covariances is a plausible assumption to make about endowments, since it is natural to assume that there are shocks (e.g., natural resource shocks) that are shared by all agents, and because the estimated covariances, reported in Shiller and Athanasoulis (1995), are almost always positive. By substitution, one finds that this Σ has an eigenvector m and that the corresponding eigenvalue is $a + bJ$. All the other eigenvalues are the same, equal to a . Thus if this is our Σ matrix and the uncertainty about the coefficients of risk aversions approaches infinity so that market designers information becomes more diffuse, it follows from Theorem 4 that the optimal first market to construct is the market for shares in the world portfolio.

The principle of exchangeability can also be generalized somewhat for our purposes into what we will call nested exchangeability, to allow for the existence of higher covariances between people within groupings of people (such as people within countries) than between people in different groupings. Suppose that $J = uV$, where u is the number of people in a grouping and V is the number of groupings. Nested exchangeability means that there is exchangeability within groupings and also between groupings, that is, that $\Sigma = aI + bu' + cD$, where c is a scalar and D is a $J \times J$ block diagonal matrix with V diagonal blocks, each consisting of a $u \times u$ matrix of ones. Then Σ has one eigenvalue equal to $a + bJ + cu$, corresponding to the eigenvector m . Furthermore, it has $V(u - 1)$ eigenvalues equal to a , and $V - 1$ eigenvalues equal to $a + cu$.

It follows from the assumptions that $a > 0$, $b > 0$, and $c \geq 0$ that the eigenvector m has the highest eigenvalue and, from Theorem 4, as the variability across agents of the risk aversion parameters goes to infinity the market for shares in the world portfolio becomes the most important new market to create. More generally, we have the following theorem:

Theorem 5. *If the variance matrix Σ is consistent with the exchangeability principle or the nested exchangeability principle, that is, if $\Sigma = aI + bu' + cD$, where if $c = 0$ exchangeability is unnested and if $c > 0$ exchangeability is nested, and if $b > 0$, and if $E(\gamma_j)$ is the same for all j , $j = 1, \dots, J$, then for sufficiently large uncertainty about the risk aversion coefficients, where sufficiently large means $E(\gamma) > (1 + (a + cu)/(bJ))EH(\Gamma)$, and where $H(\Gamma)$ is the harmonic mean of the J γ_j 's, a market for the shares in the world portfolio will be the most important market to create, that is, solves Equation (17) for $N = 1$.*

Proof. Note that the matrix of Equation (19) has an eigenvector m (corresponding to the world share) with eigenvalue $(a + bJ + cu)(E(\gamma) - EH(\Gamma))$. All other eigenvectors have the eigenvalues $aE(\gamma)$ or $(a + cu)E(\gamma)$. From Theorem 3, the share in the world portfolio is the best first market to create if its eigenvalue is largest, and the result follows immediately. ■

This theorem is an important result; it justifies the creation of a market for shares in the world portfolio. It also implies that, under the assumptions stated, both higher variability in risk aversion across individuals and larger off-diagonal elements b in the variance matrix Σ make it more likely that the world portfolio will be the first-best portfolio.

One can further generalize this result to unbalanced situations where we have groupings of agents and each group consists of a different number of agents. Suppose there are V groups indexed by $v = 1, \dots, V$ and each group has u_v individuals in it. Then we can show with sufficiently large uncertainty about the coefficients of risk aversion, the market for shares in the world portfolio is spanned by the first V markets. A discussion of this (with a proof

for the case of the uncertainty of the coefficients of risk aversion approaching infinity) is found in the appendix in the section entitled unbalanced nested exchangeability. We still find the strong conclusion that the market for shares in the world portfolio is among the most important markets in the world.

8. Preexisting Markets

The above analysis takes no account of preexisting markets, markets for some endowments, or linear combinations of endowments that already exist before the market designers begin to define new markets (contracts). Since we live in a world with existing markets, it is important to consider what effect these might have on the analysis.

One issue is whether a market for shares in the world portfolio might already exist, at least approximately. For example, if the correlation of national income with the stock market is high, then it may be possible to cross-hedge world portfolio risk with a stock market futures contract. However, we do not believe this to be the case. Stocks are claims only on the corporate profits segment of national incomes, and corporate profits are only a small fraction of these incomes. Historical data do not reveal much of a correlation between stock market returns and estimated innovations in present values of national products [see Shiller (1993a), and Bottazzi, Pesenti, and Van Wincoop (1996)].⁹

It is shown in the appendix that if there is a preexisting market, then our conclusion in Theorem 2 above, that the share in the world portfolio is not spanned by all markets that the market designers would create, no longer holds. Moreover, the appendix shows an alternative maximization problem, taking account of existing markets, that a market designer could use to define the next optimal market to create. One might conclude that this alternative maximization problem in the appendix is more relevant for market designers. However, we are somewhat inclined against this conclusion. We should not automatically assume that we are constrained by preexisting markets. History shows that preexisting derivative markets actually do sometimes wither away when another derivative market appears that serves hedgers better.¹⁰

Our analysis of preexisting markets in the appendix actually leads us to a significant conclusion about the world shares: that the market for shares in the world portfolio is a good market to create first in the sense that creating it, so that it is preexisting when other markets are then created, makes the definition of the other markets robust to uncertainty on the part of the market designers about agents' preferences. We state this as the theorem below:

⁹ However, a different view is expressed by Baxter and Jermann (1997). This disagreement within our profession is a good example of why the case with uncertainty about the variance matrix is important.

¹⁰ An example of this is the demise of the GNMA CDR futures resulting from the formation of the Treasury-bond futures [see Johnston and McConnell (1989)].

Theorem 6. *When there is a preexisting market for shares in the world portfolio, the market design problem is invariant to the amount of uncertainty the designers of markets have about the coefficients of risk aversions.*

Proof. One will notice this by comparing Equation (34) to Equation (38) in the appendix and noticing that the only difference is the expectation operator.¹¹ ■

Once the market for shares in the world portfolio is created, then the optimal markets to create thereafter (if risk aversions are expected to be the same for all agents) are defined in terms of the eigenvectors of the variance matrix of residuals when endowments are regressed on the world portfolio. All other assets have an expected price (conditional on the market designers' information) of zero (Lemma 2). Moreover, uncertainty about agents' preferences does not affect the market designers' design problem. These results may be described as implying that creating a market for shares in the world portfolio adds to the robustness of designing markets. Furthermore, constructing a world share market and residual risk markets adds to the simplicity for agents to use these markets. The world share can be used by agents to adjust the amount of systematic risk they have in their endowment after trading. The remaining contracts will be used to hedge residual or nonsystematic risk, which they can do so by solving a variance minimization problem.

9. Conclusion and Practical Implications for Contract Design

We have presented several alternative maximization problems for market designers to define optimal risk management contracts. Thus we have several alternative definitions of the optimal markets to create.

The simplest maximization problem, Equation (15), is the most restrictive: it assumes no uncertainty on the part of the market designer about individual agents' preferences. It yielded the striking conclusion that the contracts created would never allow trading shares in the world portfolio, and no linear combination of the portfolios defined in the contracts could even have non-negative quantities of all endowments. The question is, how restrictive are the assumptions in Equation (15)?

The problem with Equation (15) as a basis for defining new markets is that it assumes too much knowledge on the part of market designers. The market designer is supposed to be designing markets themselves to take account of which agent is more risk averse than another, or which agent perceives more risk than another. We must recast the problem so that market designers' lack of knowledge is represented. As a matter of historical fact, market designers

¹¹ Equation (34) is the matrix whose eigenvectors define optimal markets when there is no uncertainty in the coefficients of risk aversion and the market for world shares is a preexisting market. Equation (38) is similar except that there is uncertainty in the coefficients of risk aversion.

have found it very difficult to predict in advance of creating a new market who will want to take positions in the new market. Our representation of uncertainty about preference parameters can be regarded as a metaphor for our difficulty in predicting investor behavior.

Market designers lack of knowledge about individual agents' preferences led us to Theorem 3. Taking account of this uncertainty would be of great practical importance for market designers. If market designers assumed enormous uncertainty about preferences, so that the limiting case described in Theorem 4 applies and markets are defined in terms of eigenvectors of the variance matrix of endowments, then if there is a substantial market component in the economy they may conclude that something approximating the world portfolio would be the most important market.

Market designers' lack of knowledge about the variance matrix led us to Theorem 5. If we assume that because of uncertainty about the variance matrix the market designers adopt the exchangeability principle (or nested exchangeability principle) for defining the variance matrix, and assuming endowments are positively correlated, then even a small amount of uncertainty about preferences will result in the world portfolio always being among the markets that allow some expected welfare gain. If market designers' uncertainty about preferences is sufficiently large it will necessarily be the first-best market to create.

Of course, one may feel that the assumptions that imply the market for shares in the world portfolio is first best are not likely to capture all of our prior information about the distribution of endowments or about preferences. The exchangeability principle is not likely to be the only way of restricting the variance matrix, and uncertainty about preferences may not be so very large. Still, given arguments we have made about robustness of optimal contracts when there is a market for the world portfolio, given additional arguments we have made about the simplicity of price interpretation when there is a world portfolio, and given also a natural human desire for contracts that are simple and easily interpreted, we think that a market for shares in the world portfolio is a natural one to create.

We wish to propose to futures or security exchanges that the market for world shares be created as an experiment. While our results do not ensure the success of such a market, we feel that our results are enough reason to try, given the possibility of large welfare gains.

Appendix

Proof of Theorem 1. We may write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & A'_1 \Sigma M \Gamma M' \Sigma A_1 + \cdots + A'_N \Sigma M \Gamma M' \Sigma A_N \\ & - \lambda_1 (A'_1 \Sigma A_1 - 1) + \cdots + \lambda_N (A'_N \Sigma A_N - 1). \end{aligned} \quad (23)$$

We are requiring in this problem that the diagonal of the matrix $A'\Sigma A$ is equal to ι . The first-order conditions can be written as

$$\Sigma M \Gamma M' \Sigma A_n = \lambda_n \Sigma A_n \quad \forall n = 1, \dots, N \tag{24}$$

and

$$A'_n \Sigma A_n = 1 \quad \forall n = 1, \dots, N. \tag{25}$$

If we define Λ to be the $N \times N$ diagonal matrix with the n th diagonal element to be λ_n , we can combine the first-order conditions to obtain

$$\Sigma M \Gamma M' \Sigma A = \Sigma A \Lambda \tag{26}$$

and

$$\text{diag}(A'\Sigma A) = \iota. \tag{27}$$

Thus taking the inverse of Σ through Equation (26) gives us the result. Finally if one premultiplies Equation (26) by A' , one obtains $A'\Sigma M \Gamma M' \Sigma A = \Lambda$. The trace of the left-hand side of this is the objective function the market designer is trying to maximize. Since this equals Λ , it is diagonal and as such the designer of markets will choose the N eigenvectors corresponding to the N largest eigenvalues.¹² ■

Note that if we take a Cholesky decomposition of the variance matrix Σ , $\Sigma = C'C$, and premultiply through Equation (26) by C'^{-1} , then $C'M \Gamma M' C'$ is positive semidefinite and symmetric with eigenvectors CA . The eigenvalues of a positive semidefinite symmetric matrix are all real and nonnegative, and these are the same as the eigenvalues of $M \Gamma M' \Sigma$. Since the rank of M is $J - 1$, there are only $J - 1$ nonzero eigenvalues, and hence only $J - 1$ contracts are of any value. Thus there is no point in creating all J possible contracts; at most $J - 1$ are needed and A need have no more than $J - 1$ columns. If there is a fixed cost to creating markets, then N , the number of markets created, can be chosen optimally. We create all markets whose eigenvalues (divided by two) are greater than this cost.

One will notice in the above problem that we did not constrain the off-diagonal elements of $A'\Sigma A$ to be zero. Notice, however, that Λ is diagonal and since $C'M \Gamma M' C'$ is positive semidefinite and symmetric with eigenvectors CA , it follows that $A'\Sigma M \Gamma M' \Sigma A$ is diagonal. Since $A'\Sigma M \Gamma M' \Sigma A = A'\Sigma A \Lambda$ it must be that $A'\Sigma A$ is diagonal. Thus the constraint that the off-diagonal elements are zero are satisfied in the unconstrained problem. This was shown by Darroch (1965) and by Okamoto and Kanazawa (1968).

Proof of Theorem 2. By Equation (26) it follows that $A = M \Gamma M' \Sigma A \Lambda^{-1}$. Since $gM = 0$, it follows that $gA = 0$, that is, the A matrix is orthogonal to g . We can then show by contradiction that all $N \leq J - 1$ contracts do not span the world portfolio: if there exists a vector v such that $Av = m$, then $gAv = gm = \iota' \Gamma^{-1} \iota (\iota' \Sigma \iota)^{-0.5} (\iota' \Gamma^{-1} \Sigma \Gamma^{-1} \iota)^{-0.5} \neq 0$, which is a contradiction.¹³ ■

¹² The second-order conditions that we have a maximum are satisfied.

¹³ The proof of the theorem relies on the assumed nonsingularity of Σ . To see this, note that if all incomes were perfectly correlated, so that the rank of Σ were one, then markets for any linear combination of endowments would serve the same purpose, and so the world portfolio would be as good as any other, and not orthogonal to all optimal markets. If an agent had a riskless endowment, then Theorem 2 would not hold as well. But, singularity of Σ , that is, zero uncertainty about some linear combination of individual endowments, is something we can generally rule out. Each individual has a unique combination of circumstances that determine lifetime income, and no lifetime income is assured.

Proof of Lemma 1. By Theorem 2, g is orthogonal to all optimal contracts and $g = m$ here.¹⁴ ■

Proof of Theorem 4. Define the geometric mean of risk aversion parameters to be $G(\Gamma) = (\prod_{j=1}^J \gamma_j)^{1/J}$ and the harmonic mean as $H(\Gamma) = (\frac{1}{J} \sum_{j=1}^J \gamma_j^{-1})^{-1}$. Under the lognormal assumption, $(E(G(\Gamma))/E(\gamma)) = (\exp(\mu + (\sigma^2/2J))/\exp(\mu + (\sigma^2/2))) = \exp(-\sigma^2((J-1)/2J))$, therefore $\lim_{\sigma^2 \rightarrow \infty} (E(G(\Gamma))/E(\gamma)) = 0$. Since $H(\Gamma) \leq G(\Gamma)$ everywhere [see, e.g., Hardy, Littlewood, and Polya (1964, p. 26)], then $\lim_{\sigma^2 \rightarrow \infty} (E(H(\Gamma))/E(\gamma)) = \lim_{\sigma^2 \rightarrow \infty} Jc = 0$. Thus the limit of the matrix of Equation (20) as σ^2 goes to infinity is Σ . Since the solution of problem Equation (17) is a continuous function of the elements of the matrix of Equation (20), and since the limit of a continuous function is the function of the limit, the theorem follows. ■

Preexisting markets

Let us modify problem Equation (15) to represent that there is a single preexisting contract, where the coefficients of the endowments in the linear combination that defines this preexisting contract are given by the $J \times 1$ vector A_1 , the first column of A , which, without loss of generality, we normalize so that $A_1' \Sigma A_1 = 1$. (It is trivial to extend our results to more than one preexisting contract.) The market designers will then design $N^* = N - 1$ markets, choose $A^* = [A_1^* \ A_2^* \ \dots \ A_{N^*}^*]$, the remaining columns of A , ($A = [A_1 : A^*]$) subject to the normalization rule $A' \Sigma A = I$. Then A^* is defined by:

$$A^* \in \arg \max_{A_{n^*}^* \in \mathbb{R}^J, n^*=1, \dots, N^*} \{tr(A^{*'} \Sigma M \Gamma M' \Sigma A^*) | A^{*'} \Sigma A^* = I_{N^*}, A^{*'} \Sigma A_1 = 0\}. \tag{28}$$

Theorem 7. *The A^* matrix that solves Equation (28) has columns corresponding to the N^* eigenvectors with highest eigenvalues of*

$$\Phi M \Gamma M' \Phi' \Sigma, \tag{29}$$

where $\Phi \equiv I_J - A_1 A_1' \Sigma$.

Proof. We can write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & A_1^{*'} \Sigma M \Gamma M' \Sigma A_1^* + \dots + A_{N^*}^{*'} \Sigma M \Gamma M' \Sigma A_{N^*}^* - \lambda_1 (A_1^{*'} \Sigma A_1^* - 1) \\ & + \dots - \lambda_{N^*} (A_{N^*}^{*'} \Sigma A_{N^*}^* - 1) - \delta_1 A_1^{*'} \Sigma A_1 \\ & + \dots - \delta_{N^*} A_{N^*}^{*'} \Sigma A_1. \end{aligned} \tag{30}$$

The first-order conditions are

$$2 \Sigma M \Gamma M' \Sigma A_{n^*}^* - 2 \lambda_{n^*} \Sigma A_{n^*}^* - \delta_{n^*} \Sigma A_1 = 0 \quad \forall n^* = 1, \dots, N^* \tag{31}$$

and

$$A_{n^*}^{*'} \Sigma A_{n^*}^* - 1 = 0 \quad \forall n^* = 1, \dots, N^*, \tag{32}$$

$$A_{n^*}^{*'} \Sigma A_1 = 0 \quad \forall n^* = 1, \dots, N^*. \tag{33}$$

If we premultiply Equation (31) by A_1' , then we obtain $\delta_{n^*} = 2 A_1' \Sigma M \Gamma M' \Sigma A_{n^*}^*$. If we substitute δ_{n^*} into Equation (31), form the N^* equations $n^* = 1, \dots, N^*$ into a matrix and rearrange, we arrive at an equation in terms of eigenvectors of Equation (29). If we premultiply

¹⁴ One can obtain Theorem 2 and Lemma 1 by assuming endowments and new contracts are normally distributed and CARA utility. With more general utilities, again the optimal new markets constructed will depend on the number of markets constructed so they may not all be orthogonal to g . However, they will still never span the world portfolio.

Equation (31) by $A_{n^*}^{*}$ then we obtain $A_{n^*}^{*'} \Sigma M \Gamma M' \Sigma A_{n^*}^* = \lambda_{n^*}$ which is the elements of the expression of the function the designer of markets is trying to maximize. Thus the designer of markets chooses the columns of A^* as the N^* eigenvectors with the highest eigenvalues of Equation (29). ■

The market for world shares as a preexisting market

It is instructive to consider the problem for the market designers with the constraint that the first market is the market for shares in the world portfolio, that is, assuming that $A_1 = m$. If the share in world portfolio is already tradable, these markets are conceptually relatively simple to understand, and such simplicity might promote more effective use of the markets.

Lemma 2. *If the market for shares in the world portfolio exists (i.e., if $A_1 = m$) then all other contracts (constructed so that our normalization $A' \Sigma A = I_N$ holds) will necessarily have a zero price.*

Proof. If the first contract is the market for world shares, m , then it must be the case that the rest of the contracts $A_{n^*}^*$, $n^* = 1, \dots, N^*$ are constructed such that $A_{n^*}^{*'} \Sigma A_1 = A_{n^*}^{*'} \Sigma \iota / \sqrt{\iota' \Sigma \iota} = 0$. If this is the case, then $A_{n^*}^{*'} \Sigma \iota = 0$, and from Equation (7) the result follows. ■

Let us define the $N^* \times J$ matrix Q^* such that its j th column is the demand vector for agent j of the N^* contracts. We then have:

Theorem 8. *When $A_1 = m$ the A^* matrix that solves Equation (28) has the property that $Q^* = -A^* \Sigma M$ has columns corresponding to the N^* eigenvectors with highest eigenvalues of*

$$\Phi' \Sigma \Phi \Gamma. \tag{34}$$

Proof. Using Equation (18) and Lemma 2, the problem the market designers solve is

$$A^* \in \arg \max_{A_{n^*}^* \in \mathbb{R}^J, n^*, \dots, N^*} \{tr (A^{*'} \Sigma \Gamma \Sigma A^*) \mid A^{*'} \Sigma A^* = I_{N^*}, A^{*'} \Sigma A_1 = 0, A_1 = m\}. \tag{35}$$

Proceed as with Theorem 6. ■

$\Phi' \Sigma \Phi$ is the variance matrix of residuals when the endowments are regressed on the world endowment. If $\Gamma = I_J$, that is if everyone has the same risk aversion, then the optimal markets are defined in terms of eigenvectors of this simple variance matrix. Moreover, since $Q^{*'} = -\Sigma A^*$, the position that agent j holds of the n th contract is just the regression coefficient corresponding to the n th contract when the endowment of that agent is regressed on the vector of contract payoffs $x A^*$. These results, coupled with the above-noted zero prices for all contracts other than the share in the world portfolio, make this equilibrium a simple one to understand. Once the world share is traded, the problem agents face for orthogonal contracts is only a variance minimization problem.

Uncertainty about preferences with a preexisting world portfolio

We have seen above that if the pre-existing market is the share in the world portfolio, then all remaining contracts constructed, such that $A' \Sigma A = I$, have a zero price (Lemma 2). It is interesting to ask what the optimal contracts are if there is uncertainty about γ_j 's and the market for the world share already exists. The market designers choose A^* such that

$$A^* \in \arg \max_{A_{n^*}^* \in \mathbb{R}^J, n^*, \dots, N^*} \{E tr (A^{*'} \Sigma M \Gamma M' \Sigma A^*) \mid A^{*'} \Sigma A^* = I_{N^*}, A^{*'} \Sigma A_1 = 0, A_1 = m\}. \tag{36}$$

Using Equation (18) and noting from the constraints that $A^* \Sigma \iota = 0$, we may rewrite the market designers' problem as

$$A^* \in \underset{A_{n^*}^* \in \mathbb{R}^J, n^*, \dots, N^*}{\arg \max} \left\{ \text{tr}(A^* \Sigma E(\Gamma) \Sigma A^*) \mid A^* \Sigma A^* = I_{N^*}, A^* \Sigma A_1 = 0, A_1 = m \right\}. \quad (37)$$

Theorem 9. *The A^* matrix that solves Equation (37) has the property that $Q^* = -A^* \Sigma$ has columns corresponding to the N^* eigenvectors with highest eigenvalues of*

$$\Phi' \Sigma \Phi E(\Gamma). \quad (38)$$

Proof. Proceed as with Theorem 7. ■

This theorem shows that given the expectations of Γ , uncertainty about the γ_j 's does not affect the optimal markets when the market for the world share is a preexisting market. We know that the amount of uncertainty (Theorem 3) or diversity [see Theorem 8 from our earlier version (1997)], of the γ_j 's affects the optimal contracts if the market for the world share is not preexisting. As such, one reason to construct the market for the world share first is that the remaining markets' definitions are robust to misspecification of the uncertainty or diversity of γ_j 's.

Unbalanced nested exchangeability

In this section we show that unbalanced nested exchangeability, where not all groupings have the same number of individuals, gives us that the market for world shares is spanned by the first V markets, where V is the number of groups indexed by $v = 1, \dots, V$. Let u_v be the number of individuals in group v . There are J agents in the world with $J = \sum_{v=1}^V u_v$. The variance matrix is given by $\Sigma = aI + bu' + cD$, where a, b , and c are scalars and D is a $J \times J$ block diagonal matrix with V blocks on the diagonal, each consisting of a $u_v \times u_v$ matrix of ones. Then Σ has V eigenvectors the k th of which has the form $[\alpha_{1k} \iota'_1 \alpha_{2k} \iota'_2 \dots \alpha_{V_k} \iota'_V]$ with corresponding eigenvalues equal to $a + b(\sum_{v=1}^V u_v \alpha_{vk})^2 + c(\sum_{v=1}^V u_v^2 \alpha_{vk}^2)$ for $k = 1, \dots, V$, where ι_v is a $u_v \times 1$ vector of ones and α_{vk} are scalars defined above. Furthermore, Σ has $J - V$ eigenvalues equal to a .

It follows from the assumptions that $a > 0, b > 0$, and $c > 0$ that the V largest eigenvalues are, for $k = 1, \dots, V$, equal to $a + b(\sum_{v=1}^V u_v \alpha_{vk})^2 + c(\sum_{v=1}^V u_v^2 \alpha_{vk}^2)$ and the corresponding eigenvectors span the shares in the world portfolio, m . Now suppose the variability across agents of the risk aversion parameters goes to infinity so that the designers of markets information becomes more diffuse. From Theorem 4, the market for shares in the world portfolio becomes one of the most important new markets to create, that is, it is spanned by the first V markets that should be constructed. We state this formally in the following theorem:

Theorem 10. *If the variance matrix Σ is consistent with the unbalanced nested exchangeability principle, that is, if $\Sigma = aI + bu' + cD$, where $a > 0, b > 0$, and $c > 0$, and if $E(\gamma_j)$ is the same for all $j, j = 1, \dots, J$, then as the uncertainty of the coefficients of risk aversion goes to infinity and the market designers information becomes more diffuse, from Theorem 4, a market for shares in the world portfolio is spanned by the first V markets.*

Proof. The optimal N contracts to construct are the first N eigenvectors of Equation (19) with the corresponding largest eigenvalues. From Theorem 4, as the uncertainty of the coefficients of risk aversion approaches infinity, the optimal N contracts to construct are the first N eigenvectors of Σ with the corresponding largest eigenvalues. The Σ matrix has V eigenvectors with eigenvalues $(a + b(\sum_{v=1}^V u_v \alpha_{vk})^2 + c(\sum_{v=1}^V u_v^2 \alpha_{vk}^2))$. All other eigenvectors have the eigenvalues a . The result follows since these first V eigenvectors span the world shares as shown above. ■

Note that we can extend this result to cases where uncertainty of the coefficients of risk aversion is “sufficiently large” to obtain the result that the first V markets span the world shares. To do so, however, requires much notation and clutter so we leave it out.

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