

Eigenvalues of Random Graphs

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21.1 Introduction

In this lecture, we consider a random graph on n vertices in which each edge is chosen to be in the graph with probability one-half, independently of course. We will show that the eigenvalues of the adjacency matrix of such a graph are tightly concentrated. Curiously, the adjacency matrix eigenvalues are much more tightly concentrated than the Laplacian matrix eigenvalues.

The adjacency matrix of such a random graph may be described by choosing the values of $A(i, j)$ to be zero with probability $1/2$ and 1 with probability $1/2$, subject to $A(i, j) = A(j, i)$. Of course, we fix $A(i, i) = 0$ for all i . The expectation of every off-diagonal entry of the matrix is $1/2$. Let M denote this expected matrix, and observe that

$$M = \frac{1}{2}A_{K_n} = \frac{1}{2}(J_n - I_n),$$

where A_{K_n} is the adjacency matrix of the complete graph on n vertices, J_n is the all-1s matrix and I_n is of course the identity. From this formula, we see that M has one eigenvalue of $(n-1)/2$ and $n-1$ eigenvalues of $-1/2$. We will show that the eigenvalues of A are very close to this. In particular, we will prove that

$$\|A - M\| \leq 1.34\sqrt{n},$$

with exponentially high probability.

So, we will really focus on bounding the norm of $A - M$. As $A - M$ is a symmetric matrix, we have

$$\|A - M\| = \max_i |\lambda_i(A - M)| = \max_{\mathbf{x}} \left| \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right|.$$

Our analysis will focus on this last term.

Set

$$R = A - M,$$

and let $r_{i,j} = R(i, j)$, for $i < j$. Each $r_{i,j}$ is a random variable that is independently and uniformly distributed in $\pm 1/2$.

21.2 One Rayleigh Quotient

To begin, we fix *any* unit vector \mathbf{x} , and consider

$$\mathbf{x}^T R \mathbf{x} = \sum_{i < j} 2r_{i,j} \mathbf{x}(i) \mathbf{x}(j).$$

This is a sum of independent random variables, and so may be proved to be tightly concentrated around its expectation, which in this case is zero. There are many types of concentration bounds, with the most popular being the Chernoff and Hoeffding bounds. In this case we will apply Hoeffding's inequality.

Theorem 21.2.1 (Hoeffding's Inequality). *Let a_1, \dots, a_m and b_1, \dots, b_m be real numbers and let X_1, \dots, X_m be independent random variables such that X_i takes values between a_i and b_i . Let $\mu = \mathbf{E} [\sum_i X_i]$. Then, for every $t > 0$,*

$$\Pr \left[\sum X_i \geq \mu + t \right] \leq \exp \left(-\frac{2t^2}{\sum_i (b_i - a_i)^2} \right).$$

To apply this theorem, we view

$$X_{i,j} = 2r_{i,j} \mathbf{x}(i) \mathbf{x}(j)$$

as our random variables. As $r_{i,j}$ takes values in $\pm 1/2$, we can set

$$a_{i,j} = -\mathbf{x}(i) \mathbf{x}(j) \quad \text{and} \quad b_{i,j} = \mathbf{x}(i) \mathbf{x}(j).$$

We then compute

$$\sum_{i < j} (b_i - a_i)^2 = \sum_{i < j} 4\mathbf{x}(i)^2 \mathbf{x}(j)^2 = 2 \sum_{i \neq j} \mathbf{x}(i)^2 \mathbf{x}(j)^2 \leq 2 \left(\sum_i \mathbf{x}(i)^2 \right) \left(\sum_i \mathbf{x}(j)^2 \right) = 2,$$

as \mathbf{x} is a unit vector.

We thereby obtain the following bound on $\mathbf{x}^T R \mathbf{x}$.

Lemma 21.2.2. *For every unit vector \mathbf{x} ,*

$$\Pr_R [|\mathbf{x}^T R \mathbf{x}| \geq t] \leq 2e^{-t^2}.$$

Proof. The expectation of $\mathbf{x}^T R \mathbf{x}$ is 0. The preceding argument tells us that

$$\begin{aligned} \Pr [|\mathbf{x}^T R \mathbf{x}| \geq t] &\leq \Pr [\mathbf{x}^T R \mathbf{x} \geq t] + \Pr [\mathbf{x}^T R \mathbf{x} \leq -t] \\ &\leq \Pr [\mathbf{x}^T R \mathbf{x} \geq t] + \Pr [\mathbf{x}^T (-R) \mathbf{x} \geq t] \\ &\leq 2e^{-t^2}, \end{aligned}$$

where we have exploited the fact that R and $-R$ are identically distributed. □

21.3 Vectors near \mathbf{v}_1

You might be wondering what good the previous argument will do us. We have shown that it is unlikely that the Rayleigh quotient of any given \mathbf{x} is large. But, we have to reason about all \mathbf{x} of unit norm.

Lemma 21.3.1. *Let R be a symmetric matrix and let \mathbf{v} be a unit eigenvector of R whose eigenvalue has absolute value $\|R\|$. If \mathbf{x} is another unit vector such that*

$$\mathbf{v}^T \mathbf{x} \geq \sqrt{3}/2,$$

then

$$\mathbf{x}^T R \mathbf{x} \geq \frac{1}{2} \|R\|.$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of R and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a corresponding set of orthonormal eigenvectors. Assume without loss of generality that $\lambda_1 \geq |\lambda_n|$ and that $\mathbf{v} = \mathbf{v}_1$. Expand \mathbf{x} in the eigenbasis as

$$\mathbf{x} = \sum_i c_i \mathbf{v}_i.$$

We know that $c_1 \geq \sqrt{3}/2$ and $\sum_i c_i^2 = 1$. This implies that

$$\mathbf{x}^T R \mathbf{x} = \sum_i c_i^2 \lambda_i \geq c_1^2 \lambda_1 - \sum_{i \geq 2} c_i^2 |\lambda_i| = \lambda_1 \left(c_1^2 - \sum_{i \geq 2} c_i^2 \right) = \lambda_1 (2c_1^2 - 1) \geq \lambda_1/2.$$

□

We will bound the probability that $\|R\|$ is large by taking Rayleigh quotients with random unit vectors. Let's examine the probability that a random unit vector \mathbf{x} satisfies the conditions of Lemma 21.3.1.

Lemma 21.3.2. *Let \mathbf{v} be an arbitrary unit vector, and let \mathbf{x} be a random unit vector. Then,*

$$\Pr \left[\mathbf{v}^T \mathbf{x} \geq \sqrt{3}/2 \right] \geq \frac{1}{\sqrt{\pi n} 2^{n-1}}$$

Proof. Let B^n denote the unit ball in \mathbb{R}^n , and let C denote the cap on the surface of B^n containing all vectors \mathbf{x} such that

$$\mathbf{v}^T \mathbf{x} \geq \sqrt{3}/2.$$

We need to lower bound the ratio of the surface area of the cap C to the surface area of B^n .

Recall that the surface area of B^n is

$$\frac{n\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)},$$

where I recall that for positive integers n

$$\Gamma(n) = (n-1)!,$$

and that $\Gamma(x)$ is an increasing function for real $x \geq 1$.

Now, consider the $(n-1)$ -dimensional hypersphere whose boundary is the boundary of the cap C . As the cap C lies above this hypersphere, the $(n-1)$ -dimensional volume of this hypersphere is a

lower bound on the surface area of the cap C . Recall that the volume of a sphere in \mathbb{R}^n of radius r is

$$r^n \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

In our case, the radius of the hypersphere is

$$r = \sin(\arccos \sqrt{3}/2) = 1/2.$$

So, the ratio of the $(n-1)$ -dimensional volume of the hypersphere to the surface area of B^n is at least

$$\frac{r^{n-1} \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2} + 1)}}{\frac{n\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}} = \frac{r^{n-1}}{\sqrt{\pi n}} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n-1}{2} + 1)} \geq \frac{r^{n-1}}{\sqrt{\pi n}} \geq \frac{1}{\sqrt{\pi n} 2^{n-1}}.$$

□

21.4 The Probabilistic Argument

I'm going to do the following argument very slowly, because it is both very powerful and very subtle.

Theorem 21.4.1. *Let R be a symmetric matrix with zero diagonal and off-diagonal entries uniformly chose from $\pm 1/2$. Then,*

$$\Pr[\|R\| \geq t] \leq \sqrt{\pi n} 2^n e^{-t^2/4}.$$

Proof. Let R be a fixed symmetric matrix. By applying Lemma 21.3.2 to any eigenvector of R whose eigenvalue has maximal absolute value, we find

$$\Pr_{\mathbf{x}} \left[|\mathbf{x}^T R \mathbf{x}| \geq \frac{1}{2} \|R\| \right] \geq \frac{1}{\sqrt{\pi n} 2^{n-1}}.$$

Thus, for a random R we find

$$\begin{aligned} \Pr_{R, \mathbf{x}} \left[\|R\| \geq t \text{ and } |\mathbf{x}^T R \mathbf{x}| \geq \frac{1}{2} \|R\| \right] &= \Pr_R [\|R\| \geq t] \Pr_{R, \mathbf{x}} \left[|\mathbf{x}^T R \mathbf{x}| \geq \frac{1}{2} \|R\| \mid \|R\| \geq t \right] \\ &\geq \Pr_R [\|R\| \geq t] \frac{1}{\sqrt{\pi n} 2^{n-1}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Pr_{R, \mathbf{x}} \left[\|R\| \geq t \text{ and } |\mathbf{x}^T R \mathbf{x}| \geq \frac{1}{2} \|R\| \right] &\leq \Pr_{R, \mathbf{x}} [\|R\| \geq t \text{ and } |\mathbf{x}^T R \mathbf{x}| \geq t/2] \\ &\leq \Pr_{R, \mathbf{x}} [|\mathbf{x}^T R \mathbf{x}| \geq t/2] \\ &= \mathbf{E}_{\mathbf{x}} [\Pr_R [|\mathbf{x}^T R \mathbf{x}| \geq t/2]] \\ &\leq 2e^{-(t/2)^2}, \end{aligned}$$

where the last inequality follows from Lemma 21.2.2.

Combining these inequalities, we obtain

$$\Pr_R [\|R\| \geq t] \frac{1}{\sqrt{\pi n} 2^{n-1}} \leq e^{-(t/2)^2},$$

which implies the claimed result. \square

The probability in Theorem 21.4.1 becomes small once $e^{t^2/4}$ exceeds $\sqrt{\pi n} 2^n$. As n grows large, this happens for

$$t > 2\sqrt{\ln 2}\sqrt{n} \sim (5/3)\sqrt{n}.$$

This is a little worse than the bound that I claimed at the beginning of the lecture. The reason is that I have optimized this proof so that all the numbers that appear are nice. To get the tightest bound possible, we should choose an inner product other than $\sqrt{3}/2$ in Lemma 21.3.1, and then propagating the change through the proof. If we replace $\sqrt{3}/2$ by 0.9571, we obtain an upper bound on the norm of $1.34\sqrt{n}$.

It is known that the norm of R is unlikely to be much more than \sqrt{n} . This is proved by Füredi and Komlós [FK81] and Vu [Vu07], using a very different technique. The idea behind these papers is to consider $\text{Tr}(R^k)$ for a high power of k . They show that the expectation of this variable is unlikely to be large, and exploit the fact that

$$\|R\| \leq \left(\text{Tr}(R^k)\right)^{1/k}.$$

21.5 Question

Is there a variant of this proof that yields good bounds when the entries of A have a lower probability of being 1? For example, consider the case in which there is a number $p < 0.5$ such that each entry of A is 1 with probability p . To get a good proof in this regime, one needs a concentration inequality that is stronger than Hoeffding's.

References

- [FK81] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.
- [Vu07] Van Vu. Spectral norm of random matrices. *Combinatorica*, 27(6):721–736, 2007.