

## THE AVERAGE VALUE OF THE SMARANDACHE FUNCTION

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Given a positive integer  $n$ , let  $P(n)$  denote the largest prime factor of  $n$  and  $S(n)$  denote the smallest integer  $m$  such that  $n$  divides  $m!$

The function  $S(n)$  is known as the Smarandache function and has been intensively studied [1]. Its behavior is quite erratic [2] and thus all we can reasonably hope for is a statistical approximation of its growth, e.g., an average. It appears that the sample mean  $E(S)$  satisfies [3]

$$E(S(N)) = \frac{1}{N} \cdot \sum_{n=1}^N S(n) = O\left(\frac{N}{\ln(N)}\right)$$

as  $N$  approaches infinity, but I don't know of a rigorous proof. A natural question is if some other sense of average might be more amenable to analysis.

Erdős [4,5] pointed out that  $P(n) = S(n)$  for almost all  $n$ , meaning

$$\lim_{N \rightarrow \infty} \frac{|\{n \leq N: P(n) < S(n)\}|}{N} = 0 \quad \text{that is,} \quad |\{n \leq N: P(n) < S(n)\}| = o(N)$$

as  $N$  approaches infinity. Kastanas [5] proved this to be true, hence the following argument is valid. On one hand,

$$\lambda = \lim_{n \rightarrow \infty} E\left(\frac{\ln(P(n))}{\ln(n)}\right) \leq \lim_{n \rightarrow \infty} E\left(\frac{\ln(S(n))}{\ln(n)}\right) = \lim_{N \rightarrow \infty} \frac{1}{N} \cdot \sum_{n=1}^N \frac{\ln(S(n))}{\ln(n)}$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \cdot \left( \sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} + \sum_{P(n)<S(n)} \frac{\ln(S(n))}{\ln(n)} \right)$$

The second part vanishes:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \cdot \left( \sum_{P(n) < S(n)} \frac{\ln(S(n))}{\ln(n)} \right) \leq \lim_{N \rightarrow \infty} \frac{1}{N} \cdot \left( \sum_{P(n) < S(n)} 1 \right) = \lim_{N \rightarrow \infty} \frac{o(N)}{N} = 0$$

while the first part is bounded from above:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \cdot \left( \sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} \right) \leq \lim_{N \rightarrow \infty} \frac{1}{N} \cdot \sum_{n=1}^N \frac{\ln(P(n))}{\ln(n)} = \lim_{n \rightarrow \infty} E \left( \frac{\ln(P(n))}{\ln(n)} \right) = \lambda$$

We deduce that

$$\lim_{n \rightarrow \infty} E \left( \frac{\ln(S(n))}{\ln(n)} \right) = \lambda = 0.6243299885\dots$$

where  $\lambda$  is the famous Golomb-Dickman constant [6-9]. Therefore  $\lambda \cdot n$  is the asymptotic average number of digits in the output of  $S$  at an  $n$ -digit input, that is, 62.43% of the original number of digits. As far as I know, this result about the Smarandache function has not been published before.

A closely related unsolved problem concerns estimating the variance of  $S$ .

### References

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