Differences of the partition function

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ABSTRACT

Let $p(n)$ denote the number of unrestricted partitions of n, and let $\Delta p(n) = p(n) - p(n-1)$, $\Delta^k p(n) = \Delta(\Delta^{k-1} p)(n)$. This note answers several questions about the behavior of the k-difference $\Delta^k p(n)$ by proving that if k is large enough, there is an integer $n_0(k)$ such that $\Delta^k p(n)$ alternates in sign for $n < n_0(k)$ and is nonnegative for $n \geq n_0(k)$. It is also shown that $n_0(k) \sim \frac{6}{\pi^2} k^2 (\log k)^2$ as $k \to \infty$.

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Dedicated to Paul Erdös on the occasion of his 75th Birthday.

1. Introduction

If $f(n)$ is any function on the nonnegative integers, define its first difference Δf by $\Delta f(n) = f(n) - f(n-1)$ for $n \geq 1$, $\Delta f(0) = f(0)$. The k-th difference $\Delta^k f$ of f is then defined recursively by $\Delta^k f = \Delta (\Delta^{k-1} f)$. A few years ago, I. J. Good [5a] asked about the behavior of $\Delta^k p(n)$, where $p(n)$ denotes the number of unrestricted partitions of n. He initially conjectured [5a] that if $k > 3$, then the sequence $\Delta^k p(n)$, $n = 0, 1, \ldots,$ alternates in sign. However, computations by R. Razen and independently by I. J. Good and his associates [5b] found counterexamples to this conjecture, and led to a new conjecture, namely that for each fixed k, $\Delta^k p(n) > 0$ for n sufficiently large. I. J. Good [5b] even made the stronger conjecture that for each k, there is an $n_0(k)$ such that $\Delta^k p(n)$ alternates in sign for $n < n_0(k)$, and $\Delta^k p(n) \geq 0$ for $n \geq n_0(k)$. He also suggested that $6(k-1)(k-2) + k^3/2$ might be a good approximation to $n_0(k)$. Some further computations by R. A. Gaskins led I. J. Good to revise his conjecture about the size of $n_0(k)$, and suggest that $\pi k^{5/2}$ might be a good approximation to it [5c].

At about the same time as the first publication of I. J. Good's problem, the same question about the sign of $\Delta^k p(n)$ was also raised independently by G. E. Andrews, and was answered by H. Gupta [6]. Gupta noted that $\Delta p(n) > 0$ for all n, and gave a simple proof of the result that $\Delta^2 p(n) \geq 0$ for $n \geq 2$, while $\Delta^2 p(0) = 1$, $\Delta^2 p(1) = -1$. Gupta also noted that it can be shown easily using the Hardy-Ramanujan-Rademacher series [1,2,3,7,8] for $p(n)$ that for each k, $\Delta^k p(n) > 0$ if n is sufficiently large. In fact, this result can be obtained from some of the earliest of the Hardy-Ramanujan approximations [7] to $p(n)$:

$$p(n) = \frac{1}{2 \pi \sqrt{2}} \frac{d}{dn} (\lambda_n^{-1} \exp(C\lambda_n)) + O(\exp((C/2+\varepsilon) n^{1/2})) , \tag{1.1}$$

for every $\varepsilon > 0$, where $C = \pi(2/3)^{1/2}$ and $\lambda_n = (n - 1/24)^{1/2}$. The k-th difference of the second term on the right side of (1.1) is of the same order of magnitude as that term (for k fixed, $n \to \infty$), while the k-th difference of the first term is very close to its k-th derivative. Thus we obtain the estimate

$$\Delta^k p(n) = C_k n^{-k/2} p(n) (1 + O(n^{-1/2})) \quad \text{as} \quad n \to \infty , \tag{1.2}$$

where $C_k = (\pi/\sqrt{6})^k$. (Gupta's asymptotic estimate of $\Delta^k p(n)$ in [6] is incorrect.) Gupta's computations led him to the same conjecture as Good's about $\Delta^k p(n)$ alternating up to some $n_0(k)$ and then immediately becoming positive, but Gupta conjectured that $n_0(k) \sim k^3$ as $k \to \infty$.

Another easy proof that $\Delta^k p(n)$ is positive for large n can be obtained by applying the theorem of Bateman and Erdös [4]. They showed that if $p_A(n)$ denotes the number of partitions of n into summands taken from some set A of positive integers (repetitions allowed), then $\Delta^k p_A(n) \geq 0$ for all large n if and only if the greatest common divisor of each subset $B \subseteq A$ with $|A \setminus B| = k$ is equal to 1. The Bateman and Erdös result is far too general, though, to provide information about initial segments of $\Delta^k p_A(n)$.

This paper carries the investigation of $\Delta^k \, p(n)$ further, and largely settles the Good-Gupta conjectures. The main result is the following.

Theorem. There is a k_0 *so that if* $k \geq k_0$*, then there is an integer* $n_0(k)$ *such that* $(-1)^n \, \Delta^k \, p(n) > 0$ *for* $0 \leq n < n_0(k)$ *and* $\Delta^k \, p(n) \geq 0$ *for* $n \geq n_0(k)$*. Furthermore,*

$$n_0(k) \sim \frac{6}{\pi^2} \, k^2 \, (\log k)^2 \quad as \quad k \to \infty \, . \tag{1.3}$$

With more work it would probably be possible to establish the above result for all k. Such an extension would require replacing various O-estimates by explicit numerical bounds. We should note that the above result does not exclude the possibility that $\Delta^k \, p(n) = 0$ might occur. In fact, the proof shows that for each large k, $\Delta^k \, p(n) = 0$ can hold for at most one value of n, and it can be shown with more effort that values of k for which $\Delta^k \, p(n) = 0$ occurs for some n are very rare. It is probably true that $\Delta^k \, p(n) = 0$ has only finitely many solutions among all pairs k, n, but this conjecture seems to be hard to prove.

The asymptotic approximation (1.3) is not very accurate for small k. For example, from the computational results quoted in [5c], it appears that $n_0(30) = 15416$. Now for $k = 30$, $\pi \, k^{5/2} = 15486.49...$, while $6 \, \pi^{-2} \, k^2 \, (\log k)^2 = 6329.32...$. The proof of (1.3) can be used to obtain more accurate estimates of $n_0(k)$, however.

2. Intuitive explanation of result

If $F(z)$ denotes the generating function of $p(n)$,

$$F(z) = \sum_{n=0}^{\infty} p(n) z^n , \tag{2.1}$$

then it is well known (and easy to see) that

$$F(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-1} . \tag{2.2}$$

If we define $F_k(z)$ to be the generating function of $\Delta^k p(n)$,

$$F_k(z) = \sum_{n=0}^{\infty} \Delta^k p(n) z^n , \tag{2.3}$$

then

$$F_k(z) = (1 - z)^k F(z) = (1 - z)^k \prod_{m=1}^{\infty} (1 - z^m)^{-1} . \tag{2.4}$$

The theorem could be proved by investigating the analytic behavior of $F_k(z)$, but we will only use $F_k(z)$ to explain why the Good-Gupta conjectures are true.

The basic philosophy in the use of generating functions for asymptotic analysis is that the singularities of the function determine the behavior of the coefficients. Generally speaking, a dominant singularity (i.e., one near which the function grows faster than near other points) at 1 corresponds to a monotone increasing sequence, while a dominant singularity at -1 corresponds to an alternating sequence. The function $F(z)$ has the unit circle as its natural boundary. However, as was shown by Hardy and Ramanujan [7], $F(z)$ is most singular (i.e., grows fastest) near 1, is next most singular at -1, and is much better behaved away from those two points. This led them to the following refinement of (1.1):

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} (\lambda_n^{-1/2} \exp(C\lambda_n)) + \frac{(-1)^n}{2\pi} \frac{d}{dn} (\lambda_n^{-1} \exp(C\lambda_n/2))$$

$$+ O(\exp(n^{1/2} (C/3 + \varepsilon))) \qquad (2.5)$$

for any $\varepsilon > 0$. (Taking other points on $|z| = 1$ into account led Hardy-Ramanujan to their famous asymptotic series [7].) The first term on the right in (2.5) comes from $z = 1$, the second from $z = -1$, and the remainder is the contribution of the rest of the circle.

The importance of the fact that $z = 1$ is the dominant singularity of $F(z)$ and $z = -1$ is next most dominant is that when we study $\Delta^k p(n)$, we deal with the generating function $F_k(z) = (1-z)^k F(z)$. The effect of multiplying $F(z)$ by $(1-z)^k$ is that the singularity at $z = -1$ increases in influence, as the function is increased by about 2^k near $z = -1$. On the other hand, the singularity at $z = 1$ diminishes in influence. Since $F(z)$ grows much faster than any polynomial in $(1-z)^{-1}$ as $z \to 1$, this diminution is fairly small very close to $z = 1$, and therefore for large n, the size of $\Delta^k p(n)$ largely reflects the influence of the singularity at $z = 1$. However, for small n, this diminution is nontrivial, and allows $z = -1$ to dominate. All the other points on $|z| = 1$ make contributions that are still smaller than that of $z = -1$. The reason that the transition from alternation of signs to positivity is very sharp is that in the transition zone, the singularity at $z = 1$ begins to dominate very rapidly. Let us write

$$\Delta^k p(n) = a(n) + (-1)^n b(n) + c(n) ,$$

where $a(n)$ is the positive contribution from $z = 1$, $b(n)$ is the absolute value of the contribution from $z = -1$, and $c(n)$ is the remainder. Then in the transition region $a(n+1) - a(n)$ is about $2(b(n+1) - b(n))$, and is much larger than $c(n)$, so that once $\Delta^k p(n)$ becomes nonnegative, it stays nonnegative.

The above presents an intuitive explanation of the mechanism that causes the Good-Gupta phenomenon of alternation followed by abrupt transition to positivity. This explanation could be developed into a rigorous proof, using relatively simple analytic methods. The estimates in the transitional region between alternation of signs and positivity would in fact be fairly simple, using the rough estimates of [7]. However, the need to cover the range of small values of n requires more delicate analysis, and so the proof presented below uses the Rademacher convergent series expansion for $p(n)$ [1,2,3,8]. The explanation above presents an intuitive picture of what's happening which is not obvious from the proof below, in which the analytic behavior of the generating function shows up only indirectly in the form of the Rademacher expansion (3.3).

3. Detailed proof

We first use a very simple argument to show that for k large, $\Delta^k\, p(n)$ alternates in sign for n up to about $k/2$.

Proposition 3.1. For any $\varepsilon \in (0, 10^{-10})$ *there is a* $k_1(\varepsilon)$ *such that if* $k \geq k_1(\varepsilon)$ *and* $0 \leq n \leq (1/2 - \varepsilon)k$, *then*

$$(-1)^n\, \Delta^k\, p(n) > 0 \ .$$

Proof. Note that in the range $0 \leq n \leq (1/2 - \varepsilon)k$,

$$(-1)^n\, \Delta^k\, p(n) = \sum_{j=0}^{n} (-1)^j \binom{k}{n-j} p(j) \ .$$

Now if $0 \leq j < n$,

$$\begin{bmatrix} k \\ n-j \end{bmatrix} \begin{bmatrix} k \\ n-j-1 \end{bmatrix}^{-1} = \frac{k - n+j+1}{n-j} \geq \frac{k - n+1}{n} \geq 1 + \varepsilon .$$

By the Hardy-Ramanujan approximation (1.1), we see that $p(j+1) / p(j) < 1 + \varepsilon$ for $j \geq 2m_0 (\varepsilon)$. Hence for every $m_1 \geq m_0$ we have

(3.1)

$$\sum_{j=2m_1}^{n} (-1)^j \begin{bmatrix} k \\ n-j \end{bmatrix} p(j) \geq \sum_{m=m_1}^{\lfloor n/2 \rfloor} \left\{ \begin{bmatrix} k \\ n-2m \end{bmatrix} p(2m) - \begin{bmatrix} k \\ n-2m-1 \end{bmatrix} p(2m+1) \right\} > 0$$

since each term is positive.

To deal with the remaining sum, we note that

$$\sum_{j=0}^{2m_1-1} (-1)^j \begin{bmatrix} k \\ n-j \end{bmatrix} p(j) = \begin{bmatrix} k \\ n \end{bmatrix} \sum_{j=0}^{2m_1-1} (-1)^j \begin{bmatrix} k \\ n-j \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} p(j) .$$

Now for $0 \leq j \leq 2m_1 - 1$ and $n \leq (1/2 - \varepsilon) k$,

$$\begin{bmatrix} k \\ n-j \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} = \prod_{i=1}^{j} \frac{n - j+i}{k - n+i} = \left[\frac{n}{k} \right]^j (1 + O(k^{-1})) ,$$

(the constant in the O-notation depending on m_1 and ε), so

$$\sum_{j=0}^{2m_1-1} (-1)^j \begin{bmatrix} k \\ n-j \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} p(j) = \sum_{j=0}^{2m_1-1} (-1)^j \left[\frac{n}{k} \right]^j p(j) + O(k^{-1}) . \quad (3.2)$$

The infinite sum (2.1) for $F(z)$ does not vanish on the segment $[-(1/2 - \varepsilon), 0]$ because it has the convergent infinite product (2.2) in which all the terms are nonzero, and therefore for some $\delta = \delta(\varepsilon) > 0$, we must have $F(z) \geq \delta$ for $z \in [-(1/2 - \varepsilon), 0]$. Since the partial sums of the infinite sum in (2.1) converge to $F(z)$ uniformly on compact subsets of the unit disk, there is some m_2 such that for all

$m \geq 2\, m_2 - 1$, and all $z \in [-(1/2 - \varepsilon), 0]$,

$$\sum_{j=0}^{m} p(j) z^j \geq \delta/2 .$$

We now select $m_1 = \max(m_0, m_2)$, so that m_1 depends on ε alone, and discover from (3.2) that for $k \geq k_1(\varepsilon)$,

$$\sum_{j=0}^{2m_1 - 1} (-1)^j \left[\frac{n}{k} \right]^j p(j) + O(k^{-1}) \geq \delta/4 ,$$

which proves the proposition.

□

We next consider slightly larger values of n. First we recall the Rademacher convergent series expansion for $p(n)$ [1,2,3,8]. As before, we let

$$C = \pi\, (2/3)^{1/2} \quad , \quad \lambda_n = (n - 1/24)^{1/2} .$$

Then, for any $n \geq 1$,

$$p(n) = \frac{1}{\pi\, 2^{1/2}} \sum_{m=1}^{\infty} A_m(n) m^{1/2} \frac{d}{dn} (\lambda_n^{-1} \sinh(C m^{-1} \lambda_n)) , \tag{3.3}$$

where the $A_m(n)$ satisfy

$$A_1(n) = 1 \text{ and } A_2(n) = (-1)^n \quad \text{for} \quad n \geq 1 , \tag{3.4}$$

$$|A_m(n)| \leq m \quad \text{for all} \quad m,n \geq 1 . \tag{3.5}$$

(The $A_m(n)$ are known explicitly in terms of Dedekind sums [1,2,3,7,8].)

We define, for $m,n \geq 1$,

$$f_m(n) = m^{1/2} \frac{d}{dn} (\lambda_n^{-1} \sinh(Cm^{-1} \lambda_n)) , \tag{3.6}$$

and $f_m(0) = 0$, and we let

$$R_n = \sum_{m=3}^{\infty} A_m(n) f_m(n) , \tag{3.7}$$

so that

$$p(n) = \pi^{-1} 2^{-1/2} \left\{ f_1(n) + (-1)^n f_2(n) + R_n \right\} . \tag{3.8}$$

Lemma 3.2. For all $n \geq 1$,

$$|R_n| \leq \frac{3}{5} f_2(n) \tag{3.9}$$

and

$$|R_n| \leq 10 f_3(n) . \tag{3.10}$$

Proof of Lemma. The estimates (3.9) and (3.10) can be verified numerically for $1 \leq n \leq 50$ by computing $p(n)$, $f_1(n)$, and $f_2(n)$. (Tables of values of $p(n)$ are contained in [1,7], for example, or they can be computed using the recurrences in [1,3,7].) For $n > 50$, we use the estimate [3; pp. 191-192]

$$\left| \sum_{m=5}^{\infty} A_m(n) f_m(n) \right| \leq 2 C^2 \lambda_n^{-1} \left\{ C \lambda_n / 12 + 25^{-1} \sinh(C \lambda_n / 4) \right\}$$

together with the explicit formulas for $f_3(n)$ and $f_4(n)$ to prove (3.9) and (3.10).

□

The estimate (3.9) is tight only for very small n, while the constant 10 in (3.10) could easily be decreased with slightly more careful work.

We next investigate $\Delta^k \, p(n)$ for ranges of n not covered by Proposition 3.1.

Proposition 3.3. There are constants c_1, k_2, *and* $\varepsilon > 0$ *such that if* $k \geq k_2$, *then the following estimates hold:*

(a) For $2k/5 \leq n \leq k - 2$,

$$|\Delta^k f_1(n)| \leq c_1 \, k^{1/2} \binom{k}{n} . \tag{3.11}$$

(b) For $k-1 \leq n \leq k+1$,

$$|\Delta^k f_1(n)| \leq c_1 \, k^5 \, \exp(c_1 \, k^{1/2}) . \tag{3.12}$$

(c) For $k+2 \leq n$,

$$|\Delta^k f_1(n)| \leq c_1 \, n^{-k/10} \, \exp(c_1 \, n^{1/2}) . \tag{3.13}$$

(d) For $(1/2 - \varepsilon) k \leq n \leq k/2$,

$$|\Delta^k f_1(n)| \leq \frac{23}{10} \binom{k}{n} . \tag{3.14}$$

Proof. From the proof of Rademacher's convergent series (3.3) (see [2; p. 109], for example) we find that

$$f_1(n) = \frac{\alpha}{2 \pi i} \int_{(\beta)} t^{-5/2} \exp(t + \gamma \lambda_n^2 \, t^{-1}) \, dt , \tag{3.15}$$

where

$$\alpha \; = \; \pi^{7/2} \;\, 6^{-\,3/2} \quad , \;\; \gamma \; = \; \pi^2 / 6 \;, \tag{3.16}$$

β is any constant with $\beta \; > \; 0$, and (β) denotes the straight line from $\beta \; - \; i \, \infty$ to $\beta \; + \; i \, \infty$. Therefore, if

$$| z | \;\, e^{\gamma / \beta} \; < \; 1 \;, \tag{3.17}$$

then

$$\begin{aligned} \sum_{n=1}^{\infty} f_1 \, (n) z^n \; &= \; \frac{\alpha}{2 \, \pi \, i} \int_{(\beta)} t^{-\,5/2} \, \exp(t \; - \; \gamma/(24t)) \sum_{n=1}^{\infty} z^n \;\, e^{\gamma \, n/t} \;\, dt \\ &= \; \frac{\alpha}{2 \, \pi \, i} \int_{(\beta)} t^{-\,5/2} \;\, z \; \exp(t \; + \; \frac{23\gamma}{24t}) \;\; \frac{dt}{1 \; - \; z \; e^{\gamma/t}} \;\;, \end{aligned} \tag{3.18}$$

and so

$$\begin{aligned} G_k \, (z) \; &= \; \sum_{n=1}^{\infty} z^n \;\, \Delta^k \;\, f_1 \, (n) \\ &= \; \frac{\alpha \; (1-z)^k}{2 \, \pi \, i} \int_{(\beta)} t^{-\,5/2} \;\, z \; \exp(t \; + \; \frac{23\gamma}{24t}) \;\; \frac{dt}{1 \; - \; z \; e^{\gamma/t}} \;\; . \end{aligned} \tag{3.19}$$

The expansion (3.18) has been obtained only under the assumption (3.17), but the integral on the right hand side of (3.18) is analytic in all of $\mathbb{C} \setminus [e^{-\,\gamma/\beta} , \, \infty)$ (i.e., the entire complex plane with a slit along the positive real axis from $e^{-\,\alpha/\beta}$ to infinity removed). Thus (3.19) gives an analytic continuation of $G_k \, (z)$ to the domain $\mathbb{C} \setminus [1, \infty)$, provided that when z is real, $z \, \in \, (0,1)$, we choose $\beta \; > \; -\gamma / \log \; z$.

We now use (3.19) to obtain bounds for $\Delta^k \;\, f_1 \, (n)$. If $\mathrm{Re}(z) \; < \; 1$, $| 1-z | \; \geq \; 1/100$, we choose $\beta \; = \; 1000$, and then for $\mathrm{Re}(t) \; = \; \beta$ we have

$$\left| \frac{z \exp(t + \frac{23\gamma}{24t})}{1 - z e^{\gamma/t}} \right| \le c_2$$

for some constant $c_2 > 0$. Therefore for some $c_3 > 0$,

$$|G_k(z)| \le c_3 |1-z|^k \tag{3.20}$$

holds for all z with $\mathrm{Re}(z) < 1$, $|1-z| \ge 1/100$.

Suppose next that $\mathrm{Re}(z) < 1$, $0 < |1-z| < 1/100$. In this case we let $w = 1 - \mathrm{Re}(z)$ and $\beta = 2\gamma/w$. Then $|z| \ge 1-w$, $|e^{\gamma/t}| \le e^{w/2}$,

$$\left| 1 - z e^{\gamma/t} \right| \ge (1-w) e^{w/2} - 1 \ge w/10 ,$$

and so

$$|G_k(z)| \le \frac{c_4}{w} |1-z|^k \exp(2\gamma/w) . \tag{3.21}$$

We now use the estimates (3.20) and (3.21) to bound $\Delta^k f_1(n)$. We have

$$\Delta^k f_1(n) = \frac{1}{2\pi i} \int_S G_k(z) \frac{dz}{z^{n+1}} , \tag{3.22}$$

where S is any simple closed curve around the origin in the domain $\mathbb{C} \setminus [1,\infty)$. We will select a radius $r > 0$ later. Given r, we choose S to consist of S_1, that portion of the circle $|z| = r$ that lies to the left of the line $\mathrm{Re}(z) = 1 - (2\gamma/n)^{1/2}$ (which might be all of that circle) together with S_2, the straight line segment formed by the intersection of the disk $|z| \le r$ and the line $\mathrm{Re}(z) = 1 - (2\gamma/n)^{1/2}$ when there is such an intersection. By (3.20), we find that

$$\left| \frac{1}{2\pi i} \int_{S_1} G_k(z) \frac{dz}{z^{n+1}} \right| \le c_5 \frac{(1+r)^k}{r^n} + c_5 n^{1/2} 100^{-k} r^{-n} \exp((2\gamma n)^{1/2}) .$$

On the other hand, by (3.21) we find that when S_2 exists,

$$\left| \frac{1}{2\pi i} \int_{S_2} G_k(z) \frac{dz}{z^{n+1}} \right| \le c_6 n^{1/2} \exp((2\gamma n)^{1/2}) \int_{S_2} \frac{|1-z|^k}{|z|^{n+1}} |dz| .$$

Hence we conclude that for any $r > 0$,

$$|\Delta^k f_1(n)| \le c_7 (1+r)^k r^{-n} + c_7 n^{1/2} 100^{-k} r^{-n} \exp((2\gamma n)^{1/2})$$

$$+ c_8 n^{1/2} \exp((2\gamma n)^{1/2}) \int_0^{\omega} \frac{(2\gamma n^{-1} + v^2)^{k/2} \, dv}{(1 - 2(2\gamma/n)^{1/2} + 2\gamma/n + v^2)^{(n+1)/2}} , \tag{3.23}$$

where

$$\omega = \begin{cases} 0 & \text{if } r \le 1 - (2\gamma/n)^{1/2} , \\ (r^2 - 1 + 2(2\gamma/n)^{1/2} - 2\gamma/n)^{1/2} & \text{if } r > 1 - (2\gamma/n)^{1/2} . \end{cases} \tag{3.24}$$

For $2k/5 \le n \le k-2$, we now select $r = n/(k-n)$. We have for k sufficiently large and for $0 \le v \le \omega$,

$$\frac{(2\gamma n^{-1} + v^2)^{k/2}}{(1-2(2\gamma n^{-1})^{1/2} + 2\gamma/n + v^2)^{(n+1)/2}} \le \frac{(2\gamma n^{-1} + \omega^2)^{k/2}}{(1-2(2\gamma n^{-1})^{1/2} + 2\gamma/n + \omega^2)^{(n+1)/2}}$$

$$= \frac{\left[r^2 - 1 \right]^{k/2}}{r^{n+1}} ,$$

so that

$$|\Delta^k f_1(n)| \le c_9 (1+r)^k r^{-n} + c_{10} n^{1/2} (r^2-1)^{k/2} r^{-n} \exp((2\gamma n)^{1/2})$$

$$\le c_{11} (1+r)^k r^{-n} \le c_{12} k^{1/2} \binom{k}{n} . \tag{3.25}$$

For $k-1 \le n \le k+1$, we select $r = k$ and obtain from (3.23) the bound

$$|\Delta^k f_1(n)| \le c_{13} k^{k-n} + c_{14} k^{3/2} \exp((2\gamma k)^{1/2}) \frac{\left[k^2-1\right]^{k/2}}{k^{n+1}}$$

$$\le c_{15} k^5 \exp((2\gamma k)^{1/2}) . \tag{3.26}$$

Finally, for $k+1 < n$, we let $r \to \infty$ and obtain, for $a = 1 - 2(2\gamma n^{-1})^{1/2} + 2\gamma n^{-1}$,

$$|\Delta^k f_1(n)| \le c_{16} n^{1/2} \exp((2\gamma n)^{1/2}) \int_0^\infty \frac{(2\gamma n^{-1} + v^2)^{k/2}\, dv}{(1 - 2(2\gamma n^{-1})^{1/2} + 2\gamma/n + v^2)^{(n+1)/2}} .$$

Now the integral on the right side above is (for large k and $n \ge k+2$)

$$\le \int_0^{n^{-1/10}} \frac{(2n^{-1/5})^{k/2}\, dv}{(1 - 2(2\gamma n^{-1})^{1/2})^{(n+1)/2}} + \int_{n^{-1/10}}^\infty \frac{dv}{(1 - 2(2\gamma n^{-1})^{1/2} + v^2)^{(n+1-k)/2}}$$

$$\le 2^k n^{-k/10} \exp(c_{17} n^{1/2}) + \int_{n^{-1/5}}^\infty \frac{u^{-1/2}\, du}{(1 - 2(2\gamma n^{-1})^{1/2} + u)^{(n+1-k)/2}}$$

$$\le 2^k n^{-k/10} \exp(c_{17} n^{1/2}) + (1 - 2(2\gamma n^{-1})^{1/2} + n^{-1/5})^{-(n-1-k)/2} ,$$

and this yields the estimate

$$|\Delta^k f_1(n)| \le c_{18} n^{-k/10} \exp(c_{19} n^{1/2}) . \tag{3.27}$$

To complete the proof of the proposition, we consider $(1/2 - \varepsilon)k \le n \le k/2$, where $\varepsilon \in (0, 10^{-10})$ will be selected later. We use the same contour of integration as before, with $r = n/(k-n)$, except that we let

$$S_3 = \left\{ z \in S \colon |z+r| \le k^{-1/3} \right\} . \tag{3.28}$$

Then, by using estimates similar to those developed earlier, but bounding $|1-z|$ on $S \setminus S_3$ more carefully, we obtain

$$\left| \frac{1}{2\pi i} \int_{S \setminus S_3} \frac{G_k(z)}{z^{n+1}} \, dz \right| \le c_{20} \, r^{-n} \max_{z \in S \setminus S_3} |1-z|^k$$

$$+ \, c_{21} \, n^{1/2} \, (r^2 - 1)^{k/2} \, r^{-n} \exp((2 \gamma n)^{1/2}) \, . \tag{3.29}$$

Now for $z \in S \setminus S_3$, and k sufficiently large,

$$|1-z| \le (1+r) \, (1 - k^{-2/3}/10) \, ,$$

and so for k large,

$$\left| \frac{1}{2\pi i} \int_{S \setminus S_3} \frac{G_k(z)}{z^{n+1}} \, dz \right| \le c_{22} \, (1+r)^k \, r^{-n} \exp(- k^{1/4}) \, . \tag{3.30}$$

We next estimate the integral over S_3 by the saddle point method. Using (3.19) and interchanging orders of integration, we obtain

$$\frac{1}{2\pi i} \int_{S_3} \frac{G_k(z)}{z^{n+1}} \, dz = \frac{\alpha}{2\pi i} \int_{(\beta)} t^{-5/2} \exp\left(t + \frac{23\gamma}{24t}\right) dt \cdot g(n, z, t) \, , \tag{3.31}$$

where

$$g(n, z, t) = \frac{1}{2\pi i} \int_{S_3} \frac{z(1-z)^k}{1 - z \, e^{\gamma/t}} \, \frac{dz}{z^{n+1}} \, .$$

Making the change of variable $z = - r e^{i\theta}, - \theta_0 \le \theta \le \theta_0$, where $\theta_0 \sim r^{-1} k^{-1/3}$ as $k \to \infty$, we find that

$$g(n, z, t) = \frac{(-1)^{n-1} r^{1-n}}{2\pi} \int_{-\theta_0}^{\theta_0} \frac{(1 + r e^{i\theta})^k}{1 + r e^{i\theta + \gamma/t}} e^{-(n-1)i\theta} d\theta . \qquad (3.32)$$

We now select $\beta = 100$, say. Then γ/t is bounded for all t on the line from $\beta - i\infty$ to $\beta + i\infty$, and $1 + r \exp(i\theta + \gamma/t)$ is bounded away from 0. Furthermore,

$$1 + r e^{i\theta} = (1+r) \exp\left(\frac{ir}{1+r}\theta - \frac{r\theta^2}{2(1+r)^2} + O(|\theta|^3)\right) , \qquad (3.33)$$

where the constant in the O-term is independent of r. (Recall that $1 - 10^{-5} \le r \le 1$.) Next $kr/(1+r) = n\theta$, so

$$g(n, z, t) = \frac{(-1)^{n-1} r^{1-n} (1+r)^k}{2\pi} \int_{-\theta_0}^{\theta_0} \frac{\exp\left(-\frac{r k \theta^2}{2(1+r)^2} + O(k|\theta|^3) + i\theta\right) dt}{1 + r e^{i\theta + \gamma/t}}$$

$$(3.34)$$

$$= \frac{(-1)^{n-1} r^{1-n} (1+r)^{k+1}}{\sqrt{2\pi r k}} \frac{1 + O(k^{-1/3})}{1 + r e^{\gamma/t}} ,$$

and therefore

$$\Delta^k f_1(n) = \frac{\alpha(-1)^{n-1} r^{1-n} (1+r)^{k+1}}{\sqrt{2\pi r k}} \frac{1}{2\pi i} \int_{(\beta)} t^{-5/2} \frac{\exp\left(t + \frac{23\gamma}{24t}\right)}{1 + r e^{\gamma/t}} dt$$

$$(3.35)$$

$$+ O(k^{-5/6} (1+r)^k r^{-n}) .$$

Let

$$h(r) = \frac{1}{2\pi i} \int_{(\beta)} t^{-5/2} \frac{\exp(t + \frac{23\gamma}{24t})}{1 + r e^{\gamma/t}} dt . \tag{3.36}$$

Then $h(r)$ is a continuous function of r for $0 < r < 2$, say, and we will evaluate $h(r)$ for $r < 1$ but close to 1. Consider first $r > 1$. Then we have (using the usual Bessel function expansions that come up in Rademacher's proof)

$$\begin{aligned}
h(r) &= \frac{1}{2\pi i} \int_{(\beta)} t^{-5/2} \frac{\exp(t + \frac{23\gamma}{24t})}{r e^{\gamma/t} (1 + r^{-1} e^{-\gamma/t})} dt \\
&= \frac{1}{2\pi i} \int_{(\beta)} t^{-5/2} \exp(t + \frac{23\gamma}{24t}) \sum_{m=1}^{\infty} (-1)^{m-1} r^{-m} e^{-m\gamma/t} dt \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} r^{-m} \frac{1}{2\pi i} \int_{(\beta)} t^{-5/2} \exp(t - (m - 23/24)\gamma/t) dt \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} r^{-m} J_{3/2} (\eta_m) (\eta_m/2)^{-3/2} \\
&= \pi^{-1/2} \gamma^{-1} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} r^{-m}}{m - \frac{23}{24}} \left[\frac{\sin(\eta_m)}{\eta_m} - \cos(\eta_m) \right] ,
\end{aligned} \tag{3.37}$$

where $\eta_m = 2 \gamma^{1/2} (m - 23/24)^{1/2}$. Now

$$\left| \sum_{m=1000}^{\infty} \frac{(-1)^{m-1} r^{-m} \sin(\eta_m)}{\eta_m (m - \frac{23}{24})} \right| \le \frac{1}{2\gamma^{1/2}} \sum_{m=1000}^{\infty} \left[m - \frac{23}{24} \right]^{-3/2} \tag{3.38}$$

$$\le \frac{1}{2\gamma^{1/2}} \int_{998}^{\infty} u^{-3/2} du = \gamma^{-1/2} 998^{-1/2} \le 0.025 .$$

On the other hand, for some $v \in [m, m+1]$

$$\frac{r^{-m}\cos(\eta_m)}{m-\frac{23}{24}} - \frac{r^{-m-1}\cos(\eta_{m+1})}{m+\frac{1}{24}} = -\frac{d}{du}\left.\frac{r^{-u}\cos(\eta_u)}{u-\frac{23}{24}}\right|_{u=v} ,$$

$$= \frac{\cos(\eta_v)}{(v-\frac{23}{24})^2} + \frac{\gamma^{1/2}\sin(\eta_v)}{(v-\frac{23}{24})^{3/2}} + \frac{r^{-v}(\log r)\cos(\eta_v)}{v-\frac{23}{24}} ,$$

so for $r \in (1, 1+10^{-10})$,

$$\left|\sum_{m=1000}^{\infty}\frac{(-1)^{n-1}\cos(\eta_m)}{m-\frac{23}{24}}\right| \leq \sum_{q=500}^{\infty}\left\{\frac{1}{(2q-1)^2} + \frac{\gamma^{1/2}}{(2q-1)^{3/2}} + \frac{r^{-2q}\log r}{2q-1}\right\} \tag{3.39}$$

$$\leq \int_{498}^{\infty}\frac{du}{(2u)^2} + \int_{498}^{\infty}\frac{\gamma^{1/2}\,du}{(2u)^{3/2}} + (\log r)\int_{498}^{\infty}\frac{r^{-2u}\,du}{2u}$$

$$\leq 0.042 .$$

Therefore for $r \in (1, 1+10^{-10})$,

$$h(r) = \pi^{-1/2}\gamma^{-1}(A+B) ,$$

where

$$A = \sum_{m=1}^{998}\frac{(-1)^{m-1}r^{-m}}{m-\frac{23}{24}}\left[\frac{\sin(\eta_m)}{\eta_m} - \cos(\eta_m)\right] = 1.415972...$$

by direct calculation, and $|B| \leq 0.042 + 0.025 \leq 0.07$. Hence for $r \in (1, 1+10^{-10})$,

$$0.46 \leq h(r) \leq 0.51 , \tag{3.40}$$

Since $h(r)$ is continuous for $0 < r < 2$, we must have

$$|h(r)| \le 0.6 \quad \text{for} \quad 1-\varepsilon \le r \le 1/2 \tag{3.41}$$

if $\varepsilon < 10^{-10}$ is small enough.

We now combine all the above estimates to obtain the claims of the proposition, valid for c_1 and k large enough and ε small enough.

□

We now proceed to the proof of the theorem. We select an ε given by Proposition 3.3. Then, applying Proposition 3.1 with this value of ε, we see that $(-1)^n \Delta^k p(n) > 0$ for all n, $0 \le n \le (1/2 - \varepsilon)k$, and all $k \ge k_3 = \max(k_1(\varepsilon), k_2)$.

Next, for $k \ge k_3$ and $(1/2 - \varepsilon)k \le n \le k/2$, we have

$$\begin{aligned}(-1)^n \Delta^k p(n) &= \sum_{j=0}^{n} (-1)^j \binom{k}{n-j} p(j) \\ &= \binom{k}{n} + (-1)^n \pi^{-1} 2^{-1/2} \Delta^k f_1(n) \\ &\quad + \pi^{-1} 2^{-1/2} \sum_{j=1}^{n} \binom{k}{n-j} (f_2(j) + (-1)^{n-j} R_j) .\end{aligned} \tag{3.42}$$

By Lemma 3.2, each term in the n-term sum above is > 0, while by (3.14),

$$|\pi^{-1} 2^{-1/2} \Delta^k f_1(n)| < \frac{3}{5} \binom{k}{n} . \tag{3.43}$$

Therefore $(-1)^n \Delta^k p(n) > 0$ in this range also.

Consider now $k/2 \le n \le k-2$. In this range, in view of Lemma 3.2, it suffices to show that

$$G = \sum_{j=1}^{n} \binom{k}{n-j} f_2(j)$$

satisfies $|G| > 3 |\Delta^k f_1(n)|$. However, by (3.11) we have $3 |\Delta^k f_1(n)| < c_{23} 2^k$. On the other hand, if $J = \lfloor n - k/2 + 1 \rfloor$, then for k sufficiently large, $J + k^{1/4} \leq j \leq J + 2 k^{1/4}$,

$$\binom{k}{n-j} \geq 10^{-1} k^{-1/2} 2^k ,$$

and so

$$G \geq \frac{2^k}{10 k^{1/2}} f_2(J + \lfloor k^{1/4} \rfloor) \geq 2^k \exp(10^{-1} C k^{1/8}) , \tag{3.44}$$

which gives the desired result for $k \geq k_4 \geq k_3$. The same lower bound for G holds also for $k-1 \leq n \leq k$, and so by (3.12) we obtain the result of the theorem for that range also if $k \geq k_5 \geq k_4$.

Next, consider $n \geq k+1$. By Lemma 3.2, to obtain $(-1)^n \Delta^k p(n) > 0$ it suffices to show that if

$$H = \sum_{j=0}^{k} \binom{k}{j} f_2(n-j) ,$$

then H satisfies $H > 3 |\Delta^k f_1(n)|$. However, $f_2(m) \geq 10^{-3}$ for all $m \geq 1$, so

$$H \geq 10^{-3} \sum_{j=0}^{k} \binom{k}{j} = 10^{-3} 2^k ,$$

and by (3.12) and (3.13), we have $(-1)^n \Delta^k p(n) > 0$ for all n with $k+1 \leq n \leq 10^{-3} c_1^{-2} k^2 (\log k)^2$, provided $k \geq k_5 \geq k_4$.

Before proceeding to consider the range $n > 10^{-3} c_1^{-2} k^2 (\log k)^2$, we make the following general observation. If $f(x)$ is a $C^\infty [1/2, \infty)$ function, say, then for $x > 3/2$,

$$\Delta f(x) = f(x) - f(x-1) = \int_{x-1}^{x} f'(t)\, dt . \tag{3.45}$$

More generally, for $x > k + 1/2$,

$$\Delta^k f(x) = \int_{1/2}^{\infty} f^{(k)}(u)\, \chi_k(x-u)\, du , \tag{3.46}$$

where

$$\chi_k(t) = \chi_1 * \ldots * \chi_1(t) \tag{3.47}$$

is the k-fold convolution of the characteristic function of the unit interval,

$$\chi_1(t) = \begin{cases} 1 & 0 \le t \le 1 , \\ 0 & \text{otherwise}. \end{cases}$$

The formula (3.46) reduces to (3.45) for $k = 1$. For higher values, it is easily proved by induction. If we assume that (3.46) holds for $k - 1 \ge 1$, then (since $(\Delta g)' = \Delta g'$)

$$
\begin{aligned}
\Delta^k f(x) &= \int_{x-1}^{x} (\Delta^{k-1} f(t))' \, dt \\
&= \int_{x-1}^{x} (\Delta^{k-1} f'(t)) \, dt \\
&= \int_{x-1}^{x} dt \int_{1/2}^{\infty} f^{(k)}(u) \, \chi_{k-1}(t-u) \, du \\
&= \int_{1/2}^{\infty} f^{(k)}(u) \, du \int_{x-1}^{x} \chi_{k-1}(t-u) \, dt \\
&= \int_{1/2}^{\infty} f^{(k)}(u) \, \chi_k(x-u) \, du \, ,
\end{aligned}
$$

which proves (3.46) for k.

All that we will need to know about the $\chi_k(t)$ is that $\chi_k(t) \geq 0$, $\chi_k(t) = 0$ for $t < 0$ and $t > k$, and

$$\int_{-\infty}^{\infty} \chi_k(t) \, dt = 1 \, . \tag{3.48}$$

To deal with the remaining range, $n \geq 10^{-3} c_1^{-2} k^2 (\log k)^2$, we need to investigate the derivatives of $f_1(x)$ more precisely than before. Let $g(y) = f_1(y + 1/24)$, so that $f_1^{(r)}(x) = g^{(r)}(x - 1/24)$. We consider $r^2 \log r \leq y$. Then

$$
\begin{aligned}
g^{(r)}(y) &= \frac{d^{r+1}}{dy^{r+1}} \left\{ y^{-1/2} \sinh(C y^{1/2}) \right\} \\
&= \sum_{j=0}^{\infty} \frac{C^{2j+1} \, (j)_{r+1}}{(2j+1)!} \, y^{j-r-1} \, ,
\end{aligned} \tag{3.49}
$$

where

$$(z)_m = z(z-1) \cdots (z - m+1) \ .$$

Let a_j denote the j-th term in the sum in (3.49). By looking at the ratio a_{j+1}/a_j, we see that the maximum occurs for $j = J + O(1)$, where

$$J = \left\lfloor (r + (C^2 y + r^2)^{1/2})/2 \right\rfloor , \tag{3.50}$$

and that for $m = j-J$, $|m| \le J^{5/9}$,

$$\frac{a_j}{a_J} = \left[1 + O(J^{-1/3}) \right] \left[\frac{C^2 y}{2(2J+3)\ (J-r)} \right]^m \prod_{h=0}^{|m|-1} \left\{ \left[1 + \frac{h}{J-r} \right] \left[1 + \frac{2h}{2J+3} \right] \right\}^{-1}$$

$$= \left[1 + O(J^{-1/3}) \right] \exp \left[- \frac{m^2\ (J-r/2)}{J(J-r)} \right] ,$$

while

$$\sum_{|j-J| \ge J^{5/9}} a_j = O(J^{-1}\ a_J) \ .$$

Therefore we conclude that for $y > r^2 \log r$, $r \ge 2$,

$$g^{(r)}\ (y) = \left[\pi J \right]^{1/2} a_J\ (1 + O(y^{-1/6})) , \tag{3.51}$$

where the constant implied by the O-notation is independent of y and r, and $J = J(y,r)$ is given by (3.50). Furthermore, if in fact $y > (r+1)^2 \log (r+1)$, then

$$| J(y, r+1) - J(y,r) | = O(1) ,$$

and therefore

$$g^{(r+1)}(y) = \frac{J-r}{y}\, g^{(r)}(y)\,(1 + O(y^{-1/6}))$$

$$= \frac{C}{2y^{1/2}}\, g^{(r)}(y)\,(1 + O(y^{-1/6} + r\,y^{-1/2}))\ . \qquad (3.52)$$

Also,

$$|J(y+r,\ r) - J(y,r)| = O(1)\ ,$$

so for $0 \le t \le r$,

$$g^{(r)}(y+t) = g^{(r)}(y)\,(1 + O(y^{-1/6}))\ . \qquad (3.53)$$

We first show that if $\eta \in (0,\ 10^{-2})$ is given, then for $10^{-3}\ c_1^{-2}\ k^2\ (\log k)^2 \le n \le (1-\eta)\ 6\pi^{-2}\ k^2\ (\log k)^2$,

$$f_1^{(k)}(n) \le (1 + O(k^{-1/5}))2^k\ f_2\,(n-k)^{(1-\eta/100)}\ , \qquad (3.54)$$

and that for $(1+\eta)6\pi^{-2}\ k^2\ (\log k)^2 \le n$,

$$f_1^{(k)}(n-k) > (1 + O(k^{-1/5}))2^k\ f_2\,(n)^{(1+\eta/100)}\ . \qquad (3.55)$$

We consider only (3.55) in detail. Suppose therefore that $\eta \in (0,\ 10^{-2})$ is given, and we have

$$n \ge (1 + \eta)\ 6\pi^{-2}\ k^2\ (\log k)^2\ , \qquad (3.56)$$

where we can take k very large. We define J by (3.50) with $r = k$, $y = n-k-1/24$. Then

$$J = \frac{1}{2}\, Cn^{1/2} + \frac{1}{2}\, k + o(k) \quad \text{as} \quad k \to \infty$$

with n satisfying (3.56), and

$$f_1^{(k)}\ (n-k) \geq a_J \geq J^{-1}\ C^{2J}\ ((2J)!)^{-1}\ (J)_{k+1}\ (n-k-1)^{J-k-1}$$
$$\geq J^{-2}\ C^{2J}\ 2^{-2J}\ J^{-2J}\ e^{+2J}\ J^{k+1}\ n^{J-k-1} \cdot T\ ,$$

where

$$T = \left[1 - \frac{k+1}{n}\right]^{J-k-1} \cdot \prod_{m=1}^{k} \left[1 - \frac{m}{J}\right] \geq \exp(-\, c_{24}\ k\ (\log\ k)^{-1})\ .$$

Furthermore,

$$2^{2J}\ J^{2J}\ n^{-J} = C^{2J}\ \exp(k + o(k))\ ,$$

so

$$f_1^{(k)}\ (n-k) \geq n^{-2}\ J^k\ n^{-k}\ \exp(C\ n^{1/2}\ 1 + o(1))$$
$$\geq n^{-k/2-2}\ 2^{-k}\ C^k\ \exp(C\ n^{1/2}\ 1 + o(1)) \quad \text{as} \quad k \to \infty\ ,$$

which now implies (3.55) (subject to (3.56)) for large enough k.

Given (3.54) and (3.55), it is clear that for $k \geq k_6 = k_6\,(\eta)\,(k_6 \geq k_5)$

$$(-1)^n\ \Delta^k\ p(n) > 0 \quad \text{for} \quad 0 \leq n \leq (1-\eta) 6\pi^{-2}\ k^2\ (\log\ k)^2\ ,$$
$$\Delta^k\ p(n) > 0 \quad \text{for} \quad n \geq (1+\eta) 6\pi^{-2}\ k^2\ (\log\ k)^2\ ,$$

since by (3.46) and the monotonicity of $f_1^{(k)}\ (x)$ we have

$$f_1^{(k)}\ (n-k) \leq \Delta^k\ f_1\ (n) \leq f_1^{(k)}\ (n)\ ,$$

while

$$2^k\ f_2\ (n-k) \leq \sum_{j=0}^{k} \binom{k}{j}\ f_2\ (n-j) \leq 2^k\ f_2\ (n)\ ,$$

and by Lemma 3.2,

$$\sum_{j=0}^{k} \left| \binom{k}{j} R_{n-j} \right| < 10 \cdot 2^k f_3(n) .$$

Since this holds for every $\eta \in (0, 10^{-2})$ (with k_6 depending on η), this shows that if $n_0(k)$ exists, then $n_0(k) \sim 6\pi^{-2} k^2 (\log k)^2$ as $k \to \infty$.

At this point, to complete the proof of our theorem it only remains to show that one can choose $\eta \in (0, 10^{-2})$ so small that for $k \geq k_7 = k_7(\eta)$, $\Delta^k p(n)$ will alternate in sign and then become nonnegative and stay nonnegative as n ranges over $n_1 \leq n \leq n_2$, where

$$n_1 = \left\lfloor (1-\eta)\, 6\pi^{-2} k^2 (\log k)^2 \right\rfloor ,$$

$$n_2 = \left\lfloor (1+\eta)\, 6\pi^{-2} k^2 (\log k)^2 \right\rfloor .$$

Let

$$S(n) = \sum_{j=0}^{k} \binom{k}{j} f_2(n-j) .$$

Then we know that for any $\eta \in (0, 10^{-2})$ and k large enough (depending only on η)

$$\Delta^k f_1(n_1) < 10^{-3} S(n_1) ,$$

$$\Delta^k f_1(n_2) > 10^{-3} S(n_2) ,$$

while for any $n \in [n_1, n_2]$,

$$|\Delta^k R_n| < n^{-10} S(n) .$$

Now it is easy to see from the explicit definition of $f_2(n)$ that it is monotone increasing, and

$$f_2(n+1) \le f_2(n) + \frac{3C}{10n^{1/2}} f_2(n)$$

for large enough n, so that if k is large enough and $n \in [n_1, n_2]$, then

$$S(n) \le S(n+1) \le S(n) + CS(n)/(3n^{1/2}) .$$

On the other hand, by (3.46),

$$\begin{aligned} \Delta^k f_1(n+1) - \Delta^k f_1(n) &= \Delta^{k+1} f_1(n) \\ &= \int_{n-k-1}^{n} f_1^{(k+1)}(u)\, \chi_{k+1}(n-u)\, du \\ &\ge f_1^{(k+1)}(n-k-1) , \end{aligned}$$

and by (3.46) and (3.53), this last quantity is

$$\ge 2C(\Delta^k f_1(n))/(5\, n^{1/2}) ,$$

provided k is large enough. It is now easy to conclude the proof of the Theorem. Let N be the least integer $\ge n_1$ such that $\Delta^k f_1(N) \ge S(N)$. Then, by the above discussion,

$$\Delta^k f_1(n) + \sum_{j=0}^{k} \binom{k}{j} |R_{n-j}| < S(n)$$

for all $n < N$, $n \ge n_1$, so that $(-1)^k \Delta^k p(n) > 0$ for $n < N$. On the other hand, for $n > N, n \le n_2$,

$$\Delta^k f_1(n) > S(n) + \sum_{j=0}^{k} \binom{k}{j} |R_{n-j}| ,$$

so that $\Delta^k p(n) > 0$ for all $n > N$. Finally, $\Delta^k p(N)$ can only be negative if N is odd. This completes the proof of the theorem.

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