Sufficient Condition for a 4-Dimensional Vector Orbi-Space to Admit a Faithful Symplectic SU(2) Action

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Abstract

In this paper we state a sufficient condition for the existance of a 4-dimentional vector orb-space which admits a faithful, symplectic SU(2) action.

1 Introduction

Classifying all Hamiltonian SU(2) actions on manifolds is a hard unsolved problem. We begin by looking at SU(2) actions on vector orbi-spaces since they contain all the local information. In this paper we look at faithful symplectic SU(2) actions on 4-dimensional vector orbi-spaces and have found the following sufficient condition:

Main Theorem 1. *If Γ is a finite subgroup of the center of $SU(2)$, there is a 4-dimensional vector orbi-space V/Γ which admits a symplectic, faithful $SU(2)$ action.*

2 Background

2.1 Vector Space

A vector space V over a field F is a set together with two laws of composition:

1. $V \times V \to V$, $v, w \mapsto v + w$ (addition)
2. $F \times V \to V$, $c, v \mapsto cv$ (scaler multiplication)

and satisfying the following axioms:

1. addition makes V into a commutative group V^+.
2. scaler multiplication is associative with multiplication in F:
   $(ab)v = a(bv)$ $\forall$ $a, b \in F$ and $v \in V$
3. the element 1 acts as the identity: $1v = v$ $\forall$ $v \in V$
4. two distributive laws hold:
   $(a + b)v = av + bv$ and $a(v + w) = av + aw$ $\forall$ $a, b \in F$ and $v, w \in V$

2.2 Bilinear Form

A bilinear form is a form on a vector space V that is a function of 2 variables on V with values in the field F, $V \times V \xrightarrow{f} F$. f satisfies the bilinear axioms:

1. $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$
2. $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$
3. $f(cv, w) = cf(v, w)$
4. $f(v, cw) = cf(v, w)$

notation: $< v, w >$

2.3 Skew-symmetric Bilinear Form

Intuitively, a skew-symmetric bilinear form is one such that
$< v, w >= - < w, v >$
(since a symmetric bilinear form has $< v, w >=< w, v >$).
However, this definition, while it is useful, does not hold for characteristic 2. The universal definition is that a bilinear form is skew-symmetric if
$< v, w >= 0 \ \forall \ v \in V$

2.4 Nondegenerate Bilinear Form

A nondegenerate bilinear form is any bilinear form such that
$\forall v \in V < v, w >= 0 \ \forall w \in V$ implies that $v = 0$

2.5 Matrix Representation of a Bilinear Form

Take a basis for V with $<,>$ a bilinear form on V. Let $B = (b_1, b_2, ...b_n)$ be the basis. The matrix of the form with respect to the basis is $A = (a_{ij})$ where $a_{ij} =< b_i, b_j >$.
The standard skew-symmetric form represented as a matrix is: $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

2.6 The Symplectic Group

The symplectic group is the stabalizer of J as given above.
$SP_{2n}(\Re) = \{P \in GL_{2n}(\Re) | P^t JP = J\}$
The complex symplectic group is defined similarly. Note that all symplectic matrices have determinant 1.

2.7 Symplectic Vector Space

A symplectic vector space is a pair (v, ω) where V is a finite dimensional real vector space and ω is a nondegenerate skew-symmetric bilinear form $\omega : V \times V \rightarrow \Re$. Since the bilinear form is nondegenerate, the dimension of the symplectic vector space is always even. All symplectic vector spaces with the same dimension are isomorphic.
example: $(\Re^{2n}, \omega)$ where ω has the matrix representation J.

2.8 Quotient Space

Let V be a vector space over a field F and W be a subspace of V. Then V/W is a vector space over F and the quotient space of V by W with

1. $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$
2. $\alpha(v_1 + W) = \alpha v_1 + W$

given $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$

2.9 Symplectic Vector Orbi-Space

The quotient space V/Γ where V is a symplectic vector space and Γ is a finite subgroup of the symplectic group $SP(V)$.

2.10 Unitary Matrix

P is a unitary matrix if $P^*P = I$ where I is the identity matrix (or $P^* = P^{-1}$) and P^* is the matrix adjoint, $P^* = \overline{P}^t$.

2.11 Unitary Group U_n

$U_n = \{P | P^*P = I\}$
(This is the group of matrices representing chages of basis which leave the hermitian dot product X^*Y invariant. [1](p27))

2.12 Special Matrices

Special matrix groups are subgroups of matrix groups that have determinant 1.

2.13 Special Linear Group

The special linear group ($SL_n(\Re)$) is the group of $n \times n$ matrices with determinant 1 and entries in $\Re$. (A complex group can be defined analogously.)

2.14 Special Unitary Group $SU(n)$

$SL_n(\mathrm{C}) \cap U_n$

2.15 Group Action on a Set

A group G is said to act (or operate) on the set S if there exists a map $(g, x) \mapsto gx$ of $G \times S$ into S satisfying:

1. $1x = x, x \in S$
2. $(g_1, g_2)x = g_1(g_2x)$

([6], p72)

3 Preliminary Theorems

Theorem 1. *[9] Let $\rho : H \mapsto SP(V/\Gamma)$ be a faithful symplectic representation of a compact Lie group H on a symplectic vector orbi-space V/Γ, and let $N(\Gamma)$ denote the normalizer of Γ in $SP(V)$. The representation ρ and the short exact sequence $1 \to \Gamma \to N(\Gamma) \to SP(V/\Gamma) \to 1$ give rise to the pull-back extension $\pi : \widehat{H} \to H$ and the faithful (symplectic) pull-back representation $\widehat{\rho} : \widehat{H} \to N(\Gamma) \subset SP(V)$ so that Γ is naturally a subgroup of $\widehat{H}$, and the following diagram is exact and commutes.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \widehat{H} & \xrightarrow{\pi} & H & \longrightarrow & 1 \\ & & \| & & \downarrow \widehat{\rho} & & \downarrow \rho & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & N(\Gamma) & \xrightarrow{\mu} & SP(V/\Gamma) & \longrightarrow & 1 \end{array}$$

Conversely, given a Lie group $\widehat{H} \in SP(V)$, a symplectic representation $\widehat{\rho} : \widehat{H} \to SP(V)$ of $\widehat{H}$ on a symplectic vector space V and a finite normal subgroup Γ of $\widehat{H}$ such that $\widehat{\rho}(\Gamma) = \Gamma$, there exists a symplectic orbi-representation $\rho : H \to SP(V/\Gamma)$ of the quotient $H = \widehat{H}/\Gamma$ making the above diagram commute.

Proof. Let $\rho : H \to N(\Gamma)/\Gamma$ be a faithful symplectic representation and let $\mu : N(\Gamma) \to N(\Gamma)/\Gamma$ be defined $a \mapsto a\Gamma$. Let $\widehat{H} = \mu^{-1}(\rho(H))$.

The group $\widehat{H}$ is a subgroup of $N(\Gamma)$ as $h, k \in \widehat{H} \Rightarrow [h] \in SP(V/\Gamma), [k] \in SP(V/\Gamma)$, $\rho^{-1}([h]) \in H$ and $\rho^{-1}([k]) \in H$. Since H is a group,

$$\begin{array}{rcl} \rho^{-1}([h])\{\rho^{-1}([k])\}^{-1} \in H & \Rightarrow & [h][k]^{-1} \in SP(V/\Gamma) \\ & \Rightarrow & [h][k^{-1}] \in SP(V/\Gamma) \\ & \Rightarrow & hk^{-1} \in \widehat{H} \end{array}$$

Thus $\widehat{H}$ is a subgroup.

Furthermore $\widehat{H}$ is a Lie group. Since multiplication in the Lie group $SP(V)$ is smooth, the function $\pi : \widehat{h} \to H$ defined $a \mapsto a\Gamma$ is continuous. Therefore if we consider H to be in $SP(V)$, since H is closed in $SP(V)$ being that H is compact, $\pi^{-1}(H) = \widehat{H}$ is also closed. By SOME THEOREM $\widehat{H}$ is a Lie group.

Thus, let $\pi : \widehat{H} \to H$ be defined $a \mapsto \rho^{-1}(\mu(a))$. Then $\widehat{\rho}$ is an inclusion, it is seen that $\rho \circ \pi = \mu \circ \widehat{\rho}$, Γ is a subgroup of $\widehat{H}$ and the sequence is exact.

Conversely, given a group $\widehat{H} \subset SP(V)$, a symplectic representation $\widehat{\rho} : \widehat{H} \to SP(V)$ of $\widehat{H}$ on a symplectic vector space V and a finite normal subgroup $\Gamma \subset SP(V)$ of $\widehat{H}$ such that $\widehat{\rho}(\Gamma) = \Gamma$, we have $\widehat{\rho}(\widehat{H}) \subset N(\Gamma) \subset SP(V)$. This follows since for all $h \in \widehat{H}, h\Gamma h^{-1} = \Gamma$ and since ρ is a homomorphism $\rho(h\Gamma h^{-1} = \rho(\Gamma) = \Gamma$
$\Rightarrow \rho(h)\rho(\Gamma)\rho(h^{-1}) = \Gamma \Rightarrow \rho(h)\Gamma\rho(h)^{-1} = \Gamma \Rightarrow \rho(h) \in N(\Gamma)$.

If we let $\pi : \widehat{H} \to H$ be defined $h \mapsto h\Gamma$ and $\mu : N(\Gamma) \to SP(V/\Gamma)$ be defined $b \mapsto b\Gamma$ then we can let $\rho : H \to SP(V/\Gamma)$ be defined $[h] \mapsto \mu(\widehat{\rho}(h))$. It is obvious that $\rho \circ \pi = \mu \circ \widehat{\rho}$ and that the diagram commutes and is exact. □

Theorem 2. *If $\Gamma \subset \mathrm{Z}(G)$ (the center of G)then Γ is a normal subgroup of G.*

Proof. A subgroup N of a group G is called a normal subgroup if it has the property that for all $a \in N$ and $b \in G, bab^{-1} \in N$. So we want to show that for all $a \in \Gamma$ and $b \in G, bab^{-1} \in \Gamma$:
Pick some $b \in G$ and $a \in \Gamma$. Is $bab^{-1} \in \Gamma$? Since a is in the center of G, a commutes with b and b^{-1} since both are in G.
So $bab^{-1} = bb^{-1}a = a$ and $a \in \Gamma$. □

Theorem 3. *If $\widehat{G} = \langle G, \Gamma \rangle = \{tv : t \in G, v \in \Gamma\}$ where elements of G and Γ are represented by square matrices of the same dimension and Γ is normal in G, Γ is normal in $\widehat{G}$.*

Proof. Let $a \in \Gamma$ and $tv \in \widehat{g}$. If $tva(tv)^{-1} \in \widehat{G}$, Γ is normal in $\widehat{G}$.

$$
\begin{array}{rcll}
tva(tv)^{-1} & = & tvav^{-1}t^{-1} & \text{since } v \text{ and } t \text{ are matrices} \\
& & & vav^{-1} \in \Gamma \text{ since } v, a \in \Gamma \\
& & & \text{let } vav^{-1} = b \in \Gamma \\
& = & tbt^{-1} & \\
& = & tt^{-1}b & \Gamma \text{ is normal in } G \text{ and } t, t^{-1} \in G \\
& = & b &
\end{array}
$$

So Γ is normal in $\widehat{G}$. □

Theorem 4.

$$\varphi : \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \textit{with } a^2+b^2+a'^2+b'^2 = 1 \in SP(4,\Re) \mapsto \begin{pmatrix} a+a'i & b+b'i \\ -b+b'i & a-a'i \end{pmatrix} \in SU(2)$$

is an isomorphism.

Proof. Let

$$A = \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \text{ and } C = \begin{pmatrix} c & d & c' & d' \\ -d & c & d' & -c' \\ -c' & -d' & c & d \\ -d' & c' & -d & c \end{pmatrix}$$

with $a^2+b^2+a'^2+b'^2 = 1$ and $c^2+d^2+c'^2+d'^2 = 1$

First, $A \in SP(4,\Re)$:

$$A^tJA = \begin{pmatrix} a & -b & -a' & -b' \\ b & a & -b' & a' \\ a' & b' & a & -b \\ b' & -a' & b & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix}$$

$$= \begin{pmatrix} a' & b' & a & -b \\ b' & -a' & b & a \\ -a & b & a' & b' \\ -b & -a & b' & -a' \end{pmatrix} \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix}$$

since we have the condition $a^2+b^2+a'^2+b'^2 = 1$ we get

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J$$

So $A \in SP(4,\Re)$.

φ is a homomorphism:

We want to show that $\varphi(AC) = \varphi(A)\varphi(C)$.

$$\varphi(AC) = \varphi\left(\begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \begin{pmatrix} c & d & c' & d' \\ -d & c & d' & -c' \\ -c' & -d' & c & d \\ -d' & c' & -d & c \end{pmatrix}\right) =$$

$$\varphi\left(\left(\begin{array}{cccc} ac-a'c'-bd'-b'd' & ad+bc-a'd'+b'c' & ac'+bd'+a'c-b'd & ad'-bc'+a'd+b'c \\ -bc-ad-b'c'+a'd' & -bd+ac-b'd'-a'c' & -bc'+ad'+b'c+a'd & -bd'-ac'+b'd-a'c \\ -a'c+b'd-ac'-bd' & -a'd-b'c-ad'+bc' & -a'c'-b'd'+ac-bd & -a'd'+b'c'+ad+bc \\ -b'c-a'd+bc'-ad' & -b'd+a'c+bd'+ac' & -b'c'+a'd'-bc'ad & -b'd'-a'c'-bd+ac \end{array}\right)\right)$$
$$= \left(\begin{array}{cc} (ac-a'c'-bd-b'd')+(ac'+a'c+bd'-b'd)i & (ad+bc-a'd'+b'c')+(ad'-bc'+a'd+b'c)i \\ (-ad-bc+a'd'-b'c')+(ad'-bc'+a'd+b'c)i & (ac-a'c'-bd-b'd')-(ac'+a'c+bd'-b'd)i \end{array}\right)$$
$$= \left(\begin{array}{cc} a(c+c'i)+a'(-c'+ci)+b(-d+d'i)+b'(-d'-di) & a(d+d'i)+a'(-d+di)+b(c-c'i)+b'(c'+ci) \\ a(-d+d'i)+a'(d'+d')+b(-c-c'i)+b'(-c'+ci) & a(c-c'i)+c'(-c'-ci)+b(-d-d'i)+b'(-d'+di) \end{array}\right)$$
$$= \left(\begin{array}{cc} (a+a'i)(c+c'i)+(b+b'i)(-d+d'i) & (a+a'i)(d+d'i)+(b+b'i)(c-c'i) \\ (a-a'i)(-d+d'i)+(-b+b'i)(c+c'i) & (a-a'i)(c-c'i)+(-b+b'i)(d+d'i) \end{array}\right)$$
$$= \left(\begin{array}{cc} a+a'i & b+b'i \\ -b+b'i & a-a'i \end{array}\right)\left(\begin{array}{cc} c+c'i & d+d'i \\ -d+d'i & c-c'i \end{array}\right)$$
$$= \varphi\left(\left(\begin{array}{cccc} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{array}\right)\right)\varphi\left(\left(\begin{array}{cccc} c & d & c' & d' \\ -d & c & d' & -c' \\ -c' & -d' & c & d \\ -d' & c' & -d & c \end{array}\right)\right)$$
$= \varphi(A)\varphi(C)$
so φ is a homomorphism.

φ is obviously injective.

Is φ surjective?
Pick some element $\left(\begin{array}{cc} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{array}\right) \in SU(2)$. We want to show that there is some element in $SP(4,\Re)$ which maps to $\left(\begin{array}{cc} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{array}\right)$. Pick $\left(\begin{array}{cccc} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{array}\right)$ such that $a+a'i=\alpha$ and $b+b'i=\beta$ then since $\left(\begin{array}{cc} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{array}\right) \in SU(2)$, $\det\left(\begin{array}{cc} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{array}\right) = 1 = a^2+b^2+a'^2+b'^2$ which is the condition on φ. So φ is surjective.
So φ is an isomorphism. □

4 Main Theorem

Main Theorem 1. *If Γ is a finite subgroup of the center of $SU(2)$, there is a 4-dimensional vector orbi-space V/Γ which admits a symplectic, faithful $SU(2)$ action.*

Proof. Let $\widetilde{\Gamma}$ be a finite subgroup of the center of $SU(2)$ and φ as described in Theorem 4. Let $\Gamma = \varphi^{-1}(\widetilde{\Gamma}) \in SP(4,\Re)$.
Let $SU(2) = \left\{\left(\begin{array}{cccc} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{array}\right) \in SP(4,\Re)\right\}$.
Let $\widehat{SU(2)} = \langle SU(2),\Gamma\rangle = \{tv : t \in SU(2), v \in \Gamma\}$. $\widehat{SU(2)} \in SP(4,\Re)$ since $\Gamma \subset SP(4,\Re)$ and $SU(2) \subset$

$SP(4,\Re)$. Γ is normal in $\widehat{SU(2)}$ by Theorems 2 and 3 since $\Gamma \subset Z(SU(2))$.

Thus by Theorem 1, since we have a symplectic representation of $\widehat{SU(2)}$ on the vector orbi-space $\Re^4/\Gamma$, we have the following diagram which is exact and commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \widehat{SU(2)} & \xrightarrow{\pi} & SU(2) & \longrightarrow & 1 \\ & & \| & & \downarrow \widehat{\rho} & & \downarrow \rho & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & N(\Gamma) & \xrightarrow{\mu} & SP(V/\Gamma) & \longrightarrow & 1 \end{array}$$

Thus there is a symplectic representation of $SU(2)$ on the vector orbi-space $\Re^4/\Gamma$. Thus the vector orbi-space admits an $SU(2)$ action. $\square$

References

[1] Michael Artin. *Algebra.* Prentice-Hall, Inc., Upper Saddle River, New Jersey, 1991.

[2] Theodor Brocker and Tammo tom Dieck. *Graduate Texts in Mathematics: Representations of Compact Lie Groups.* Springer-Verlag New York Inc., 1985.

[3] Morton L. Curtis. *Matrix Groups.* Springer-Verlag New York Inc., second edition, 1984.

[4] Marshall Hall Jr. *The Theory of Groups.* The Macmillan Company, New York, 1959.

[5] Kenneth Hoffman and Ray Kunze. *Linear Algebra.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1961.

[6] Nathan Jacobson. *Basic Algebra I.* W. H. Freeman and Company, New York, second edition, 1985.

[7] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology.* Oxford University Press Inc., New York, second edition, 1998.

[8] Joseph J. Rotman. *Graduate Texts in Mathematics: An Introduction to the Theory of Groups.* Springer-Verlag New York Inc., fourth edition, 1995.

[9] Noel Watson. Categorization of 4 dimensional symplectic vector orbi-spaces.

[1] [2] [3] [4] [5] [6] [7] [8] [9]