Solutions to Homework 1: Geometric Computation

Solution 1:

(i) The dot product of two normalized vectors yields the cosine of the angle between them. Thus, the angle θ in degrees is given by:

$$\widehat{u} \leftarrow \frac{\vec{u}}{\sqrt{\vec{u}\cdot\vec{u}}}, \qquad \widehat{v} \leftarrow \frac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}}, \qquad \theta \leftarrow \frac{180}{\pi}\arccos(\widehat{u}\cdot\widehat{v}).$$

(ii) The cross product yields a vector that is perpendicular to its arguments. Therefore, the answer is $\widehat{w}$ given by

$$\vec{w} \leftarrow \vec{u}\times\vec{v}, \qquad \widehat{w} \leftarrow \frac{\vec{w}}{\sqrt{\vec{w}\cdot\vec{w}}}.$$

(iii) The offset vector of c with respect to p is the sum of $-\vec{u}$ scaled by f, and the *up vector* (the y-axis unit vector) $\vec{e}_y = (0,1,0)$ scaled by h. Thus, we have

$$c \leftarrow p - f\vec{u} + h\vec{e}_y.$$

(iv) Recall that positive rotations are counterclockwise, so a 5-minute clockwise rotation is a rotation by $-360/12 = -30$ degrees. In radians this is $\theta = -30(\pi/180) = -\pi/6$. Plugging this into the matrix (given in class) that achieves a rotation by an angle of θ (in radians) and doing a few simplifications (using the facts that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$), we obtain the desired rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos(-\frac{\pi}{6}) & -\sin(-\frac{\pi}{6}) & 0 \\ \sin(-\frac{\pi}{6}) & \cos(-\frac{\pi}{6}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) & 0 \\ -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(v) We can do this in three steps. First, apply a translation that maps c to the origin, then apply the rotation from (iv), and finally translate the origin back to c. Define:

$$T(c) = \begin{pmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the transformation that maps c back to the origin is $T(-c)$. The final result is the product $T(c)R(\theta)T(-c)$.

Solution 2: The point aab is one-third along the way from a to b, and therefore $aab = \frac{2}{3}a + \frac{1}{3}b$. Analogously, $bcc = \frac{1}{3}b + \frac{2}{3}c$. We can apply analogous formulas for the other points around the boundary. Observe that abc is midway between aab and bcc, so we have $abc = \frac{1}{2}aab + \frac{1}{2}bcc$. Simplifying yields $abc = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$. In summary:

$$
\begin{array}{cc}
aab = \frac{2}{3}a + \frac{1}{3}b & abb = \frac{1}{3}a + \frac{2}{3}b \\
bbc = \frac{2}{3}b + \frac{1}{3}c & bcc = \frac{1}{3}b + \frac{2}{3}c \\
aac = \frac{2}{3}a + \frac{1}{3}c & acc = \frac{1}{3}a + \frac{2}{3}c \\
\multicolumn{2}{c}{abc = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c.}
\end{array}
$$

There are a number of equally valid ways to express these. For example, aab can also be written as $a + \frac{1}{3}(b - a)$.

Solution 3: We stated that it is okay to assume that the point lies in the first quadrant. This solution works no matter which quadrant the projectile is shot from.

As discussed in class, the projectile's x- and z-coordinates vary linearly with time t and the y-coordinate follows an inverted parabola. Letting $g = 9.8\text{m/sec}^2$ denote the gravitational acceleration, we have

$$
x(t) \;=\; p_x + t\,v_{0,x}, \qquad z(t) \;=\; p_z + t\,v_{0,z}, \qquad y(t) \;=\; -\frac{g}{2}t^2 + v_{0,y}t + h.
$$

(See the lecture notes for the derivation of $y(t)$.) Let t_g and t_w the times when the projectile hits the ground ($y = 0$) and when it hits the wall ($x = 0$), respectively. We will compute these and then select the smaller positive value between these two. (We consider only positive solutions, since we are uninterested in what happens in the past.)

First, let us consider the time t_g when the projectile will hit the ground. (The solution is the same one given in the lecture notes.) Let $a = g/2$, $b = -v_{0,y}$, and $c = -h$. We seek the smallest positive value of t such that $y(t) = at^2 + bt + c = 0$. (We have intentionally negated the coefficients so that $a > 0$.) By the quadratic formula we have

$$
t_g \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \;=\; \frac{v_{0,y} \pm \sqrt{v_{0,y}^2 + 2gh}}{g}.
$$

As mentioned in class, the desired root comes by selecting the + option from the $\pm$, and it is guaranteed to be positive under the reasonable assumption that the projectile starts above the ground.

Second, the time to hit the wall is determined by considering the x-coordinate of the projectile's position. We seek the value of t such that $x(t) = p_x + t\,v_{0,x} = 0$, that is

$$
t_w \;=\; \frac{-p_x}{v_{0,x}}.
$$

We ignore t_w if either $v_{0,x} = 0$ (meaning it is moving parallel to the wall and so never hits it) or if $t_w < 0$ (meaning that the projectile is moving away from the wall and does not hit it in the future). Otherwise, the first contact with either floor or wall occurs at time $t^* = \min(t_g, t_w)$. If t_g is the smaller, we return

$$
(x(t_g), y(t_g), z(t_g)) \;=\; (p_x + t_g\,v_{0,x},\ 0,\ p_z + t_g\,v_{0,z}).
$$

On the other hand, if t_w is the smaller, we return

$$
(x(t_w), y(t_w), z(t_w)) \;=\; (0,\ -\frac{g}{2}t_w^2 + v_{0,y}t_w + h,\ p_z + t_w\,v_{0,z}).
$$

Solution to the Challenge Problem: At time t, the positions of the two disk centers are $p(t) = p + t\vec{u}$ and $q(t) = q + t\vec{v}$, respectively. The disks intersect if their centers are within distance the sum of their radii, that is, 2 of each other. Distances are a bit messy to deal with because of the square roots, but we can equivalently test whether the *squared distances* between the two disks is ever smaller than $2^2 = 4$. Let $\vec{x} = p - q$ and let $\vec{w} = \vec{u} - \vec{q}$. It follows that the vector from $q(t)$ to $p(t)$ is

$$p(t) - q(t) \;=\; (p + t\vec{u}) - (q + t\vec{v}) \;=\; \vec{x} + t\vec{w}.$$

Using the fact that the squared length of a vector $\vec{z} = \vec{z} \cdot \vec{z}$, it follows that the squared distance between the disk centers at time t is

$$(p(t) - q(t)) \cdot (p(t) - q(t)) \;=\; (\vec{x} + t\vec{w}) \cdot (\vec{x} + t\vec{w}) \;=\; (\vec{w} \cdot \vec{w})t^2 + 2(\vec{x} \cdot \vec{w})t + (\vec{x} \cdot \vec{x}).$$

To determine the time of intersection, we could set this to 4 and then solve the resulting quadratic function of t using the quadratic formula. (We'll leave this as an exercise.)

Another approach is to apply calculus to determine the value of t that minimizes the distance between the two centers. Let us assume that $\vec{u}$ and $\vec{v}$ are not parallel. Letting $f(t)$ denote the squared distance function, we can compute the derivative of the squared distance function with respect to t, which yields

$$\frac{df}{dt} \;=\; 2t(\vec{w} \cdot \vec{w}) + 2(\vec{x} \cdot \vec{w}).$$

To compute the minimum, we set the derivative to zero and solve for t, which yields

$$t^* = -\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}.$$

(How do we know that this is a minimum, and not a maximum? Observe that the second derivative is $d^2f/dt^2 = 2(\vec{w} \cdot \vec{w}) = 2\|\vec{w}\|^2$, which cannot be negative. Since the second derivative is not negative, $f(t^*)$ is a minimum.)

To determine whether there is an intersection in the future, we test whether $t^* \geq 0$. If so, then if $f(t^*) \leq 4$, the disks collide at or before time t^*. If $t^* < 0$, then they are close as they can possibly be at time $t = 0$. We check whether $f(0) \leq 4$, and if so we report that the disks collide even before the start moving.