Incomplete higher order Gauss sums

Ronald Evans [a] Marvin Minei [b] Bennet Yee [c]

[a] *Department of Mathematics, 0112*
University of California at San Diego
La Jolla, California 92093-0112

[b] *Department of Mathematics*
University of California at Irvine
Irvine, California 92697-3875

[c] *Department of Computer Science & Engineering, 0114*
University of California at San Diego
La Jolla, California 92093-0114

Abstract

We consider the classical incomplete higher order Gauss sums

$$S_m(B) = \sum_{j=0}^{B} \exp(2\pi i j^m/N), \qquad 0 \le B < N-1,$$

where N is large. In 1976, D. H. Lehmer analyzed the beautiful spirals appearing in the directed graph

$$S_m(0) \to S_m(1) \to S_m(2) \to \ldots$$

in the complex plane, for $m = 2$. In the process, he obtained sharp uniform estimates for the sums $S_2(B)$. Sullivan and Zannier expanded on Lehmer's work in 1992 by giving an asymptotic formula for $S_2(B)$ valid for $B < N(1/2 - \varepsilon)$.

For general $m > 1$, the directed graphs still exhibit spirals, but with more complicated behavior than in the quadratic case. We analyze this behavior and in the process obtain sharp asymptotic estimates for certain incomplete Gauss sums $S_m(B)$.

Key words: incomplete Gauss sums, Gauss sums, graphs, spirals, condensation points, asymptotic expansions, Springer numbers, Euler numbers

---

Email addresses: revans@ucsd.edu (Ronald Evans), mminei@uci.edu (Marvin Minei), bsy@cs.ucsd.edu (Bennet Yee).

1 Introduction

Let N be large (not necessarily an integer) and fix $m > 1$. For integers $B \geq 0$, define the exponential sums

$$S_m(B) = \sum_{j=0}^{B} e^{2\pi i j^m/N}. \tag{1.1}$$

If N is an integer, $S_m(N-1)$ is a classical Gauss sum of order m. For $0 \leq B < N-1$, $S_m(B)$ is an *incomplete* Gauss sum of order m.

To analyze the growth of the incomplete quadratic Gauss sums $S_2(B)$ for integer N, D. H. Lehmer [10] sketched directed graphs

$$0 \to S_2(0) \to S_2(1) \to S_2(2) \to \cdots \to S_2(N-1) \tag{1.2}$$

in the complex plane, where the first arrow represents an edge (unit vector) connecting vertex 0 to vertex $S_2(0)$, the second arrow connects $S_2(0)$ to $S_2(1)$, and so on.

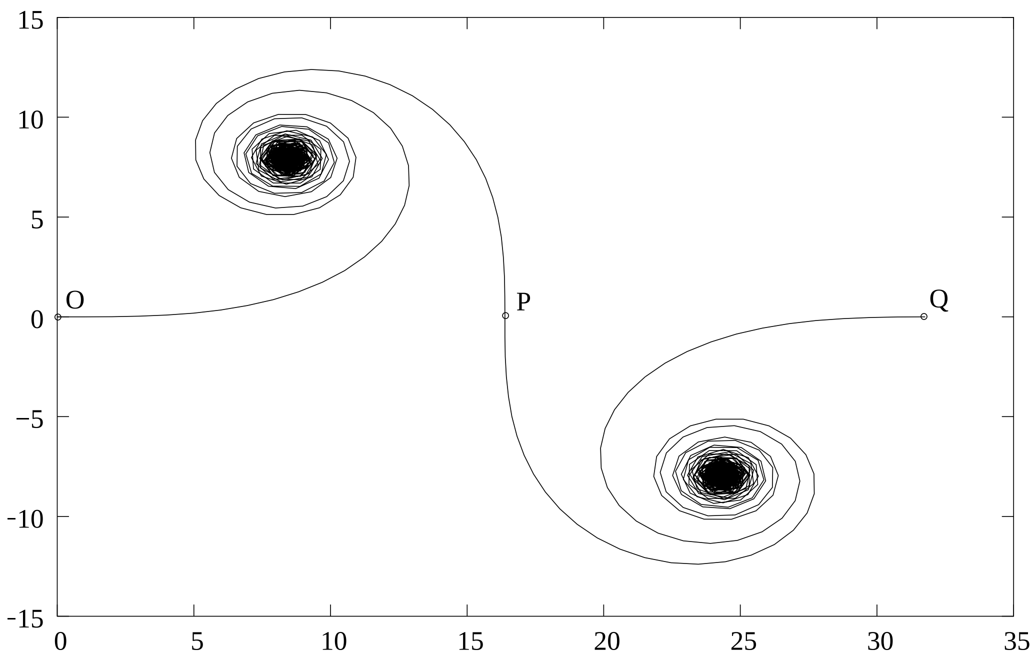

Fig. 1. $m = 2$, $N = 1009$

For example, Figure 1 gives the graph (1.2) for $N = 1009$. The graph starts at the origin $O = 0$, ends at the point $Q = S_2(N-1) = N^{1/2}$, and has the point $P = S_2((N-1)/2) = (1+N^{1/2})/2$ in the middle. The points O, P, and Q are “spiral endpoints” for the four spirals on the graph. The spirals close in upon the two “condensation points” $S_2([N/4])$ and $S_2([3N/4])$.

Using the geometry of spirals, Lehmer showed that for large N, almost all the vertices $S_2(B)$ in the first half of the graph are in the vicinity of the single condensation point $S_2([N/4])$, with

$$S_2([N/4]) = \sqrt{N}(1+i)/4 + O(1). \tag{1.3}$$

As an application, sharp uniform estimates for incomplete quadratic Gauss sums were obtained of the shape

$$S_2(B) \leq c\sqrt{N}, \qquad 0 \leq B \leq N/2, \tag{1.4}$$

where c is an explicit constant.

Motivated by applications in the theory of optics, Sullivan and Zannier [14, p. 55] expanded on Lehmer’s work by proving that if $B, N \to \infty$ in such a way that B is an integer satisfying $B/N < \rho < 1/2$ (where ρ is constant), then

$$\begin{aligned} S_2(B) &= \sqrt{N}(1+i)/4 \;+\; 1/2 \qquad (1.5)\\ &\;+\; \exp(2\pi i B^2/N)(1 - i\cot(2\pi B/N))/2 \;+\; O(N^2 B^{-3}). \end{aligned}$$

Note that for $B = [N/4]$, (1.5) gives an estimate for the condensation point $S_2([N/4])$ which is more precise than that in (1.3).

The first object of this paper is to extend the asymptotic estimate (1.5) of Sullivan and Zannier to the incomplete Gauss sums $S_m(B)$ of order m, for each fixed $m > 1$. This provides solutions to both Research Problems #9 and #10 posed in the book of Evans, Berndt, and Williams [1, pp. 496–497].

In the last twenty-five years, many papers have given estimates for sums of the form

$$\sum_{j=0}^{B} e^{\pi i \tau j^m}, \qquad 0 < \tau. \tag{1.6}$$

The case $\tau = 2/N$ yields our sum $S_m(B)$, and so our interest is when $\tau \to 0$. Loxton [11] investigated (1.6) in the case $m = 1/2$; his method does not extend to the values $m > 1$ considered here. The techniques of most of the papers that deal with $m > 1$ (including the present one) involve some sort of combination of Poisson summation, van der Corput’s method, or stationary phase; there is nevertheless a large variance in the nature of the hypotheses, results, and refinement of error analysis. One of the earliest papers dealing with $m = 2$ is that of Fiedler, Jurkat, and Körner [6]. Their error term is too large for our purposes when $\tau \to 0$. The recent paper of Coutsias and Kazarinoff [3] for $m = 2$ also gives too large an error term in this context, although it is sharp for selected subsequences of

τ approaching 0. Among the papers dealing with more general $m > 1$, we mention those of Deshouillers [5] and Moore and van der Poorten [12]. The former provides asymptotic analysis for fixed values of τ rather than for $\tau \to 0$. For interesting connections of (1.6) with a variety of areas in mathematics and physics, see Oskolkov [13], Berry and Goldberg [2], and the references within [3].

The following graph for the incomplete Gauss sums of order m is analogous to Lehmer's graph in (1.2):

$$S_m(0) \to S_m(1) \to S_m(2) \to \dots. \tag{1.7}$$

Picturesque spirals appear for each $m > 1$, just as in the case $m = 2$, but it is harder to pinpoint the spiral locations in the general case. Traversing the graph

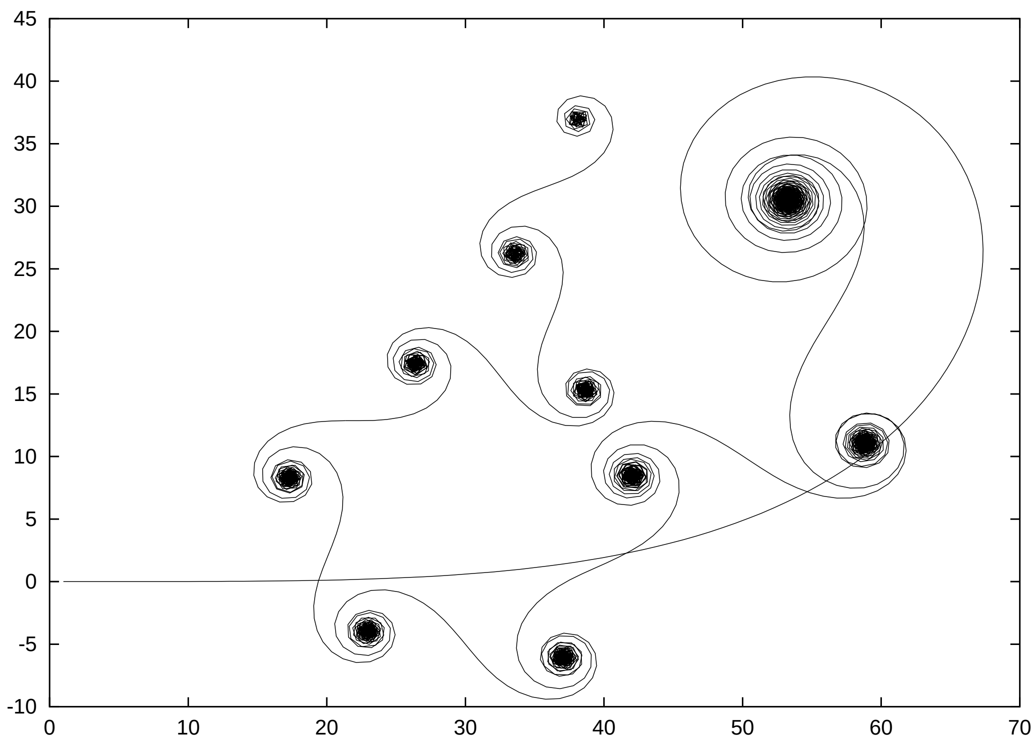

Fig. 2. $m = 3$, $N = 2000000$

(1.7) for large N, one encounters within $O(N^{1/(m-1)})$ edges a primary condensation point followed by an unlimited number of additional condensation points, called "satellites". The primary point and the satellites are the focus of this paper. We illustrate by sketching a portion of the graph (1.7) for $m = 3$, $N = 2000000$ (Figure 2). This graph starts at $1 = S_3(0)$ and continues through 2516 edges, long enough to show the primary condensation point (at roughly $54 + 31i$) and 9 satellites. In general, for large N, the graph (1.7) will distinctly show L condensation points if it continues through at least $((L - 1/2)N/m)^{1/(m-1)}$ edges, for any fixed positive integer L.

A second object of this paper is to provide asymptotic formulas that accurately locate the points in the neighborhood of each satellite. These formulas lead to sharp estimates for certain incomplete Gauss sums $S_m(B)$. The estimates are especially precise for quadratic "quarter Gauss sums" $S_2([N/4])$.

As will be seen from Theorem 8 below, the distance from the origin to the primary condensation point is asymptotic to $N^{1/m}$ times a positive constant, while the distance from the primary point to the k-th satellite (for any fixed k) is $O(N^{1/(2m-2)})$. Thus for large N, the primary condensation point is much closer to its k-th satellite than it is to the origin, when $m > 2$. This is not the case when $1 < m < 2$. For Theorem 7 shows that the distance from the primary point to its first satellite is asymptotic to $N^{1/(2m-2)}$ times a positive constant, which is much more than the distance from the primary point to the origin, when $1 < m < 2$.

We now summarize the main results of this paper.

In Section 2, we identify the specific vertices of the graph (1.7) which are "condensation points" and "spiral endpoints". It is shown why we can designate $S_m(B_k)$, $k = 1,3,5,\ldots$, as condensation points, and $S_m(B_k)$, $k = 0,2,4,\ldots$, as spiral endpoints, where for $k \geq 0$,

$$b_k = (kN/(2m))^{1/(m-1)}, \qquad B_k = [b_k]. \tag{1.8}$$

In Section 3, we prove the following theorem, which gives good estimates for incomplete Gauss sums of order m in the vicinity of either of the two spirals surrounding the primary condensation point $S_m(B_1)$.

Theorem 1 *Fix $m > 1$. If $B, N \to \infty$ in such a way that B is an integer satisfying*

$$v = v(B,N) := mB^{m-1}/N < 1, \tag{1.9}$$

then

$$\begin{aligned} S_m(B) = &\left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right) + \frac{1}{2} \\ &+ \exp(2\pi i B^m/N)(1 - i\cot(\pi v))/2 \\ &+ O\left(\frac{1+vB^{2-m}}{B(1-v)^3} + \frac{1}{Bv^2}\right). \end{aligned} \tag{1.10}$$

In (1.10) and the sequel, implied constants depend only on m. See the end of Section 3 for a slight improvement of the error term when $1 < m < 2$.

Consider the special case of Theorem 1 where $m = 2$. When v is bounded away from 1, we recover the main result (1.5) of Sullivan and Zannier [14, p. 55]. While our result (1.10) holds for all $v < 1$, the result (1.5) of Sullivan and Zannier does *not* hold for all $v < 1$, despite what is stated in [14, Observation 1, p. 56]. This is easily verified by considering the example $N = 2B+1$ for large even integers B.

Choosing $B = B_1$ (where B_1 is defined in (1.8)), we have, by (1.9),

$$v = 1/2 + O(N^{-1/(m-1)}), \qquad \cot(\pi v) = O(N^{-1/(m-1)}), \tag{1.11}$$

and so Theorem 1 yields the following precise estimate for the location of the primary condensation point $S_m(B_1)$.

Corollary 2 *Fix $m > 1$. As $N \to \infty$, we have*

$$S_m(B_1) = \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right) \tag{1.12}$$
$$+ \; 1/2 \; + \; \exp(2\pi i B_1^m/N)/2 \; + \; E(N),$$

where the error term $E(N)$ satisfies

$$E(N) = \begin{cases} O(N^{-1/(m-1)}), & \text{if} \quad m \geq 2, \\ O(1/N), & \text{if} \quad 1 < m < 2. \end{cases} \tag{1.13}$$

Note that in the case $m = 2$, Corollary 2 is in agreement with (1.3) and (1.5).

In the example with $m = 3$, $N = 2000000$, the primary condensation point is at $53.7784 + 30.6392i$ (cf. Figure 2), while the three main terms on the right side of (1.12) sum to $53.7781 + 30.6401i$. (All decimals are truncated.)

Let $d = d(N)$ be any function such that $dN^{1/m}$ is an integer with

$$d \to \infty, \qquad d = o(N^{1/(m^2-m)}). \tag{1.14}$$

Then by (1.11) with $B = dN^{1/m}$, we have

$$v = d^{m-1}/N^{1/m} = o(1), \quad \cot(\pi v) = o(N^{1/m}), \quad B^{-1}v^{-2} = o(N^{1/m}). \tag{1.15}$$

Thus Theorem 1 yields:

Corollary 3 *Fix $m > 1$ and suppose that (1.14) holds. Then as $N \to \infty$,*

$$S_m(dN^{1/m}) \sim \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right). \tag{1.16}$$

The significance of Corollary 3 is as follows. While the number of summands in the incomplete Gauss sum $S_m(B_1)$ exceeds $(N/(2m))^{1/(m-1)}$, a comparison of Corollaries 2 and 3 shows that the main contribution to the sum $S_m(B_1)$ already comes from just the first $dN^{1/m}$ terms, no matter how slowly d grows with N. The terms remaining in the tail combine to contribute relatively little, due to enormous cancellation from the effect of the vortex surrounding the primary condensation point.

Lehmer [10] and Loxton [11] explained this cancellation geometrically by showing that there is a disk of small radius that contains a vast proportion of the spiral surrounding the primary condensation point. We do not appeal to their geometric argument in this paper, but include it in the Appendix, since it provides an interesting perspective, and because it can be used for explicit calculations such as the determination of c in (1.4).

In (1.15), $v \to 0$. We now discuss some consequences of Theorem 1 when $v \to 1$. Fix α with $0 < \alpha < 1/(m-1)$. Let $\gamma = \gamma(N)$ be any function such that γN^{α} is an integer with

$$\gamma \to \infty, \qquad \gamma = o\left(N^{-\alpha+1/(m-1)}\right). \tag{1.17}$$

Recall from (1.8) that $B_2 = [(N/m)^{1/(m-1)}]$. A straightforward application of Theorem 1 yields the following asymptotic formula as $N \to \infty$:

$$S_m(B_2 - \gamma N^{\alpha}) = \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right) \, + \, R(N), \tag{1.18}$$

where

$$R(N) = \begin{cases} o\left(N^{-\alpha+1/(m-1)} + N^{-3\alpha+2/(m-1)}\right), & \text{if} \quad m \geq 2, \\ o\left(N^{-\alpha+1/(m-1)} + N^{-3\alpha+(4-m)/(m-1)}\right), & \text{if} \quad 1 < m < 2. \end{cases} \tag{1.19}$$

In particular, if either

$$m \geq 2, \ \alpha \geq \frac{m+1}{3m(m-1)} \quad \text{or} \quad 1 < m < 2, \ \alpha \geq \frac{3m+1-m^2}{3m(m-1)}, \tag{1.20}$$

then

$$S_m(B_2 - \gamma N^{\alpha}) \sim \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right). \tag{1.21}$$

In Section 4, we sharpen the results of Theorem 1 in the quadratic case $m = 2$. Theorem 4 below gives a precise version of a generalization of (1.5) suggested by Sullivan and Zannier [14, Observation 2, p. 56]. We later apply this theorem to prove an estimate for quarter Gauss sums in Theorem 5.

Theorem 4 *As $B, N \to \infty$ in such a way that B is an integer satisfying $v = 2B/N < 1$, we have*

$$
\begin{aligned}
S_2(B) &= \sqrt{N}(1+i)/4 + 1/2 + \exp(2\pi i B^2/N)/2 \\
&+ \frac{\exp(2\pi i B^2/N)}{2\pi i} \sum_{j=0}^{J-2} \frac{1}{j!} \left(\frac{1}{2\pi i N} \right)^j F^{(2j)}(v) \qquad (1.22) \\
&+ O(B^{1-J}(1-v)^{1-2J} + B^{1-J} v^{-J})
\end{aligned}
$$

for any fixed positive integer J, where

$$F(v) = \pi \cot(\pi v). \qquad (1.23)$$

For $J = 2$, Theorem 4 reduces to the case $m = 2$ of Theorem 1. The derivatives $F^{(n)}(v)$ appearing in (1.22) are polynomials in $x = \cot(\pi v)$ of degree $n+1$, for every positive integer n. For example,

$$
\begin{aligned}
F'(v) &= -\pi^2(1+x^2) \qquad (1.24) \\
F''(v) &= 2\pi^3(x+x^3) \\
F^{(3)}(v) &= -2\pi^4(1+4x^2+3x^4) \\
F^{(4)}(v) &= 8\pi^5(2x+5x^3+3x^5).
\end{aligned}
$$

Properties of these polynomials are discussed in [8], [9].

In the case $m = 2$, the primary condensation point is the quadratic "quarter Gauss sum"

$$Q_N := S_2([N/4]) = S_2(B_1). \qquad (1.25)$$

By Corollary 2, this quarter Gauss sum satisfies

$$Q_N = \frac{(1+i)}{4}\sqrt{N} + \frac{1}{2} + \frac{1}{2}\exp\left(\frac{2\pi i [N/4]^2}{N}\right) + O\left(\frac{1}{N}\right). \qquad (1.26)$$

Formula (1.26) is made more precise by the following asymptotic expansion for Q_N proved in Section 4.

Theorem 5 *Write*

$$h := N - 4[N/4], \qquad (1.27)$$

so that $0 \le h < 4$. For $n = 0, 1, 2, \ldots$, define the monic polynomials $S_{2n}(y)$ of degree $2n$ by the generating function

$$\frac{\cos(2x - yx)}{\cos(2x)} = \sum_{n=0}^{\infty} (-1)^n S_{2n}(y) \frac{x^{2n}}{(2n)!}. \qquad (1.28)$$

Then as $N \to \infty$,

$$Q_N = \sum_{r=0}^{[N/4]} e^{2\pi i r^2/N} = \frac{(1+i)}{4}\sqrt{N} + \frac{1}{2} \tag{1.29}$$
$$+ \frac{1}{2} e^{2\pi i(N-2h)/16} \sum_{r=0}^{R-1} \frac{1}{r!}\left(\frac{\pi i}{8N}\right)^r S_{2r}(h) + O(N^{-R})$$

for any fixed positive integer R.

In the case $R = 1$, Theorem 5 reduces to (1.26).

The first few polynomials $S_{2n}(y)$ are given in the table below.

n	$S_{2n}(y)$	$S_{2n}(1)$	$S_{2n}(2)/4^n$
0	1	1	1
1	$y^2 - 4y$	-3	-1
2	$y^4 - 8y^3 + 64y$	57	5
3	$y^6 - 12y^5 + 320y^3 - 3072y$	-2763	-61
4	$y^8 - 16y^7 + 896y^5 - 28672y^3 + 278528y$	250737	1385

The numbers

$$S_{2n} := S_{2n}(1) \tag{1.30}$$

in the table were investigated by Glaisher [7] but are often called (signed) Springer numbers. Note that $S_{2n} = S_{2n}(1) = S_{2n}(3)$, since $S_{2n}(y) = S_{2n}(4-y)$ by definition (1.28). The numbers

$$E_{2n} := S_{2n}(2)/4^n \tag{1.31}$$

in the table are the well known (signed) Euler numbers. Springer numbers and their relationship with Euler numbers are discussed in [9].

From Theorem 5, we have the following asymptotic expansions for large integers N in fixed congruence classes modulo 4.

As $N \to \infty$ with $N \equiv 1 \pmod 4$,

$$Q_N - \frac{(1+i)}{4}\sqrt{N} - \frac{1}{2} \sim e^{2\pi i(N-2)/16} \sum_{r=0}^{\infty} \frac{1}{2\, r!}\left(\frac{\pi i}{8N}\right)^r S_{2r}. \tag{1.32}$$

As $N \to \infty$ with $N \equiv 3 \pmod 4$,

$$Q_N - \frac{(1+i)}{4}\sqrt{N} - \frac{1}{2} \sim e^{2\pi i(N-6)/16} \sum_{r=0}^{\infty} \frac{1}{2\, r!}\left(\frac{\pi i}{8N}\right)^r S_{2r}. \tag{1.33}$$

As $N \to \infty$ with $N \equiv 2 \pmod 4$,

$$Q_N - \frac{(1+i)}{4}\sqrt{N} - \frac{1}{2} \sim\ e^{2\pi i(N-4)/16} \sum_{r=0}^{\infty} \frac{1}{2\,r!} \left(\frac{\pi i}{2N}\right)^r E_{2r}. \tag{1.34}$$

Theorem 6 below shows that in the case that $N \equiv 0 \pmod 4$, Q_N can be evaluated in closed form. An elementary proof is given in Section 4.

Theorem 6 *If* $N \equiv 0 \pmod 4$, *then*

$$Q_N = \sqrt{N}(1+i)/4 + (1+i^{N/4})/2. \tag{1.35}$$

In Section 3, we estimated incomplete Gauss sums of order m in the vicinity of the spirals surrounding the primary condensation point $S_m(B_1)$. In Section 5, we estimate incomplete Gauss sums of order m in the vicinity of the spirals surrounding each satellite $S_m(B_{2h+1})$, where h is a fixed positive integer. For this purpose, define the sums

$$S_m(A,C) = \sum_{j=A+1}^{C} e^{2\pi i j^m/N}, \tag{1.36}$$

for integers A, C. In Theorem 1, we estimated sums of the form

$$S_m(B_0,C) = S_m(C) - 1, \qquad 0 = B_0 < C < B_2. \tag{1.37}$$

Theorem 7 below, proved in Section 5, estimates the related sums

$$S_m(A,B_{2k}), \qquad S_m(B_{2k},C) \tag{1.38}$$

for each positive integer k, where the integers A, C satisfy, for some small positive constant ε,

$$B_{2k-2+\varepsilon} < A < B_{2k-\varepsilon} < B_{2k+\varepsilon} < C < B_{2k+2-\varepsilon}. \tag{1.39}$$

For brevity, write

$$x = x(N) = (kN/m)^{1/(m-1)}, \tag{1.40}$$

so that by (1.8), $[x] = B_{2k}$.

Theorem 7 *Fix $m > 1$, fix a positive integer k, and assume that (1.39) holds. Then as $N \to \infty$, each of the sums $S_m(A,B_{2k})$, $S_m(B_{2k},C)$ equals $L(N) + O(1)$, where*

$$L(N) := N^{1/(2m-2)} \left\{ \left(\frac{m}{k}\right)^{\frac{m-2}{2m-2}} \frac{(1+i)}{(8m^2-8m)^{1/2}}\, e^{2\pi i k x(1-m)/m} \right\}, \tag{1.41}$$

with x as defined in (1.40). In particular,

$$S_m(B_{2k},B_{2k+1}) = L(N) + O(1) \tag{1.42}$$

and

$$S_m(B_{2k-1}, B_{2k}) = L(N) + O(1). \tag{1.43}$$

Viewed as a complex vector, $S_m(B_{2k-1}, B_{2k})$ connects the satellite $S_m(B_{2k-1})$ to the spiral endpoint $S_m(B_{2k})$, while $S_m(B_{2k}, B_{2k+1})$ connects this spiral endpoint $S_m(B_{2k})$ to the next satellite $S_m(B_{2k+1})$. For any fixed positive integer h, the h-th satellite $S_m(B_{2h+1})$ may be computed by

$$S_m(B_{2h+1}) = S_m(B_1) + \sum_{k=1}^{h} S_m(B_{2k-1}, B_{2k+1}). \tag{1.44}$$

It follows from (1.44), Theorem 7, and Corollary 2 that each satellite $S_m(B_{2h+1})$ can be located by the asymptotic formula below.

Theorem 8 *Fix $m > 1$. For any fixed positive integer h,*

$$\begin{aligned} S_m(B_{2h+1}) = {} & \\ & \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left(\frac{2\pi i}{4m}\right) \\ & + \sum_{k=1}^{h} N^{1/(2m-2)} \left(\frac{m}{k}\right)^{\frac{m-2}{2m-2}} \frac{2(1+i)}{(8m^2-8m)^{1/2}} e^{2\pi i k x(1-m)/m} \\ & + O(1), \end{aligned} \tag{1.45}$$

where x is defined in (1.40).

Let $D = D(N)$ be any function such that $DN^{1/(2m-2)}$ is an integer with

$$D \to \infty, \qquad D = o(N^{1/(2m-2)}). \tag{1.46}$$

We prove the following result in Section 5:

Theorem 9 *Fix $m > 1$ and fix a positive integer k. With D as in (1.46),*

$$S_m(B_{2k}, B_{2k} + DN^{1/(2m-2)}) \sim L(N) \tag{1.47}$$

and

$$S_m(B_{2k} - DN^{1/(2m-2)}, B_{2k}) \sim L(N) \tag{1.48}$$

as $N \to \infty$, where $L(N)$ is defined in (1.41).

The significance of (1.47) is as follows. While the number of summands in the sum $S_m(B_{2k}, B_{2k+1})$ exceeds a positive constant times $N^{1/(m-1)}$, a comparison of (1.42) and (1.47) shows that the main contribution to the sum $S_m(B_{2k}, B_{2k+1})$ already comes from just the first $DN^{1/(2m-2)}$ terms, no matter how slowly D grows with N. The remaining terms combine to contribute relatively little, due to cancellation from the effect of the vortex surrounding the satellite $S_m(B_{2k+1})$.

2 Spiral endpoints and condensation points

Define B_k as in (1.8). In this section, we look at the graph (1.7) and explain why we can designate the vertices $S_m(B_k)$, $k = 1,3,5,\ldots$, as "condensation points", and the vertices $S_m(B_k)$, $k = 0,2,4,\ldots$, as "spiral endpoints".

For each vertex $V = S_m(j)$, $j > 0$, consider the change in argument

$$\triangle(V) = 2\pi(j+1)^m/N \; - \; 2\pi j^m/N \tag{2.1}$$

between the two edges that intersect at V. When $\triangle(V)$ is near an even multiple of π, the edge vectors at V have nearly the same direction (i.e., the "spiral curvature" is nearly 0), and so V will be near a spiral endpoint. When $\triangle(V)$ is near an odd multiple of π, the edge vectors at V have nearly opposite directions, and so V will be near a condensation point. As we traverse the graph from a spiral endpoint to a condensation point, our path is a (polygonal) spiral winding in a counterclockwise direction. Proceeding from a condensation point to a spiral endpoint, the path is a spiral unwinding clockwise.

Let W be the vertex on a spiral whose two edge vectors are essentially perpendicular. It is interesting to note that all of the points on this spiral between its condensation point and W are constrained to lie in a single disk of radius < 1.25. This is a consequence of Theorem 10 in the Appendix.

As indicated above, we are interested in finding those vertices V for which $\triangle(V)$ is nearly a multiple of π. Accordingly, we set

$$\triangle(V) = k\pi \tag{2.2}$$

for $k = 0,1,2,\ldots$ and solve (2.2) for j, temporarily ignoring the fact that the solution j may not be an integer. By (2.1) – (2.2) and the mean value theorem,

$$kN/2 = (j+1)^m - j^m = m(j+\delta)^{m-1} \tag{2.3}$$

for some $\delta \in (0,1)$. Thus the real solution j to (2.2) is given by $j = b_k - \delta$, in the notation of (1.8). This solution $b_k - \delta$ differs from the integer B_k by less than 1. We thus see that for each positive integer k, $\triangle(S_m(B_k))$ nearly equals $k\pi$; more precisely, as $N \to \infty$,

$$\triangle\big(S_m(B_k)\big) = k\pi + O\left(N^{-1/(m-1)}\right). \tag{2.4}$$

Moreover, for each positive integer k,

$$(k-1)\pi < \triangle\big(S_m(j)\big) < k\pi, \quad \text{for} \quad B_{k-1} < j < B_k. \tag{2.5}$$

In view of (2.4), we designate the vertices $S_m(B_k)$ as condensation points when

k is odd, and as spiral endpoints when k is even. The primary condensation point is $S_m(B_1)$, and the satellite condensation points are $S_m(B_3), S_m(B_5), S_m(B_7), \ldots$.

3 The primary condensation point

The object of this section is to prove Theorem 1, which estimates incomplete Gauss sums of order m in the vicinity of the primary condensation point. The proof starts out much like that of Sullivan and Zannier [14], commencing with the Poisson summation formula; however, a more elaborate error analysis is required. Throughout the proof, N and B are large, B is an integer, and $v = mB^{m-1}/N < 1$. For brevity, write

$$e(x) := \exp(2\pi ix). \tag{3.1}$$

Proof of Theorem 1. By the Poisson summation formula [4, p .14],

$$S_m(B) = \sum_{r=0}^{B} e(r^m/N) = \frac{1}{2} + \frac{1}{2}\, e(B^m/N) + \sum_k I_k, \tag{3.2}$$

where

$$\sum_k = \lim_{K\to\infty} \sum_{k=-K}^{K}$$

and

$$I_k := \int_0^B e(w^m/N + kw)dw. \tag{3.3}$$

First consider the case $k = 0$. By [15, section 3.127],

$$\begin{aligned} I_0 &= \int_0^\infty e(u^m/N)du - \int_B^\infty e(u^m/N)du \\ &= \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) e\left(\frac{1}{4m}\right) - \int_B^\infty e(u^m/N)du. \end{aligned} \tag{3.4}$$

Integrating by parts J times, we obtain

$$\begin{aligned} &-\int_B^\infty e(u^m/N)du = Be(B^m/N) \sum_{j=0}^{J-1} (2\pi iBv)^{-j-1} \prod_{1\le d\le j} (dm-1) \\ &- \left(\frac{N}{2\pi im}\right)^J \prod_{1\le d\le J} (dm-1) \int_B^\infty u^{-Jm} e(u^m/N)du. \end{aligned} \tag{3.5}$$

Thus,

$$-\int_B^\infty e(u^m/N)du = \tag{3.6}$$
$$Be(B^m/N)\sum_{j=0}^{J-1}(2\pi iBv)^{-j-1}\prod_{1\le d\le j}(dm-1)+O(N^JB^{1-Jm}).$$

By (3.4) and (3.6), we have, for any positive integer J,

$$I_0 = \left(\frac{N}{2\pi}\right)^{1/m}\Gamma\left(\frac{m+1}{m}\right)e\left(\frac{1}{4m}\right)+ \tag{3.7}$$
$$Be(B^m/N)\sum_{j=0}^{J-1}(2\pi iBv)^{-j-1}\prod_{1\le d\le j}(dm-1)+O(N^JB^{1-Jm}).$$

We now investigate I_k for $k\neq 0$. Make the change of variable $w = Bu$ in (3.3) to obtain

$$I_k = B\int_0^1 e(Bg(u))du, \tag{3.8}$$

where

$$g(u) := vu^m/m+ku. \tag{3.9}$$

Integrating by parts in (3.8), we obtain

$$I_k = \frac{e(Bg(u))}{2\pi ig'(u)}\bigg|_0^1 + \frac{v(m-1)}{2\pi i}\int_0^1\frac{u^{m-2}e(Bg(u))}{g'(u)^2}\,du. \tag{3.10}$$

Since

$$g'(u) = vu^{m-1}+k, \tag{3.11}$$

it follows from (3.10) that

$$I_k = \frac{e(Bg(u))}{2\pi ig'(u)}\bigg|_0^1 + O\left(\frac{vB^{1-m}}{(|k|-1/2)^2}\right)+\frac{v(m-1)}{2\pi i}\int_{1/B}^1\frac{u^{m-2}e(Bg(u))}{g'(u)^2}\,du. \tag{3.12}$$

Integrating by parts in (3.12), we obtain

$$I_k = O\left(\frac{vB^{1-m}}{(|k|-1/2)^2}\right)+\frac{e(Bg(u))}{2\pi ig'(u)}\bigg|_0^1 \tag{3.13}$$
$$+\ \frac{v(m-1)u^{m-2}e(Bg(u))}{(2\pi i)^2Bg'(u)^3}\bigg|_{1/B}^1$$
$$+\ \frac{v(m-1)}{(2\pi i)^2B}\int_{1/B}^1\frac{e(Bg(u))}{g'(u)^4}(v(2m-1)u^{2m-4})du$$
$$+\ \frac{v(m-1)}{(2\pi i)^2B}\int_{1/B}^1\frac{e(Bg(u))}{g'(u)^4}(k(2-m)u^{m-3})du.$$

Therefore, summing yields

$$\sum_{k\neq 0} I_k = O(vB^{1-m}) + \frac{e(B^m/N)}{2\pi i}\sum_{k\neq 0}\frac{1}{k+v} + \frac{v(m-1)e(B^m/N)}{(2\pi i)^2 B}\sum_{k\neq 0}\frac{1}{(k+v)^3} + \frac{v^2(m-1)(2m-1)}{(2\pi i)^2 B}\sum_{k\neq 0}\int_{1/B}^{1} f_k(u)du + \frac{v(m-1)(2-m)}{(2\pi i)^2 B}\sum_{k\neq 0} k\int_{1/B}^{1} g_k(u)du, \tag{3.14}$$

where

$$f_k(u) = \frac{u^{2m-4}e(Bvu^m/m+Buk)}{(k+vu^{m-1})^4} \tag{3.15}$$

and

$$g_k(u) = \frac{u^{m-3}e(Bvu^m/m+Buk)}{(k+vu^{m-1})^4}. \tag{3.16}$$

Define the function

$$M(B) := \max(1, B^{2-m}). \tag{3.17}$$

Thus $M(B) = 1$ when $m \geq 2$ and $M(B) = B^{2-m}$ when $1 < m < 2$. By (3.15), we have

$$\int_{1/B}^{1} f_k(u)du \ll M(B)\int_0^1 \frac{u^{m-2}}{(|k|-vu^{m-1})^4}\,du \ll \frac{M(B)}{v(|k|-v)^3}, \tag{3.18}$$

where the last inequality follows because, with the change of variable

$$y = |k| - vu^{m-1}, \tag{3.19}$$

we have

$$\int_0^1 \frac{u^{m-2}}{(|k|-vu^{m-1})^4}\,du \ll \frac{1}{v}\int_{|k|-v}^{|k|} y^{-4}\,dy \ll \frac{1}{v(|k|-v)^3}. \tag{3.20}$$

We now examine $\int_{1/B}^{1} g_k(u)du$, with $m \neq 2$. (If $m = 2$, this integral is immaterial since the last term in (3.14) vanishes.) By (3.16),

$$\int_{1/B}^{1} g_k(u)\,du \ll \int_{1/B}^{1}\frac{u^{m-3}}{(|k|-vu^{m-1})^4}\,du \ll \int_{1/B}^{1/2}\frac{u^{m-3}}{(|k|-v(1/2)^{m-1})^4}\,du + 2\int_{1/2}^{1}\frac{u^{m-2}}{(|k|-vu^{m-1})^4}\,du. \tag{3.21}$$

By (3.20) and (3.21),

$$\int_{1/B}^{1} g_k(u)du \ll \frac{M(B)}{(|k|-v(1/2)^{m-1})^4} + \frac{2}{v(|k|-v)^3}. \tag{3.22}$$

Therefore, by (3.18) and (3.22), formula (3.14) yields

$$\sum_{k\neq 0} I_k = \frac{e(B^m/N)}{2\pi i} \sum_{k\neq 0} \frac{1}{k+v} + O\left(\frac{1+vB^{2-m}}{B(1-v)^3}\right). \tag{3.23}$$

Combining (3.23) with the case $J=2$ of (3.7), we obtain

$$\sum_k I_k = \left(\frac{N}{2\pi}\right)^{1/m} \Gamma\left(\frac{m+1}{m}\right) e\left(\frac{1}{4m}\right) \tag{3.24}$$
$$+ \frac{e(B^m/N)}{2\pi i} \sum_k \frac{1}{k+v} + O\left(\frac{1+vB^{2-m}}{B(1-v)^3}\right) + O\left(\frac{1}{Bv^2}\right).$$

Theorem 1 now follows from (3.24) and (3.2), in view of the well known partial fraction decomposition [15, section 3.22]

$$\sum_k \frac{1}{k+v} = \pi \cot(\pi v). \tag{3.25}$$

■

In some cases, we can improve the error term in Theorem 1. For example, suppose that $1 < m < 2$ and that $y = y(B,N)$ satisfies $(1-v)^3 \leq y < 1$ and $By \to \infty$. Then by using $1/(By)$ instead of $1/B$ in (3.12), we readily see that the numerator $1 + vB^{2-m}$ in the error term of (1.10) can be replaced by the smaller quantity $1 + v(yB)^{2-m}$.

4 Primary points in the quadratic case

This section is devoted to sharpening Theorem 1 and Corollary 2 in the quadratic case $m = 2$. We begin by proving Theorem 4, which reduces to Theorem 1 when $m = J = 2$. Throughout, N and B are large, B is an integer, and $v = 2B/N < 1$.

Proof of Theorem 4. Let k be a nonzero integer and let $m = 2$. Integrating by parts J times in (3.8) yields

$$I_k = \frac{1}{2\pi i}\sum_{j=0}^{J-1}\left(\frac{v}{4\pi iB}\right)^j \frac{(2j)!}{j!}\,\frac{e(Bg(u))}{(k+vu)^{2j+1}}\Bigg|_0^1 + \frac{B(2J)!}{J!}\left(\frac{v}{4\pi iB}\right)^J \int_0^1 \frac{e(Bg(u))}{(k+vu)^{2J}}\,du, \tag{4.1}$$

where $g(u) = vu^2/2 + ku$. (We do not know an elegant counterpart to (4.1) for general m.) It follow from (4.1) that for any positive integer J,

$$\sum_{k\neq 0} I_k = \frac{e(Bv/2)}{2\pi i}\sum_{j=0}^{J-1}\frac{(2j)!}{j!}\left(\frac{v}{4\pi iB}\right)^j \sum_{k\neq 0}\frac{1}{(k+v)^{2j+1}} + \frac{B(2J)!}{J!}\left(\frac{v}{4\pi iB}\right)^J \int_0^1 e(Bvu^2/2)\sum_{k\neq 0}\frac{e(Buk)}{(k+vu)^{2J}}\,du, \tag{4.2}$$

where the interchange of integration and summation in (4.2) is justified by absolute convergence. Since $v < 1$, we have

$$\sum_{k\neq 0}\frac{e(Buk)}{(k+vu)^{2J}} = O((1-vu)^{-2J}) \tag{4.3}$$

uniformly for $u \in [0,1]$. Also,

$$\int_0^1 \frac{du}{(1-vu)^{2J}} = O(v^{-1}(1-v)^{1-2J}). \tag{4.4}$$

By (4.2) – (4.4) and the fact that $v = 2B/N$,

$$\sum_{k\neq 0} I_k = \frac{e(B^2/N)}{2\pi i}\sum_{j=0}^{J-1}\frac{(2j)!}{j!}\left(\frac{1}{2\pi iN}\right)^j \sum_{k\neq 0}\frac{1}{(k+v)^{2j+1}} + O(B^{1-J}v^{J-1}(1-v)^{1-2J}). \tag{4.5}$$

Since

$$\sum_{k\neq 0}\frac{1}{(k+v)^{2J-1}} = O((1-v)^{1-2J}), \tag{4.6}$$

the term for $j = J-1$ in (4.5) can be absorbed into the error term in (4.5). By (3.7) with $m = 2$,

$$I_0 = \frac{(1+i)}{4}\sqrt{N} + \frac{e(B^2/N)}{2\pi iv}\sum_{j=0}^{J-1}\frac{(2j)!}{j!}(4\pi iBv)^{-j} + O(N^J B^{1-2J}). \tag{4.7}$$

Since $4\pi iBv = 2\pi iNv^2$, (4.7) can be rewritten as

$$I_0 = \frac{(1+i)\sqrt{N}}{4} + \frac{e(B^2/N)}{2\pi i}\sum_{j=0}^{J-1}\frac{(2j)!}{j!}\frac{(2\pi iN)^{-j}}{v^{2j+1}} + O(N^J B^{1-2J}). \tag{4.8}$$

The term for $j = J - 1$ in (4.8) may be absorbed into the error term. Adding (4.5) and (4.8) we obtain

$$\sum_k I_k = \frac{(1+i)\sqrt{N}}{4} + \frac{e(B^2/N)}{2\pi i} \sum_{j=0}^{J-2} \frac{(2j)!}{j!} (2\pi i N)^{-j} \sum_k \frac{1}{(k+v)^{2j+1}} \tag{4.9}$$
$$+ O(B^{1-J}(1-v)^{1-2J} + B^{1-J} v^{-J}).$$

By (1.23) and (3.25),

$$\sum_k \frac{1}{(k+v)^{n+1}} = \frac{(-1)^n F^{(n)}(v)}{n!}, \quad n = 0, 1, 2, \ldots, \tag{4.10}$$

so Theorem 4 follows from (3.2), (4.9), and (4.10). ■

We next apply Theorem 4 to prove Theorem 5, which gives a precise estimate for the quarter Gauss sum $Q_N = S_2([N/4])$. Let N be large, and recall from (1.27) the definition $h := N - 4[N/4]$.

Proof of Theorem 5. Define

$$T_N := \left(Q_N - \frac{(1+i)\sqrt{N}}{4} - \frac{1}{2} \right) 2 e^{2\pi i (2h-N)/16}. \tag{4.11}$$

To prove Theorem 5, we must show that

$$T_N = \sum_{r=0}^{R-1} \frac{1}{r!} \left(\frac{\pi i}{8N} \right)^r S_{2r}(h) + O(N^{-R}) \tag{4.12}$$

for every fixed positive integer R. By Theorem 4 with $B = (N-h)/4$ and $J = R+1$,

$$T_N = e^{2\pi i h^2/(16N)} \left(1 + \frac{1}{\pi i} \sum_{j=0}^{R-1} \frac{1}{j!} \left(\frac{1}{2\pi i N} \right)^j F^{(2j)} \left(\frac{1}{2} - \frac{h}{2N} \right) \right) \tag{4.13}$$
$$+ O(N^{-R}).$$

Therefore, it suffices to prove the following equality of formal power series in $t = 1/N$:

$$e^{2\pi i h^2 t/16} \left(1 + \frac{1}{\pi i} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{t}{2\pi i} \right)^j F^{(2j)} \left(\frac{1}{2} - \frac{ht}{2} \right) \right) \tag{4.14}$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\pi i t}{8} \right)^r S_{2r}(h).$$

Since $F(x) = \pi \cot(\pi x)$, we can rewrite (4.14) as

$$e^{2\pi i h^2 t/16}\left(1 - i\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-i\pi t}{2}\right)^{j}\tan^{(2j)}\left(\frac{\pi h t}{2}\right)\right) \tag{4.15}$$
$$= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\pi i t}{8}\right)^{r} S_{2r}(h).$$

For brevity, write

$$D_k = \frac{d^k}{dt^k}. \tag{4.16}$$

To prove (4.15), we will show that the coefficients of $t^r/r!$ are the same on both sides, i.e., we will show that

$$S_{2r}(h)(\pi i/8)^r - (2\pi i h^2/16)^r \tag{4.17}$$
$$= -i\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-i\pi}{2}\right)^{j} D_r\left(t^j\, e^{2\pi i h^2 t/16}\tan^{(2j)}\left(\frac{\pi h t}{2}\right)\right)\Bigg|_{t=0}.$$

Since $D_k t^j\big|_{t=0}$ is nonzero only when $k = j$, (4.17) is equivalent to

$$S_{2r}(h) - h^{2r} = A_r, \tag{4.18}$$

where

$$A_r := -i\left(\frac{8}{\pi i}\right)^{r}\sum_{j=0}^{r}\left(\frac{-i\pi}{2}\right)^{j}\binom{r}{j} D_{r-j}\left(e^{2\pi i h^2 t/16}\tan^{(2j)}\left(\frac{\pi h t}{2}\right)\right)\Bigg|_{t=0}. \tag{4.19}$$

The rightmost factor in (4.19) equals

$$\sum_{a=0}^{r-j}\binom{r-j}{a}\left(\frac{2\pi i h^2}{16}\right)^{a}\tan^{(r+j-a)}(0)\left(\frac{\pi h}{2}\right)^{r-j-a}. \tag{4.20}$$

Making the change of variable $s = r + j - a$, we see that

$$A_r = \sum_{s=0}^{2r}\sum_{j=0}^{r} 2^{2s-2j} h^{2r-s}\binom{r}{j}\binom{r-j}{s-2j}(-1)^{(s+1)/2}\tan^{(s)}(0). \tag{4.21}$$

(Note that $\tan^{(s)}(0)$ vanishes when s is even). It is a well-known fact (proved by comparing the coefficients of x^s in $(1+x)^{2r}$ and in $(x^2+2x+1)^r$) that

$$\binom{2r}{s} = \sum_{j=0}^{r}\binom{r}{j}\binom{r-j}{s-2j}2^{s-2j}. \tag{4.22}$$

Thus, (4.21) simplifies to

$$
\begin{aligned}
A_r &= \sum_{s=0}^{2r} \binom{2r}{s} h^{2r-s} 2^s (-1)^{(s+1)/2} \tan^{(s)}(0) \qquad (4.23)\\
&= (-1)^r \sum_{s=0}^{2r} \binom{2r}{s} D_{2r-s}(\sin hx) D_s(\tan 2x)\Big|_{x=0}\\
&= (-1)^r D_{2r}(\sin hx \tan 2x)\Big|_{x=0}.
\end{aligned}
$$

Since also

$$h^{2r} = (-1)^r D_{2r}(\cos hx)\Big|_{x=0}, \qquad (4.24)$$

it follows from (4.23) and (4.24) that (4.18) is equivalent to

$$S_{2r}(h)(-1)^r = D_{2r}(\cos hx + \sin hx \tan 2x)\Big|_{x=0}. \qquad (4.25)$$

The desired result (4.25) now follows from (1.28) with $y = h$. ∎

Suppose now that N is any positive integer multiple of 4, and write $N = 4M$. We close this section by giving an elementary proof of Theorem 6, which provides an exact formula for the quarter Gauss sum $Q_N = Q_{4M} = S_2(M)$.

Proof of Theorem 6. We have

$$Q_N = \sum_{j=0}^{M} e(j^2/N) = (T+1)/2, \qquad (4.26)$$

where

$$T := \sum_{j=-M}^{M} e(j^2/N) = \sum_{j=0}^{2M} e((j-M)^2/N). \qquad (4.27)$$

Thus

$$Q_N = (1 + i^M U)/2, \qquad (4.28)$$

where

$$U := e(-M^2/N)T = \sum_{j=0}^{2M} e(j^2/N)(-1)^j. \qquad (4.29)$$

Gauss's evaluation of the classical quadratic Gauss sum is [1, p. 15]

$$G_n := \sum_{j=0}^{n-1} e(j^2/n) = \sqrt{n}(1+i)(1+i^{-n})/2. \qquad (4.30)$$

Separating even and odd j in (4.29), we have

$$
\begin{aligned}
U &= \sum_{j=0}^{M} e(j^2/M) \;-\; \frac{1}{2} \sum_{j=0}^{2M-1} e((2j+1)^2/N) \qquad (4.31)\\
&= (G_M + 1) - (G_N - 2G_M)/2 = 1 + 2G_M - G_N/2.
\end{aligned}
$$

Theorem 6 now follows from (4.28), (4.31), and (4.30). ∎

5 The satellite condensation points

Fix a positive integer k. We begin this section with a proof of Theorem 7, which estimates incomplete Gauss sums of order m in the vicinity of each of the two spirals which meet at the spiral endpoint $S_m(B_{2k})$. The condensation points of these two spirals are the satellites $S_m(B_{2k-1})$ and $S_m(B_{2k+1})$. Throughout this section, $m > 1$ is fixed and N is large.

Proof of Theorem 7. We will prove only that

$$S_m(B_{2k}, C) = L(N) + O(1), \tag{5.1}$$

since the proof that $S_m(A, B_{2k}) = L(N) + O(1)$ is completely analogous. Define

$$x := (kN/m)^{1/(m-1)}, \qquad y := ((k+1-\varepsilon/2)N/m)^{1/(m-1)}, \tag{5.2}$$

so that

$$[x] = B_{2k}, \qquad [y] = B_{2k+2-\varepsilon}. \tag{5.3}$$

By (1.39),

$$B_{2k} \le x < B_{2k+\varepsilon} < C < B_{2k+2-\varepsilon} \le y. \tag{5.4}$$

Define the function

$$f(w) := w^m/N. \tag{5.5}$$

Then $f'(w)$ is increasing on the interval $[B_{2k}, y]$, and for a sufficiently small constant η, we see that k is the only integer in the interval $[f'(B_{2k}) - \eta, f'(C) + \eta]$. It therefore follows from a theorem of van der Corput [16, section 4.7] that

$$S_m(B_{2k}, C) = \sum_{j=B_{2k}+1}^{C} e^{2\pi i j^m/N} = \int_{B_{2k}}^{C} e^{2\pi i (f(w)-kw)}\, dw \;+\; O(1). \tag{5.6}$$

(Note that the van der Corput estimate in [16, section 4.7] does not require the limits of integration to be constant, in contrast with the stationary phase estimate [17, (3.22)] which we will use below.)

By (5.4), $f'(C)$ and $f'(y)$ are bounded away from k. Therefore, the integration by parts formula

$$\int e^{2\pi i(f(w)-kw)}\, dw = \frac{e^{2\pi i(f(w)-kw)}}{2\pi i(f'(w)-k)} + \int \frac{e^{2\pi i(f(w)-kw)} f''(w)}{2\pi i(f'(w)-k)^2}\, dw \tag{5.7}$$

shows that

$$\int_C^y e^{2\pi i(f(w)-kw)}\,dw \;=\; O(1). \tag{5.8}$$

Combining (5.6) and (5.8), we see that

$$S_m(B_{2k},C) = \int_x^y e^{2\pi i(f(w)-kw)}\,dw \;+\; O(1). \tag{5.9}$$

With the change of variables $w = xt$, (5.9) becomes

$$S_m(B_{2k},C) = x\int_1^\beta e^{ixH(t)}\,dt \;+\; O(1), \tag{5.10}$$

where β is the constant $\beta := y/x > 1$ and

$$H(t) := 2\pi k(t^m/m - t). \tag{5.11}$$

Since $H'(1) = 0$ and $H'(t) > 0$ in the interval $(1,\beta)$, we can apply the method of stationary phase [17, (3.22)] to approximate the integral in (5.10). The result is

$$x\int_1^\beta e^{ixH(t)}\,dt \;=\; L(N) \;+\; O(1), \tag{5.12}$$

as $N \to \infty$. Combining (5.10) and (5.12), we obtain the desired result (5.1). ∎

We conclude this section by proving Theorem 9, which shows that the main contribution to the sum $S_m(B_{2k}, B_{2k+1})$ already comes from just the first $DN^{1/(2m-2)}$ terms, where $D = D(N)$ satisfies (1.46).

Proof of Theorem 9. We will prove only (1.47), since the proof of (1.48) is completely analogous. Define

$$f(w) := w^m/N, \qquad x := (kN/m)^{1/(m-1)}, \qquad z := B_{2k} + DN^{1/(2m-2)}. \tag{5.13}$$

Comparing (1.42) and (1.47), we see that it suffices to prove that

$$S_m(z, B_{2k+1}) = o(\sqrt{x}). \tag{5.14}$$

Now, $f'(w)$ is increasing on the interval $[z, B_{2k+1}]$, and for a sufficiently small constant η, we see that k is the only integer in the interval $[f'(z) - \eta, f'(B_{2k+1}) + \eta]$. It therefore follows from the aforementioned theorem of van der Corput [16, section 4.7] that

$$S_m(z, B_{2k+1}) = \sum_{j=z+1}^{B_{2k+1}} e^{2\pi i j^m/N} = \int_z^{B_{2k+1}} e^{2\pi i(f(w)-kw)}\,dw \;+\; O(1). \tag{5.15}$$

It remains to prove that

$$\int_{z}^{B_{2k+1}} e^{2\pi i(f(w)-kw)}\,dw \;=\; o(\sqrt{x}). \tag{5.16}$$

Clearly

$$1/(f'(B_{2k+1})-k) < 3, \tag{5.17}$$

and it is easily seen from the mean value theorem that

$$1/(f'(z)-k) = 1/(f'(z)-f'(x)) = O(\sqrt{x}/D). \tag{5.18}$$

Thus, the integration by parts formula (5.7) shows that

$$\int_{z}^{B_{2k+1}} e^{2\pi i(f(w)-kw)}\,dw \;=\; O(\sqrt{x}/D). \tag{5.19}$$

This proves the desired estimate (5.16). ■

6 Appendix

Lehmer [10, Theorem 3] employed the principle that as the graph starts spiraling inwards, it remains confined to a disk. This highly intuitive principle is not trivial to justify. Indeed, Lehmer himself offered no proof. It was not until 7 years later that a proof was published, by Loxton [11, p. 159]. Although we have not directly appealed to this geometric argument, it has provided motivation for this paper and others on exponential sums. Loxton's proof is so short and ingenious that we take the liberty of reproducing it here.

Theorem 10 *Let $a(0), a(1), a(2), \ldots$ be a sequence of real numbers, and define $\delta(n) = a(n+1) - a(n)$ for $n \geq 1$. Let t, u be integers with $1 \leq t < u$, and assume that for $t \leq n < u$,*

$$0 < a(n+1) - a(n) = \delta(n) \leq \pi \text{ and } \delta(n) \text{ is increasing}. \tag{6.1}$$

Define

$$I(n) := \sum_{j=0}^{n} e^{ia(j)}, \qquad n \geq 0. \tag{6.2}$$

Let Γ_t be the circle in the complex plane passing through the three points $I(t-1)$, $I(t)$, and $I(t+1)$. Let Γ be the closed disk whose center is the same as that of Γ_t and whose radius is $1/2$ more than that of Γ_t. Then Γ is a disk of diameter $1 + \operatorname{cosec}(\delta(t)/2)$ which contains all of the points $I(n)$, $t-1 \leq n \leq u$.

PROOF. By (6.1), Γ_{t+1} lies inside Γ_t, except for the arc between $I(t)$ and $I(t+1)$, which, at its center, is a distance $h(t)$ outside Γ_t, where

$$h(t) := (\tan(\delta(t+1)/4) - \tan(\delta(t)/4))/2. \tag{6.3}$$

The circle Γ_{t+2} lies inside Γ_{t+1}, except for the arc between $I(t+1)$ and $I(t+2)$, which, at its center, is a distance $h(t+1)$ outside Γ_{t+1}. Continuing in this way, we see that the points $I(n)$, $t-1 \le n \le u$, all lie in the closed disk whose boundary is that circle concentric to Γ_t with a radius exceeding that of Γ_t by the amount

$$\sum_{n=t}^{u-2} h(n) = (\tan(\delta(u-1)/4) - \tan(\delta(t)/4))/2. \tag{6.4}$$

Finally, by (6.1), the right side of (6.4) is less than $1/2$, and so the points $I(n)$, $t-1 \le n \le u$, all lie in the disk Γ. ∎

In the graph

$$I(t-1) \to I(t) \to I(t+1) \to \cdots \to I(u), \tag{6.5}$$

the second edge makes an angle $\delta(t) = \triangle(I(t))$ with respect to the first edge. If $\delta(n)$ were to equal $\delta(t)$ for all $n \ge t$, then every point $I(n)$ with $n \ge t$ would lie on the circle Γ_t. As it is, the graph (6.5) spirals inward, approaching a condensation point in Γ. Theorem 10 shows for example that once $\delta(n)$ exceeds $\pi/2$, the graph remains confined to a disk of diameter $\le 1+\sqrt{2}$.

References

[1] B.C. Berndt, R. J. Evans, and K. S. Williams, Gauss and Jacobi Sums, Wiley-Interscience, N.Y., 1998.

[2] M. V. Berry and J. Goldberg, Renormalisation of curlicues, *Nonlinearity* **1**(1988), 1–26.

[3] E. Coutsias and N. Kazarinoff, The approximate functional formula for the theta function and Diophantine Gauss sums, *Trans. Amer. Math. Soc.*, **350**(1998), no. 2, 615–641.

[4] H. Davenport, Multiplicative Number Theory, third ed., Springer-Verlag, N.Y., 2000.

[5] J.-M. Deshouillers, Geometric aspect of Weyl sums, *Elementary and analytic theory of numbers* (Warsaw, 1982), 75–82, Banach Center Publ., **17**, PWN, Warsaw, 1985.

[6] H. Fiedler, W. Jurkat, and O. Körner, Asymptotic expansions of finite theta series, *Acta Arith.*, **32**(1977), no. 2, 129–146.

[7] J. W. L. Glaisher, On the coefficients in the expansions of $\cos x/\cos 2x$ and $\sin x/\sin 2x$, *Quarterly J. of Pure and Appl. Math.*, **45**(1914), 187–222.

[8] M. E. Hoffman, Derivative polynomials for tangent and secant, *Amer. Math. Monthly*, **102**(1995), 23–30.

[9] M. E. Hoffman, Derivative polynomials, Euler polynomials, and associated integer sequences, *Electron. J. Combin.*, **6**(1999), Research Paper 21, 13pp.

[10] D. H. Lehmer, Incomplete Gauss sums, *Mathematika*, **23**(1976), 125–135.

[11] J. H. Loxton, The graphs of exponential sums, *Mathematika*, **30**(1983), 153–163.

[12] R. Moore and A. van der Poorten, On the thermodynamics of curves and other curlicues, Miniconference on Geometry and Physics (Canberra, 1989), 82–109, *Proc. Centre Math. Anal. Austral. Nat. Univ.*, **22**, Canberra, 1989.

[13] K. Oskolkov, Schrödinger equation and oscillatory Hilbert transforms of second degree, *J. Fourier Anal. Appl.*, **4**(1998), no. 3, 341–356.

[14] F. Sullivan and U. Zannier, A note on incomplete quadratic Gauss sums, *Boll. Un. Mat. Ital. A*, **6**(1992), 49–57.

[15] E.C. Titchmarsh, The Theory of Functions, second ed., The Clarendon Press, Oxford University Press, N.Y., 1939.

[16] E.C. Titchmarsh, The Theory of the Riemann Zeta-function, second ed., The Clarendon Press, Oxford University Press, N.Y., 1986.

[17] R. Wong, Asymptotic Approximations of Integrals, *Classics in Applied Mathematics*, **34**, SIAM, Philadelphia, PA, 2001.