PLANAR GRAPHS AND WAGNER'S AND KURATOWSKI'S THEOREMS

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Abstract. This is an expository paper in which we rigorously prove Wagner's Theorem and Kuratowski's Theorem, both of which establish necessary and sufficient conditions for a graph to be planar.

Contents

1. Introduction and Basic Definitions

The planarity of a graph—its ability to be embedded in a plane—is a deceptively meaningful property which, through various theorems, can tell us many other things about a graph. Among other properties, planar graphs were famously found to be 4-colorable. Of course, to use such theorems to determine whether a graph has these properties, we must first determine whether that graph is planar. Wagner's and Kuratowki's theorems show that there are simple and easily testable characterizations of planarity, but proving that they work is much less simple. Before we can do that, we must establish some definitions.

Definition 1.1. An **undirected graph** is an ordered pair $G = (V, E)$, where V is a set of vertices, and E is a set of edges, where each edge is a set of two vertices. We will mostly refer to these simply as **graphs**, as this paper does not deal with directed graphs.

Definition 1.2. A graph H is a **subgraph** of G if $H = (W, F)$ for some $W \subseteq V$ and $F \subseteq E$.

Definition 1.3. Two vertices $u, v \in V$ are **adjacent** in G if $\{u, v\} \in E$. This is often written as $u \sim v$.

Definition 1.4. A vertex v is **incident** to an edge e if $v \in e$.

Definition 1.5. The **degree** of v, $\deg(v)$, is the number of edges that v is incident to.

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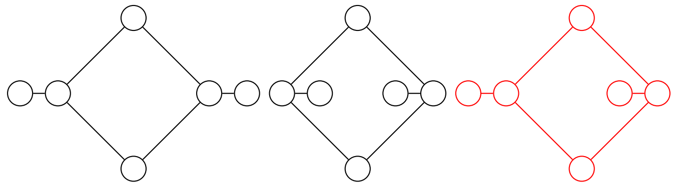

FIGURE 1. Three planar embeddings of the same graph. The first two embeddings are isomorphic, with the outer face of the first graph corresponding to the inner face of the second and vice versa. The third embedding is not isomorphic to either of the first two, even though the graph is the same.

Definition 1.6. A graph G is a **subdivision** of H if we can obtain H by taking G, removing vertices with degree 2, and drawing new edges between the two vertices that were adjacent to each vertex removed this way. The vertices that remain are called the **principal vertices** of G, and the ones that were removed are the **subdivision vertices**.

Definition 1.7. Let $G = (V, E)$ be a graph, let $u, v \in V$, and let $e = \{u, v\} \in E$. The **edge contraction** of e is the graph G/e, in which u and v are combined into a single vertex, which is adjacent to every vertex that was adjacent to u or v in G.

Definition 1.8. A **graph minor** of G is a graph that is formed by deleting vertices, deleting edges, and/or contracting edges of G.

Definition 1.9. A graph is **planar** if it can be drawn on a plane, with each vertex represented as a distinct point and each edge represented as a simple curve of finite length with its endpoints at its two incident vertices, in such a way that no two edges intersect, except where their endpoints touch at a shared incident vertex.

Definition 1.10. A **face** of a planar embedding is a connected component of the complement of the graph. An **outer face** is a face with infinite area. Any other face is an **inner face**. A planar embedding of a finite planar graph can only have one outer face. Having more would require either an infinite edge or an infinite number of edges to form the boundary between them.

Definition 1.11. The **degree** of a *face* is the number of vertices incident to edges that bound that face.

Definition 1.12. A subgraph H completely bounds a face if H contains exactly the edges that are adjacent to the face and their incident vertices.

Definition 1.13. Two planar embeddings of a graph are **isomorphic** if there exists a bijection between them such that each face of one embedding is incident to the same vertices as the corresponding face of the other embedding.

Definition 1.14. A **complete graph** on n vertices, written as K_n, is a graph containing n vertices, and an edge that connects every pair of vertices.

Definition 1.15. A **complete bipartite graph**, written as $K_{m,n}$, is graph of the form $G = (V_1 \cup V_2, E)$, where V_1 and V_2 are disjoint sets of m and n vertices respectively, and $E = \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\}$.

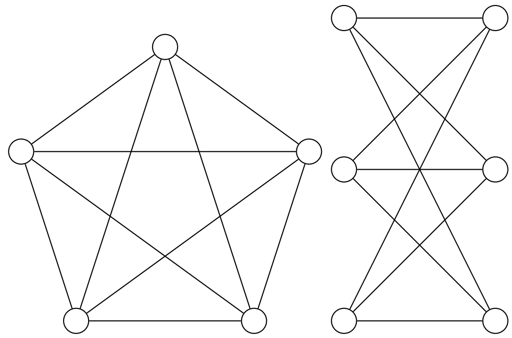

FIGURE 2. K_5 (left) and $K_{3,3}$ (right).

Definition 1.16. A **path of length** n is a graph P_n on $n+1$ vertices such that $P_n = (V, E)$ where $V = \{v_1, \ldots, v_{n+1}\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n]\}$. We call this a path from v_1 to v_{n+1}.

Definition 1.17. A graph G is **connected** if for any two vertices u, v in G, there exists a path from u to v that is a subgraph of G.

Definition 1.18. G is k**-connected** if either $G = K_{k+1}$ or G is connected and the smallest set of vertices that can be removed from G (removing associated edges as well) to leave a disconnected graph has cardinality k.

2. Kuratowski's Theorem

In 1930, Kazimierz Kuratowski proved a theorem that provides a way to tell whether a graph is planar simply by checking whether it contains a particular type of subgraph.

Definition 2.1. A **Kuratowski subgraph** is a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Lemma 2.2. *If G is planar, every subgraph of G is planar.*

Proof. Let H be a subgraph of G. Take a plane drawing of G and remove all edges and vertices not in H. This process can't create edge crossovers, so we now have a plane drawing of H. □

Lemma 2.3. *If H is a subdivision of G and H is planar, then G is planar.*

Proof. Let H be a subdivision of G. Take a plane drawing of H and remove all vertices that are not in G. These vertices must have degree 2, so we attach the two edges that were adjacent to each one at the site of the vertex removal, where they have a shared endpoint, to form a plane drawing of G. □

Note that we can draw an edge between two vertices if and only if they share an incident face. If we draw the edge between u and v through a face that is bounded by a cycle with vertices, ordered clockwise, $u \ldots v \ldots u$ (not all faces are bounded by cycles, but the ones we deal with now will be), the face is divided into two faces, bounded by cycles $u \ldots vu$ and $v \ldots uv$.

Lemma 2.4. *In a planar embedding of a graph that is at least 2-connected, every face is completely bounded by a cycle.*

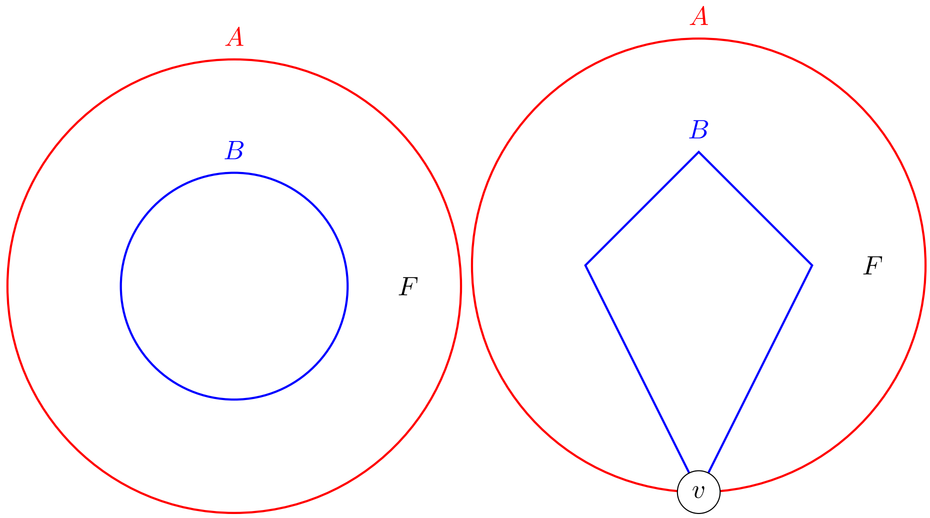

FIGURE 3. Left: Case 1, Right: Case 2

Proof. Let G be a graph that is at least 2-connected, and let F be a face in a planar embedding of G. Suppose for contradiction that F is not completely bounded by a cycle. There are 2 ways this can happen:

(1) F has multiple boundaries not connected by edges along F. Let A be the set of vertices in one boundary of F, and let B be the set of vertices in the other boundary. (See Figure 3. The individual vertices are not shown because only the general structure is important). Then there is no path between A and B, as such a path would have to go through F. Thus A and B are part of separate components, so G is disconnected. Contradiction.

(2) F is completely bounded by a closed walk that passes through some vertex v more than once. Then the set of vertices that the walk passes through between the first and second times it passes through v form a "bubble," which cannot connect to the rest of the graph without dividing F. Thus v is a cut vertex, which separates G into components A and B (see Figure 3), so G is at most 1-connected. Contradiction.

Thus F must be completely bounded by a cycle. □

Lemma 2.5. K_5 *is not planar.*

Proof. Let a, b, c, d, and e be the vertices of K_5. K_5 contains the cycle $abcdea$. Any embedding of this cycle must be isomorphic to a pentagon. We can then add 2 edges through the inner face, which splits it into three faces bounded by 3-cycles, to which we can't add any more edges because all their incident vertices are already adjacent. We can similarly add at most 2 edges through the outer face. This gives us a total of 9 edges, but we need 10 for K_5. Thus we cannot embed K_5 in a plane. □

Lemma 2.6. $K_{3,3}$ *is not planar.*

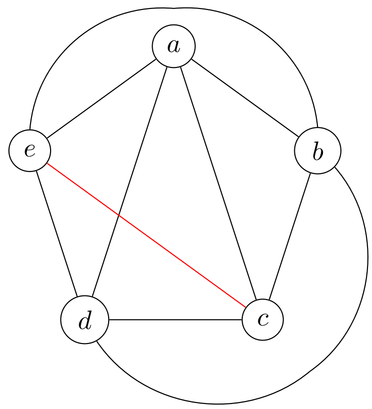

FIGURE 4. An attempt at a planar embedding of K_5. Note that there is no way to draw an edge between c and e without intersecting another edge.

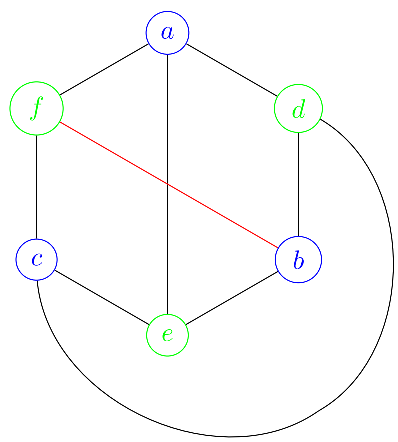

FIGURE 5. An attempt at a planar embedding of $K_{3,3}$.

Proof. Let a-f be the vertices of $K_{3,3}$ such that each vertex in $\{a, b, c\}$ is adjacent to each vertex in $\{d, e, f\}$. The graph contains the cycle $adbecfa$, which must have an embedding isomorphic to a hexagon. This leaves us with three more edges to draw, ae, bf, and cd. We can draw one of these through the center of the hexagon, but that divides the face so that neither of the other edges can be drawn there (see Figure 5). We can draw another through the outer face, but that face becomes divided in the same way. Thus we cannot draw all 9 edges, and we therefore cannot embed $K_{3,3}$ in a plane. □

Thus containing a Kuratowski subgraph is a sufficient condition for a graph to be nonplanar, but we still need to prove it is a necessary one.

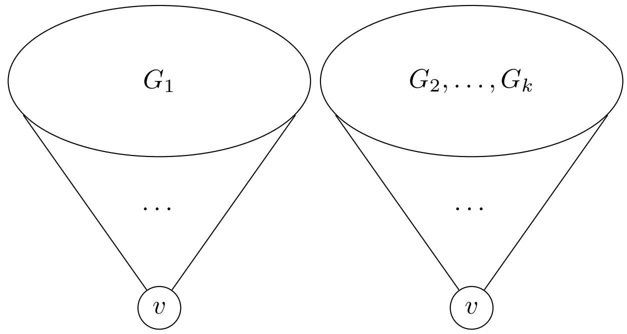

FIGURE 6. Plane drawings of H and H', under the assumption that G is only 1-connected. The number of edges connecting the components to v is not specified or important.

Lemma 2.7. *Let e be an edge of a planar graph G. There exists a plane drawing of G such that e bounds the outer face.*

Proof. Take any plane drawing of G and project it onto a sphere. Then rotate the sphere and project back onto the plane, starting from a point directly opposite a point in a face bounded by e. That face then becomes the outer face.

Note that for any vertex v that is incident to e, in every plane drawing where e bounds the outer face, v will be incident to the outer face. □

Now we will assume for contradiction that there exists at least one nonplanar graph that doesn't contain a Kuratowski subgraph. Let G be such a graph on the smallest possible number of vertices. Then removing a vertex from G leaves a planar graph.

Lemma 2.8. *G is at least 3-connected and has at least 5 vertices.*

Proof. We don't need to worry about K_1, K_2, K_3, and K_4 for this proof because they're all planar, so we don't have to deal with those special cases for k-connected graphs.

Suppose G is disconnected. Then G is made up of at least 2 separate components, which are all smaller than G, and therefore planar. Then we can make plane drawings of the components next to each other and we have a plane drawing of G, but that means G is planar, so this is a contradiction.

Suppose G is only 1-connected. Then there exists a cut-vertex v that, when removed, separates G into at least 2 components, labeled $G_1, \ldots, G_k$. Let H be the subgraph of G consisting of G_1 and v, and let H' be the subgraph containing $G_2, \ldots, G_k$ and v. Both H and H' are smaller than G, and therefore planar, so we can draw planar embeddings of both with v incident to the outer face. Then we can arrange them so they share a single v, and thus we have a plane drawing of G (this may require squeezing the edges somewhat so the components don't overlap, but this is always possible). Contradiction.

Suppose G is only 2-connected. Then there there exist subgraphs $G_1, \ldots, G_k$ that are separated by the removal of two vertices, v and u. Every such subgraph must contain at least one point adjacent to u and one point adjacent to v, or else removing the vertex not adjacent to one of the subgraphs would disconnect G.

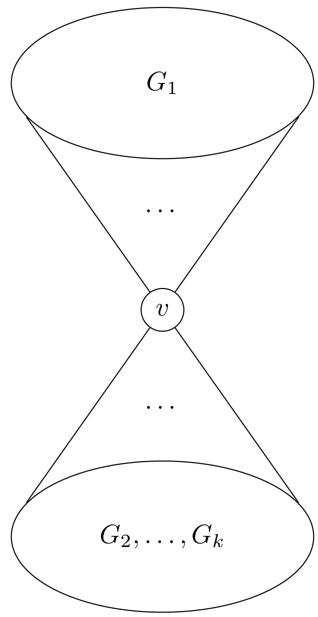

FIGURE 7. The resulting planar embedding of a 1-connected G.

Let H be the subgraph consisting of u, v, and G_1 and let H' be the subgraph consisting of u, v, and all the separable components except G_1. Since H and H' are both smaller than G, they are planar. Now we need to consider two cases:

(1) $u \sim v$.

Then we can make planar embeddings of H and H' with the edge $\{u, v\}$ bounding the outer face. These can be arranged together so they share the edge without other parts of the graphs overlapping, which creates a planar embedding of G.

(2) $u \not\sim v$.

Suppose for contradiction that adding the edge $\{u, v\}$ creates a Kuratowski subgraph in H. Then we can consider a subgraph of G that consists of the rest of this Kuratowski subgraph and a path from u to v through some G_i for $i \neq 1$. By removing the vertices between u and v on this path and attaching their incident edges, it becomes apparent that this graph is just a subdivision of the Kuratowski subgraph we created, and so it is itself a Kuratowski subgraph. But G doesn't contain a Kuratowski subgraph, so this is a contradiction. We can similarly show that adding this edge does not create a Kuratowski subgraph in H'.

Thus H and H' with the edge $\{u, v\}$ added are smaller than G and do not contain Kuratowski subgraphs, which means they are planar. We can attach them as in case 1, and then remove the edge $\{u, v\}$ to create a planar embedding of G.

Therefore G is planar. Contradiction.

Thus G must be at least 3-connected. The only 3-connected graph with less than 5 vertices is K_4, which is planar, so G must have at least 5 vertices. $\square$

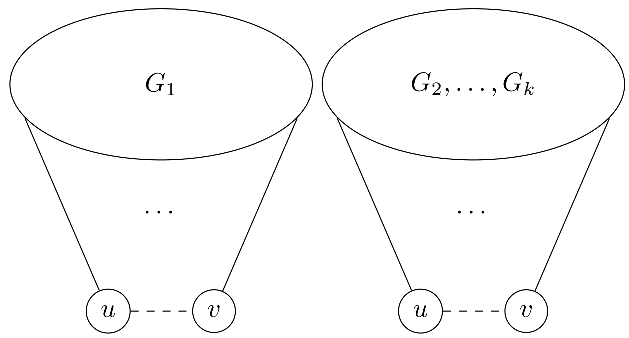

FIGURE 8. Plane drawings of H and H', for a 2-connected G. The edge $\{u, v\}$ may or may not exist.

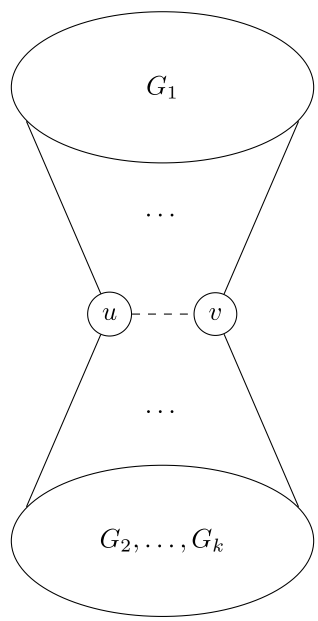

FIGURE 9. The resulting plane drawing of a 2-connected G.

Lemma 2.9. *G contains an edge e such that G/e is at least 3-connected.*

Proof. Suppose not. Then for all $e \in E$, G/e contains a pair of vertices that can be removed to disconnect the graph. One of these must be the one created by fusing the vertices incident to e, or else removing the two would disconnect G, which is not allowed because G is 3-connected. Let $e = \{a, b\}$ be an edge in G, and let ab be the combination of a and b in G/e. Then there exists some vertex c such that ab and c constitute a separating set of G/e. Let a, b, and c be such that removing all three from G (or equivalently, removing ab and c from G/e) creates the largest possible component, which we will call A. We will call the rest of the separated graph B.

Clearly a, b, and c must each have at least one adjacent vertex in A and one in B, else we would be able to find a smaller separating set, contradicting the 3-connectedness of G. Therefore there exists some $d \in B$ that is adjacent to c. Then we can find some x such that $\{c, d, x\}$ separates G.

Suppose $x \in A \cup \{a, b\}$. Then $A \setminus x$ must be connected. Otherwise, the components of $A \setminus x$ would need to be connected by a path through the rest of the graph (since G is 3-connected). Since $a \sim b$, a and b are in the same component, so the path between the components must go through a or b, and through c, since that is the only other connection between A and B. But removing $\{x, c\}$ disconnects G, which is not allowed because G is 3-connected.

Suppose $x \in B$. Then, since none of the cut vertices are in $A \cup \{a, b\}$, that set is still connected, and is therefore a component. But $A \cup \{a, b\}$ is larger than A, which contradicts the part where we said $\{a, b, c\}$ is the cut that produces the largest component.

Thus G must contain an edge that can be contracted without reducing the graph's connectedness. □

Lemma 2.10. *For any edge e in G, G/e does not contain a Kuratowski subgraph.*

Proof. Let $e = \{x, y\}$ and suppose G/e does contain a Kuratowski subgraph, which we will call K. Clearly the vertex xy formed by the edge contraction must be part of K, else K would be the same in G. For any $v \in G/e$, we define $\deg_K(v)$ to be the number of vertices in K adjacent to v. For any $v \in K$, $\deg_K(v)$ can only be 4 (principal vertices of K_5), 3 (principal vertices of $K_{3,3}$), or 2 (subdivision vertices). Now we need to consider some cases:

(1) All vertices in K that are adjacent to xy in G/e are adjacent to x in G (an analogous argument applies if they are all adjacent to y). Then we can find an isomorphic Kuratowski subgraph in G using all the same vertices, except substituting x for xy.

(2) All vertices in K that are adjacent to xy in G/e are adjacent to x (this can also be applied analogously with x and y switched) in G except one, which we will call a. Then a must be adjacent to y in G, so we can draw our Kuratowski subgraph on G as we did in the previous case, except since x and a are not adjacent, we instead draw a path from x through y to a. This creates a graph that is like K, but with y as an extra subdivision vertex. This is still a Kuratowski subgraph.

Since every vertex adjacent to xy in G/e must be adjacent to either x or y in G, these cases cover every possibility except one very specific but interesting one:

(3) $\deg_K(xy) = 4$, and the edges are such that exactly two of the vertices in K that are adjacent to xy in G/e are adjacent to x in G, and only the other two are adjacent to y. This can only happen if K is a subdivision of K_5, and if we try to reconstruct our Kuratowski subgraph on G, in a manner similar to what we did in the last case, we get something that looks like the following: There may be subdivision vertices not shown in the diagram, but regardless, we can observe that this graph contain $K_{3,3}$ (or a subdivision thereof) (see Figure 10). Thus G contains a Kuratowski subgraph.

Thus G/e cannot contain a Kuratowski subgraph. In fact, since the only property of G used in this proof is that G contains no Kuratowski subgraph, this result shows

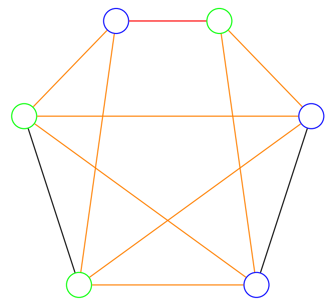

FIGURE 10. A graph that can be contracted (along the red edge) to K_5, as in case 3. The orange edges, along with the red edge, form a subgraph isomorphic to $K_{3,3}$.

that applying an edge contraction to *any* graph that doesn't contain a Kuratowski subgraph leaves a graph that also has no Kuratowski subgraph. □

Now we can finally prove Kuratowski's Theorem.

Theorem 2.11. *A graph G is planar if and only if G does not contain a Kuratowski subgraph.*

Proof. The forward direction is simple. If G is planar, then G clearly cannot contain a Kuratowski subgraph, as Kuratowski graphs are not planar.

For the converse, suppose for contradiction that G, as we have defined it, exists. Let $e = \{x, y\}$ be an edge that can be contracted while leaving the graph 3-connected. By Lemma 2.10, since G does not contain a Kuratowski subgraph, G/e doesn't, either. Since G is defined to be the smallest nonplanar graph without a Kuratowski subgraph, and G/e is smaller than G, this means G/e is planar.

Consider a plane drawing of G/e, and then remove the vertex xy and its associated edges. This combines all faces incident to xy into one face, which, because $(G/e) \setminus xy$ is 2-connected, must be bounded by a cycle, which we will call C. If we put xy back, all vertices adjacent to xy are in C, which means in G, all vertices adjacent to x or y (besides x and y themselves) are in C. Let X and Y be the sets of vertices in C adjacent to x and y respectively. Then we need to look at a few cases of those adjacent vertices.

(1) $X \cap Y = \emptyset$ and the sets do not interlace (i.e. there do not exist vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that their order, counting clockwise on C from x_1, is $x_1 y_1 x_2 y_2$). Then we can place x on the side of the face with all the x_i, and y on the side with all the y_i, then draw lines from x to every $x_i \in X$, and from y to every $y_i \in Y$. These lines don't need to cross, so G is planar.

(2) $|X \cap Y| = 1, 2$ and X and Y do not interlace. Then we can arrange x and y as in case 1, but with one or two vertices in C adjacent to x and y. This still does not force edges to overlap, so G is planar.

(3) $|X \cap Y| \geq 3$. Let $a, b, c \in X \cap Y$. Then x and y are adjacent to each other and to a, b, and c. We can also divide C into three disjoint paths, one between a and b, one between b and c, and one between c and a. We then observe that this is a subdivision of K_5. Thus G contains a Kuratowski subgraph.

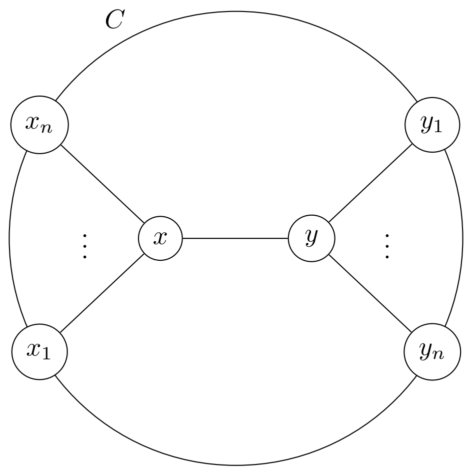

FIGURE 11. A plane drawing of the relevant part of G in case 1.

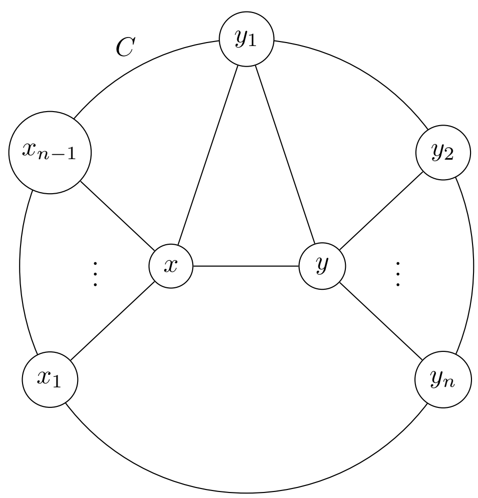

FIGURE 12. A plane drawing of the relevant part of G in the case where $|X \cap Y| = 1$. Note that y_1 is also x_n. For $|X \cap Y| = 2$, the drawing is similar, but with $x_1 = y_n$ at the bottom like y_1 is at the top.

(4) X and Y interlace. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ in the clockwise order $x_1y_1x_2y_2$. Then we can divide C into 4 internally disjoint paths connecting x_1 and x_2 to y_1 and y_2. Then x, y, x_1, x_2, y_1, and y_2 are the principal vertices of a subdivision of $K_{3,3}$. Thus G contains a Kuratowski subgraph.

Thus any case of G leads to a contradiction, so every nonplanar graph contains a Kuratowski subgraph. □

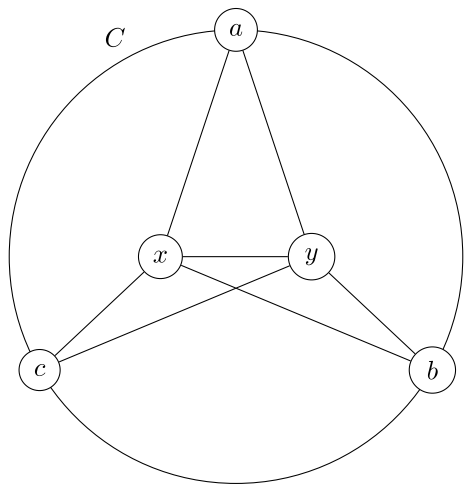

FIGURE 13. The relevant part of G in case 3. A subdivision of K_5.

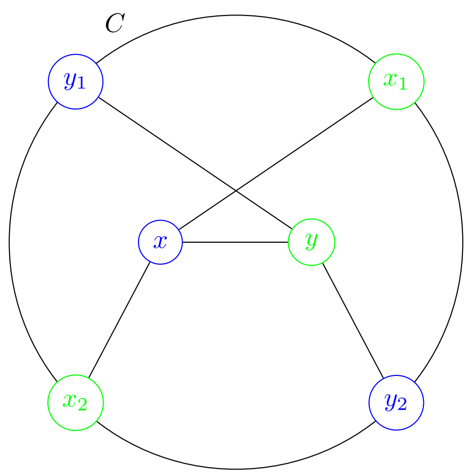

FIGURE 14. The relevant part of G in case 4. A subdivision of $K_{3,3}$.

3. Wagner's Theorem

In 1937, Klaus Wagner came up with another characterization of planar graphs that is, in fact, equivalent to Kuratowski's Theorem.

Theorem 3.1. *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a graph minor.*

Proof. Suppose G contains K_5 or $K_{3,3}$ as a graph minor. Then, as shown in Lemma 2.10, G contains a Kuratowski subgraph, and is therefore nonplanar.

Suppose G is nonplanar. Then G contains a Kuratowski subgraph, K. If we take an edge in K incident to a subdivision vertex and contract it, we are left with a Kuratowski graph with one less subdivision vertex. We can repeat this process until all subdivision vertices are annihilated, leaving us with either K_5 or $K_{3,3}$, and whichever of those remains is a graph minor of G. □

This ends our proofs of necessary and sufficient conditions for graph planarity. This problem, however, is just a specific case of the larger problem concerning graph embedding in arbitrary topological spaces, where the embeddable graphs may be different. We can embed K_5 and $K_{3,3}$ on a torus, for example. Graphs themselves can also be thought of as a specific case of ordered pairs of the form (V, E), where V is a set and E is a set of subsets of V, where in a graph we have the restriction that for each $e \in E$, e contains exactly two elements.

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