A De Bruijn-Erdős Theorem for Asteroidal-Triple Free Graphs*

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Abstract

A classical result of De Bruijn and Erdős shows that any given n points are either collinear or determine at least n distinct lines in the plane. Chen and Chvátal conjectured that a generalization of this result in all metric spaces, with appropriate definitions of lines, holds as well. In this note, we show that the Chen-Chvátal generalization holds for Asteroidal-Triple free graphs, where lines are defined using the notion of betweenness that rises naturally from the underlying convex geometry defined by this graph class.

The Sylverster-Gallai theorem asserts that any set of n points is either collinear or there is a line that goes through exactly two distinct points. One of the corollaries of this result is the De Bruijn-Erdős theorem; which asserts that any n collinear points in the plane determine at least n distinct lines [10]. This in fact holds in settings where distances and angles are not needed; just the notions of lines and points on lines in ordered geometries. In particular, there is a close relationship between ordered geometries and the notion of betweenness defined by Menger [14]. Betweenness is a ternary relation, that relates the "placement" of a point x between two other points p and q. We write $[pxq]$ to say that x lies between p and q, and write $(E, \mathcal{B})$ to denote the set system defined on the ground set E of points, and a betweenness relation $\mathcal{B}$ defined on the elements of E. In this setting, one can define a *line* $\overline{pq}$ between any two distinct points p and q as follows:

$$\overline{pq} = \{p, q\} \cup \{x : [xpq] \in \mathcal{B} \vee [pxq] \in \mathcal{B} \vee [pqx] \in \mathcal{B}\} \tag{1}$$

Thus the definition of a line varies with every notion of betweenness. For instance, in Euclidean space, betweenness translates naturally to the Euclidean metric:

$$[pxq] \iff p, q, x \text{ are distinct points and } dist(p, x) + dist(x, q) = dist(p, q),$$

However in arbitrary spaces, metric betweenness leads to other types of lines. In fact, it can lead to families of lines with different behaviour. For instance, a line can be properly contained in another line, as shown in the example below [6]. Given

$$dist(u, v) = dist(v, x) = dist(x, y) = dist(y, z) = dist(z, u) = 1$$
$$dist(u, x) = dist(v, y) = dist(x, z) = dist(y, u) = dist(z, v) = 2,$$

the line $\overline{vy} = \{v, x, y\}$ is properly contained in the line $\overline{xy} = \{v, x, y, z\}$.

Chen and Chvátal conjecture in [6] the following

Conjecture 0.1. *Every metric space on $n \geq 2$ points either has at least n distinct lines or has a line that contains all the points.*

Clearly, the definition of a line varies with that of betweenness. Let's call a line *universal* if it contains all the points. And let's say that a metric space satisfies the De Bruijn-Erdős property if it either has a universal line or at least n distinct lines. Conjecture 0.1 has been proven for special cases where the notion of betweenness is well understood. In particular, it was shown that the conjecture holds for posets and chordal graphs, and other graph classes, see for instance [1, 2, 8]. For many of these cases, the notion of betweenness rises naturally from

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an underlying structure of the graph class or the combinatorial objects. In particular, chordal graphs admit a monophonic convexity, and it is indeed what is used to define the betweenness and the lines used to prove the conjecture in [2]. Posets admit a double shelling antimatroid, whose corresponding convex geometry is precisely captured by the betweenness used to prove the Chen-Chvátal conjecture in [1]. We discuss these two examples in more detail below.

In [1], a notion of triangle betweenness is used to show that graphs satisfy the Chen-Chvátal conjecture. However the definition of a line that rises from this betweenness does not capture any underlying structure of the graph class. We hope to illustrate to the reader that for a given graph class, an underlying convex geometry and antimatroid structure are precisely what should be used to prove that a graph class has the De Bruijn-Erdős property. Based on this connection, we show that asteroidal triple free graphs also have the De Bruijn-Erdős property, using a recently discovered underlying convex geometry for this graph class [5].

Let $(V, \mathcal{B})$ be a tuple with a betweenness relation defined on V. For a pair $a, c \in V$, let $I(a, c)$, the interval of a, c, denote the set of elements $b \in V$ such that $[abc]$. Thus $\mathcal{B}$ is the set of all triples a, b, c for which $b \in I(a, c)$. In [7], Chvátal defined a *strict betweenness* as the ternary betweenness relation with the extra property that for all $(a, b, c) \in \mathcal{B} \implies (c, b, a) \in \mathcal{B}$ and a, b, c are distinct elements.

In the same work [7] and in [3], it was shown that strict betweenness defines what we call a convexity space. A *convexity space* is a tuple $(V, \mathcal{N})$, where V is a finite ground set, and $\mathcal{N}$ is a collection of subsets of V such that $\emptyset \in \mathcal{N}$, $V \in \mathcal{N}$, and $\mathcal{N}$ is closed under intersection. The elements of $\mathcal{N}$ are called convex sets. One can define a convex hull of a subset $S \subseteq V$ in the natural way as $\tau(S)$ being the intersection of all the convex sets that contain S.

If a convexity space satisfies the following condition, known as *the anti-exchange property*, then it is called *a convex geometry*.

The Anti-Exchange Property:

Let $S \subseteq V$ and $a, b, \in V$ such that $a, b \notin \tau(S)$.
Then $a \in \tau(S \cup \{b\}) \implies b \notin \tau(S \cup \{a\})$.

Convex geometries have appeared in the literature under different names and aspects, the most famous one being antimatroids. Given a convex geometry $(V, \mathcal{N})$, then the set system $(V, \mathcal{F})$ where $\mathcal{F} = \{X : X = V \backslash Y, Y \in \mathcal{N})$ is an antimatroid. We call the sets $X \in \mathcal{F}$ feasible sets. Antimatroids capture various combinatorial properties on graph classes that have led to a number of algorithms. In particular, various eliminations orderings on graph classes are basic words of some well defined antimatroid, and the feasible sets of the antimatroids are precisely the suffixes of these elimination orderings.

A well studied antimatroid is the one that rises from chordal graphs. These are graphs where the largest induced cycle is a triangle. We refer the read to [11] for more on this graph class. Chordal graphs are characterized by a vertex ordering known as a *perfect elimination ordering*, or PEO. An ordering $\sigma = v_1, v_2, \ldots, v_n$ is a PEO if for all $i \in [n], v_i$ is simplicial in $\sigma[v_1, \ldots, v_{i-1}]$; Meaning the neighbourhood of v_i to its left in σ induces a clique. Thus v_n is a simplicial vertex in G. The set system $(V, \mathcal{F})$ whose ground set are the vertices of a chordal graph, and its feasible sets are the suffixes of PEOs form an antimatroid. Given such an antimatroid, the corresponding convex geometry is the tuple $(V, \mathcal{N})$ where $S \subseteq V$ is convex if for all chordless paths between the vertices in S are also in S. That is, for all $a, c \in S$, if b lies in a chordless ac path in G, then $b \in S$. This is known as the monophonic convexity. Indeed the set system $(V, \mathcal{N})$ is a convex geometry if and only if G is chordal.

A second well studied antimatroid is the double shelling antimatroid of posets. Let $P(V, \prec)$ be a poset. The double shelling antimatroid on P is the set system whose feasible sets are unions of ideals and filters of P. It is easy to see that the corresponding convex geometry is a set system $(V, \mathcal{N})$, where a set $S \in \mathcal{N}$ is convex if for all $a, c \in V$, every b that satisfies $a \prec b \prec c$ or $c \prec b \prec a$ is also in S.

For more on these antimatroids/convex geometries, and for other examples, we refer the reader to the monograph *Greedoids* by Korte, Lovász, and Schrader [13].

Both examples above illustrate the betweenness relationship in their corresponding convex geometries. For chordal graphs, a convex set contains all the vertices in a chordless ac path. That is all the vertices b between a and c. In the poset, every b between a and c in the poset is also in the convex set. Thus in both cases, we have the following

$$S \text{ is convex if} \qquad \forall a, c \in S, b \in I(a,c) \implies b \in S$$

And to use the line notation above, we write $[abc]$ for both b is in a chordless ac path in a the chordal graph G, or b is comparable to both a, c in the poset P. Using the line definition in (1), Beaudou et al. proved in [2] that the Chen-Chvátal conjecture holds for all metric spaces induced by connected chordal graphs. This line definition makes sense for this graphs since it uses its underlying convex geometry structure. Similarly, in [1], Aboulker et al. proved that posets also satisfy the Chen-Chvátal conjecture for the betweenness that rises from the double shelling antimatroid, where a line is defined as:

$$\overline{ac} = \{a, c, \} \cup \{b : b \text{ is comparable to both } a \text{ and } c\}$$

Once again, this makes sense for posets since the notion of line rises naturally from the underlying convex geometry.

The question of whether the Chen-Chvátal conjecture holds for general finite metric spaces is still open. But we do know that it does not hold for 3-uniform hypergraphs, as shown in [6] by Chen and Chvátal. In [1], Aboulker et al. used the following notion of line:

$$\overline{ac} = \{a, c\} \cup \{b : a, b, c \text{ forms a triangle in } G\} \tag{2}$$

to show that the conjecture holds for graphs. However for various families, there are well defined notions of lines that rise naturally from underlying convex geometries. For a number of graph classes that admit such structures, the line definition in (2) does not capture the true betweenness defined by such convex geometries. Thus to continue on the line of work that uses monophonic convexity on chordal graphs and double shelling convexity on posets to prove the Chen-Chvátal conjecture, we use the convex geometry defined on a large graph class, AT-free graphs, to show that this graph class has the De Bruijn-Erdős property.

Let $G(V, E)$ be a graph. Given a path P in G and a vertex u not on P, we say that vertex u misses P if u does not have any neighbour on P. An *asteroidal triple* in G is an independent triple x, y, z of vertices such that there is a path between every pair of the triple that avoids the neighbourhood of the third. A graph $G(V, E)$ is *asteroidal-triple free*, or AT-free for short, if it does not contain an ansteroidal triple. Introduced by Lekkerkerker and Boland to characterize interval graphs, AT-free graphs have been well studied, see for instance [9, 12]. They form a large graph class that contains interval, cographs, permutation, and cocomparability graphs.

Given a pair of non-adjacent vertices x, y in an AT-free graph, we say that a vertex z is in the interval of x, y (and write $z \in I_G(x, y)$) if $zx, zy \notin E$, and there is a zx path that y misses, and a zy path that x misses. If a vertex z is in no interval, then we say that z is *admissible*.

One can see from this definition already that if $z \in I_G(x, y)$, then vertex z is between x and y. Indeed, this betweenness has been used recently to show that AT-free graphs also have an underlying convex geometry [4]. In particular, for an AT-free graph $G(V, E)$, define a set system $(V, \mathcal{N})$ where $S \in \mathcal{N}$ is convex if for all $x, y \in S$, if there exists a non-admissible vertex $z \in V$ such that $z \in I_G(x, y)$ then $z \in S$. In [4], Chang et al. showed recently that this set system is indeed a convex geometry on AT-free graphs. Notice that if $z \in I_G(x, y)$ then $x \notin I_G(z, y)$ and $y \notin I_G(x, z)$ for otherwise the triple x, y, z would form an asteroidal triple. Thus using the convex geometry and strict betweenness defined by AT-free graphs, the definition of the line in (1) reduces to

$$\begin{aligned} \overline{xy} &= \{x, y\} \cup \{z : [xzy]\} \\ &= \{x, y\} \cup \{z : z \in I_G(x, y)\} \end{aligned}$$

We give a short proof that AT-free graphs have the De Bruijn-Erdős property.

Theorem 0.2. *A connected AT-free graph on $n \geq 2$ vertices either induces at least n distinct lines or has a universal line.*

Proof. Let's call a line *trivial* if it only contains its end points, i.e., $\overline{xy} = \{x, y\}$. We begin by a simple remark: If G is AT-free graph on n vertices, then unless $n = 2$, G will never have a universal line. To see this, it suffices to notice that for a line $\overline{ab}$ to be universal, it must contain all of V. However, for all $a' \in N(a), b' \in N(b)$, $a', b' \notin I_G(a, b)$.

We first consider an extreme case: If G has no AT-free intervals, an induced cycle on five vertices for instance, then G has roughly n^2 lines. Indeed every pair of vertices defines a distinct line. And so G has $\binom{n}{2} \approx n^2$ lines. We in fact get more than n lines if $n \geq 3$. Thus again for $n = 2$, the points are collinear.

let G be an arbitrary AT-free graph. For every vertex $v \in V$, and every neighbour $u \in N(v)$, the pair u, v defines a trivial line $\overline{uv}$. Thus every graph has m trivial distinct lines, where m is the number of edges of G. Since G is connected $m \geq n - 1$. Thus we have $l_1, \ldots, l_{n-1}$ distinct lines.

If G is a clique, then similar to the extreme case above, we get $\approx n^2$ lines. If G is not a clique, then there exists a pair of vertices a, b such that $ab \notin E$. Either $I_G(a, b) = \emptyset$ or not. If $I_G(a, b) = \emptyset$, then the line $\overline{ab}$ is trivial and is distinct from all $l_1, \ldots, l_{n-1}$ lines defined by the m edges, since $ab \notin E$. If $I_G(a, b) \neq \emptyset$, then there exists vertex c between a and b and thus $[acb]$, and $\overline{ab} \subseteq \{a, b, c\}$, which is distinct from all $l_1, \ldots, l_{n-1}$ lines.

In all cases we get at least n lines. $\square$

For various definitions of lines, one can show whether a graph class satisfies the Chen-Chvátal conjecture. One can argue however that a definition that rises from underlying structure imposed the graph class is the right one to use, as was the case for monophonic convexity in chordal graphs [2], double shelling antimatroid in posets [1], and now the convexity on AT-free graphs as shown above. We ask whether other graph classes have an underlying convex geometry, and whether such structure leads to natural notions of lines that can be used to prove De Bruijn-Erdős type results. Indeed, one way to see this is to look at the graph classes that admit elimination orderings. Such an ordering would be a basic word of an underlying antimatroid. Thus the complement sets of the suffixes would define a convex geometry. The question then is to identify the type of betweenness that rises from these convex geometries in order to come up with the right notion of a line.

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