Graph searches and geometric convexities in graphs

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Overview. In an attempt to understand graph searching on cocomparability graphs has been so successful, one quickly notices that the orderings produced by these traversals are precisely words of some antimatroids or convex geometries. The notion of antimatroids and convex geometries have appeared in the literature under various settings; in this work, we focus on the graph searching setting, where we discuss some known geometries on cocomparability graphs, and then present new structural properties on AT-free graphs in the hope of exploring whether the algorithms on cocomparability graphs can be lifted to this larger graph class.

1 A Primer on Convexity

1. Convexity spaces. A tuple $(V, \mathcal{N})$, where V is a finite ground set, is a convexity space if $\mathcal{N}$ is a collection of subsets of V such that $\emptyset \in \mathcal{N}$, $V \in \mathcal{N}$, and $\mathcal{N}$ is closed under intersection. The elements of $\mathcal{N}$ are called **convex sets**. One can define a convex hull of a subset $S \subseteq V$ in the natural way as $\tau_{\mathcal{N}}(S)$ being the intersection of all the convex super-sets of S. Formally: For a ground set V, the map $\tau : 2^V \rightarrow 2^V$ is a **closure operator** if it satisfies the following conditions:

- $\tau(\emptyset) = \emptyset$
- $\forall X \subseteq V$, $X \subseteq \tau(X)$.
- $\forall X, Y \subseteq V$ with $X \subseteq Y$ then $\tau(X) \subseteq \tau(Y)$.
- $\forall X \subseteq V$, $\tau(\tau(X)) = \tau(X)$.

2. Extreme Points. With every closure operator, it is natural to talk of *extreme points* as the set of points one can delete and still maintain a convex set. To this end, we define an *extreme point operator* as a map from sets $S \subset V$ and extreme points of S. Formally: An extreme point operator is a map $ext : 2^V \rightarrow 2^V$ where $\forall A \subseteq V$, $ext(A) = \{x \in A : x \notin \tau(A\text{-}x)\}$. An element in $ext(A)$ is called an **extreme element** in A.

3. Betweenness. There is a close relationship between convexity spaces and the notion of *abstract betweenness* defined by Menger [16]. Betweenness is a ternary relation, that relates the "placement" of a point z between two other points a and b. For sake of simplicity we say z belongs to the interval $I[a, b]$ when z is between a and b. As noticed by Chvátal in [6], several abstract betweennesses can be defined on graphs. Given a betweenness relation, one can easily deduce a convexity as follows:

$$C \subseteq V(G) \text{ is convex if } \forall a, b \in C \text{ and } \forall z \in I[a, b], z \in C$$

4. Convex Geometries. If a convexity space satisfies the following condition, known as *the anti-exchange property*, then it is called *a convex geometry*. [The Anti-Exchange Property] Let $S \subseteq V$ and $a, b, \in V$ such that $a, b \notin \tau(S)$. Then $a \in \tau(S \cup \{b\}) \implies b \notin \tau(S \cup \{a\})$. For a convex geometry $(V, \mathcal{N})$, we define the **convex sets** of $\mathcal{N}$ as $X \subseteq V$ such that $\tau_{\mathcal{N}}(X) = X$. Introduced first by Edelman and Jamison in 1985 [12], convex geometries have appeared in the literature under different names and aspects, the most famous one being *antimatroids*.

5. Antimatroids. Given a convex geometry $(V, \mathcal{N})$, then the set system $(V, \mathcal{F})$ where $\mathcal{F} = \{X : X = V \backslash Y, Y \in \mathcal{N})$ is an antimatroid. We call the sets $X \in \mathcal{F}$ feasible sets. If the reader is more familiar with greedoids, then the following definition of antimatroids might be more suitable. A set system $\mathcal{A} = (V, \mathcal{F})$ is a antimatroid if:

- $\emptyset \in \mathcal{F}$
- $\forall X \in \mathcal{F}, X \neq \emptyset, \exists x \in X$ such that $X \backslash \{x\} \in \mathcal{F}$
- $\forall X, Y \in \mathcal{F}, X \subsetneq Y, \exists x \in X \backslash Y$ such that $Y \cup \{x\} \in \mathcal{F}$

A set system is called **accessible** if it satisfies the first two conditions of the above definition. The third condition is sometimes refered to as the *augmentation property*.

6. Basic Words. An antimatroid can also be viewed as a language $\mathcal{L}$ over an alphabet Σ, when $\mathcal{L}$ satisfies the following conditions:

- The prefix of every string $\alpha \in \mathcal{L}$ is also in $\mathcal{L}$.
- For any two strings $\alpha, \beta \in \mathcal{L}$, if α differs from β by at least one character s, then $\beta \cdot s \in \mathcal{L}$ – this is just the augmentation property.
- Every character $s \in \Sigma$ appears in at least one string of $\mathcal{L}$, and no character appears more than once in a given string.

By the augmentation property, there exists always strings in $\mathcal{L}$ that consist of all the characters in Σ. We call these strings the **basic words** of $\mathcal{L}$ – (or of the antimatroid).

1.1 Elimination Schemes and Antimatroids

Antimatroids capture various combinatorial graph properties useful algorithmically. In particular, various eliminations orderings on graph classes are basic words of some well defined antimatroid, and the feasible sets of the antimatroids are precisely the suffixes of these elimination orderings.

A well studied antimatroid is the one that rises from chordal graphs, from PEOs more specifically. The set system $(V, \mathcal{F})$ whose ground set are the vertices of a chordal graph, and its feasible sets are the suffixes of PEOs form an antimatroid [14]. Given such an antimatroid, the corresponding convex geometry is the tuple $(V, \mathcal{N})$ where $S \subseteq V$ is convex if for all chordless paths between the vertices in S are also in S. That is, for all $a, c \in S$, if b lies in a chordless ac path in G, then $b \in S$. This is known as the *monophonic convexity*. Indeed the set system $(V, \mathcal{N})$ is a convex geometry if and only if G is chordal [14].

Another well studied antimatroid is the one that rises from double shellings of posets. Let $P(V, \prec)$ be a poset. The double shelling antimatroid on P is the set system whose feasible sets are unions of ideals and filters of P. It is easy to see that the corresponding convex geometry is a set system $(V, \mathcal{N})$, where a set $S \in \mathcal{N}$ is convex if for all $a, c \in V$, every b that satisfies $a \prec b \prec c$ or $c \prec b \prec a$ is also in S.

In this work, we study convex geometries that rise from graph searches. Motivated by the success `LexDFS` in particular has had on cocomparability graphs, when `LexBFS` has failed, we initially get the following: For a graph $G(V, E)$, let $\mathcal{F}$ be the set of end vertices of `LexDFS` cocomparability orderings ending at a vertex f. The set system $(V, \mathcal{F})$ is an antimatroid. We omit the proof of this claim because we later prove a more general result below. This property is also not "satisfying" since the `LexDFS` cocomparability orderings are just linear extensions, and we already know of the shelling antimatroid of posets – which we note is different than the double shelling one mentioned above. For a given graph property $\mathcal{P}$, we say that a vertex ordering $\tau = v_1, v_2, \ldots, v_n$ is $\mathcal{P}$-perfect if $\forall v_i, \mathcal{P}(v_i)$ is true for v_i in $G[v_i, \ldots, v_n]$. We say that $\mathcal{P}$ is hereditary if it satisfies the property that $\forall v \in V(G)$, and $\forall H$ subgraph of G, if $\mathcal{P}(v)$ holds in G, then if $v \in V(H)$, $\mathcal{P}(v)$ holds in H as well. Examples of Theorem 1 include property $\mathcal{P}(x) = \{d(x) \leq k\}$ for instance, or $\mathcal{P}(x) = \{x$ is not the middle vertex of a $P_4\}$, or $\mathcal{P}(x) = \{x$ is a, true or false, twin$\}$. Using the above definition, we observe the following simple but key theorem, which generalizes the classical antimatroid of simplicial elimination orderings.

Theorem 1. If $\mathcal{P}$ is a hereditary property of a graph class $\mathcal{G}$, then the set $\mathcal{N}$ of prefixes of $\mathcal{P}$-perfect orderings form an antimatroid $(V, \mathcal{N})$.

Proof. We use Definition 1 of an antimatroid to prove this claim. It is easy to see that $(V, \mathcal{N})$ is an accessible set system. We'll that the augmentation property holds as well.
Let $\sigma = x_1, \ldots, x_n$ and $\tau = y_1, \ldots, y_n$ be two $\mathcal{P}$-perfect orderings of a graph G.
Let $X = x_1, \ldots x_h$ and $Y = y_1, \ldots y_k$ be two prefixes of respectively σ and τ.
Suppose $X \setminus Y \neq \emptyset$, and let x_i be the first element in X which does not belong to Y. Therefore $x_i \in \{y_{k+1}, \ldots y_n\}$, say $x_i = y_j$ with $k+1 \leq j \leq n$.
Let $Y' = \{y_1, \ldots, y_k, x_i\}$; we show that Y' is a prefix of some $\mathcal{P}$-perfect ordering of G. By the choice of x_i as the first element in X not in Y, it follows that $\{x_i, y_{k+1}, \ldots y_n\} \subseteq \{x_i, \ldots x_n\}$. Since $\mathcal{P}$ is a hereditary property, $\mathcal{P}(x_i)$ holds on the subgraph $G[\{x_i, y_{k+1}, \ldots y_n\}]$; and thus, the following ordering $\tau' = y_1, \ldots, y_k, x_i, y_{k+1}, \ldots, y_{j-1}, y_{j+1}, \ldots y_n$ is a $\mathcal{P}$-perfect ordering of G. Therefore Y' is a valid prefix. $\square$

Consider a generic graph search $\mathcal{S}$ on a graph $G(V, E)$. If we consider the sequence of visited vertices as follows: $V_0 = \{x_0\}, V_1 = \{x_0, x_1\}, \ldots V_n = \{V\}$, then this set system yields a convexity on $V(G)$ by defining the closure operator as $\forall A \subseteq V$, $\tau(A) = V_i$, where i is the smallest index for which $A \subseteq V_i$. Let's call this convexity an $\mathcal{S}$-convexity.

Lemma 1. If G is connected and $\mathcal{S}$ is a generic search then its $\mathcal{S}$-convexity is a geometric convexity.

Proof. To prove the claim, it suffices to show the anti-exchange property. Consider A a convex set and $x, y \notin \tau(A)$. Suppose $x \in \tau(A + y)$, then vertex x was visited before vertex y since $\mathcal{S}$ is a generic search. And thus $y \notin \tau(A + x)$. $\square$

As a consequence, since complements of convex geometries are antimatroids, the suffixes of generic graph searches considered as sets form an antimatroid; this includes the `LexDFS` cocomparability orderings of Property 1.1, and raises the very natural question:. Pr $For which graph searches and which graph classes is this$ S$-convexity$

Theorem 2. Let $\mathcal{S}$ be a generic search and $\mathcal{G}$ an hereditary class of graphs. The three following properties are equivalent for a graph convexity $\mathcal{C}$:

(i) At each step of the search $\mathcal{S}$, the set of visited vertices is a convex set of $\mathcal{C}$.

(ii) The last vertex visited by $\mathcal{S}$ is extreme for the $\mathcal{C}$ convexity.

(iii) The suffixes of the search $\mathcal{S}$ considered as sets form an antimatroid.

Proof. The proof is quite obvious, it suffices to use the definition of S-convexity and the fact that the feasible sets of an antimatroid are the compelement sets of its corresponding convex geometry. $\square$

If there is a convexity $\mathcal{C}$ and a graph search $\mathcal{S}$ for which Theorem 2 holds, we say that $\mathcal{S}$ is **compatible** with the $\mathcal{C}$ convexity. As a consequence, for a hereditary class of graphs, to check the compatibility of a given graph search with a convexity, it is enough to test whether the last vertex is always extreme for this convexity.

let's examine some properties for the graph convexities described above; in particular on graph searches when applied to cocomparability and AT-free graphs, using the following two convexities below. By convention, an extreme vertex for V, will be called an extreme vertex of G.

- **cc-convexity**: Cocomparability convexity : $I_{cc}[x,y] = \{z | x,y,z$ is an independent triple and $\forall\sigma$ cocomparability ordering of G, either $x \prec_\sigma z \prec_\sigma y$ or $y \prec_\sigma z \prec_\sigma x\}$.
- **2p-convexity**: 2-paths convexity: $I_{2p}[x,y] = \{z | x,y,z$ is an independent triple and $\exists P,Q$ two induced paths, such that P from z to x avoids $N[y]$, and Q from z to y avoids $N[x]\}$.

We present the following theorem which connects these two type of convexities. Notice however, that the converse of the theorem is not true as can be seen with any long chain.

Theorem 3. If G is a cocomparability graphs, then $z \in I_{2p}[x,y] \implies z \in I_{cc}[x,y]$.

Proof. Let σ be a cocomparability ordering of G such that $z \prec_\sigma x \prec_\sigma y$. That is $z \notin I_{cc}[x,y]$. Suppose $z \in I_{2p}[x,y]$; thus xyz is an independent triple of G. Since there exists a path in G from z to y avoiding the neighbourhood of x; consider the first edge uv going across x in this path. Such an edge must exist since $z \prec_\sigma x \prec_\sigma y$. Necessarily $ux, xv \notin E$ since x misses this yz path, and therefore uv is an umbrella over x contradicting σ being a cocomparability ordering. If z is after x and y in σ, then we the path from z to x instead. □

2 New Properties on AT-free Graphs

Recall that a vertex v is said to be *admissible* if for every pair y,z such that x,y,z are independent, y hits every xz-path or z hits every xy-path. An *admissible elimination ordering* – AEO – is an elimination ordering $\sigma = v_1, v_2, \dots, v_n$ where every v_i is admissible in $G[v_1, \dots, v_i]$. Two classical results on AT-free graphs state:

Theorem 4. [9] Every `LexBFS` ordering on an AT-free graph is an AEO. In particular, the end vertex of a `LexBFS` traversal is always admissible.

Theorem 5. [9] Let x be the end vertex of a `LexBFS`(G) AT-free, and y the end vertex of a `LexBFS`(G) which starts at x, then (x,y) is a dominating pair in G.

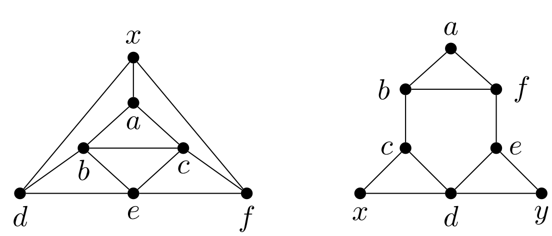

Figure 1: The sun$^+$ (left) and the rocket (right).

Notice that the existence of one single admissible elimination ordering does not characterize AT-free graphs – as can be seen in Fig. 1. Both graphs contain an asteroidal triple: ace, yet both graphs admit an admissible elimination ordering. The sun$^+$ graph has a `LexBFS` ordering $\sigma_S = x, a, b, c, d, e, f$ that is an AEO [10]. The rocket graph has $\sigma_R = a, b, c, d, e, f, g, h$ as an AEO, and vertex h is the only admissible vertex in the graph, yet vertex h will never be the end vertex of a `LexBFS` or even a BFS. So if we want to use Theorem 4 to build a recognition algorithm for AT-free graphs, we need to check *all* `LexBFS` orderings of the input graph. This indeed is was proven by Corneil and Köhler:

Theorem 6. [8] G is AT-free if and only if every `LexBFS` search is compatible with the $2p$-convexity.

Another way to recognize AT-free graphs is by checking if the input graph has an AT-free ordering. Recall this is the vertex ordering characterization of this graph class: [10] An ordering σ is an AT-free ordering if for every $z \in I_{2p}[x, y]$, either $z \prec_\sigma x$ or $z \prec_\sigma y$. One way to tie `LexBFS` orderings – and thus AEO's– to AT-free orderings is the following:

Lemma 2. Let σ be a LexBFS ordering of G ending at x. There exists an AT-free ordering τ of G where x is the end vertex of τ.

Proof. Suppose not. Consider $\tau = u_1, \ldots, x, \ldots, u_n$ an AT-free ordering of G. Let y be the left most vertex to the right of x in τ.

It's easy to see that if $xy \in E$, we can swap x and y in τ and create a new AT-free order τ' of G. Suppose $xy \notin E$. Suppose further that swapping x and y would create a bad triple: that is, there exists some vertex z such that $x \in I_{2p}[z, y]$ and $z, y \prec_{\tau'} x$. This means in τ, we have $z \prec_\tau x \prec_\tau y$ which contradicts x being admissible. Therefore it suffices to repeatedly swap x in new orderings τ' until it is at the end of an AT-free ordering of G. □

However, in [10], Corneil and Stacho gave an example of an AT-free graph for which no `LexBFS` ordering produces an AT-free ordering. Recently (2018), Beisegel [3] gave a variant of BFS that produces AT-free orderings.

Lemma 3 (The Private Neighbour Path). Let σ, π be two `LexBFS` orderings of G where $\pi = \texttt{LexBFS}^+(\sigma)$. For any two vertices a, b such that $a \prec_\sigma b$ and $a \prec_\pi b$, there exists an induced path Q starting at a, where all the vertices (except possibly a) in Q are non adjacent to b.

Let $\mathrm{label}(v)$ be the label of vertex v as assigned by the `LexBFS` algorithm, when v was visited.

Proof. Consider the ordering of a and b in π, at the time a was chosen by `LexBFS`, either a and b had equal labels or $\mathrm{label}(a) \geq_{\text{lexico}} \mathrm{label}(b)$. If a and b had equal lexicographic labels then the $^+$ rule would have broken ties by choosing b first since $a \prec_\sigma b$. Therefore $\mathrm{label}(a) \geq_{\text{lexico}} \mathrm{label}(b)$; that is there exists a private neighbour of a with respect to b, call it a_1, in π such that $a_1 \prec_\pi a \prec_\pi b$ and $a_1 a \in E, a_1 b \notin E$. Choose a_1 to be the left most such private neighbour of a in π.

If $a_1 \prec_\sigma b$ then $a_1 \prec_\sigma b$ and $a_1 \prec_\sigma b$, and by the argument above, there exists a_2, a private neighbour of a_1 with respect to b in π such that $a_2 \prec_\pi a_1 \prec_\pi a \prec_\pi b$. By the choice of a_1 as the left most private of a in π with respect to b, it follows that $a_2 a \notin E$. Again, chose a_2 to be the left most such private neighbour of a_1 in π.

By repeating this argument, and by the finiteness of the graph, we eventually get to a vertex a_j that has a private neighbour a_{j+1} with respect to b in π, and flips ordering with b from σ to π, that is:

$$a_{j+1} \prec_\pi a_j \prec_\pi \ldots \prec_\pi a_1 \prec_\pi a \prec_\pi b \text{ and}$$
$$a \prec_\sigma a_1 \prec_\sigma \ldots \prec_\sigma a_j \prec_\sigma b \prec_\sigma a_{j+1}$$

Thus the induced path constructed by concatenating the edges $a_{j+1}a_j \cdot a_j a_{j-1} \cdot \ldots \cdot a_2 a_1 \cdot a_1 a$ avoids b with the exception of possibly ab in E. □

A well known property on cocomparability graphs says that the end vertex of an arbitrary $\texttt{LexBFS}$ ordering on G cocomparability is a sink in a poset [15]; as a corollary we get the following:

Corollary 2.1. *Consider a sequence of $\texttt{LexBFS}^+$ orderings on a cocomparability graph:*

$$\sigma_1 = \texttt{LexBFS}(G, a) = a \dots x \qquad \sigma_2 = \texttt{LexBFS}^+(\sigma_1) = x \dots y \qquad \sigma_3 = \texttt{LexBFS}^+(\sigma_2) = y \dots z$$

Then $\operatorname{dist}(x, z) \leq 1$.

Proof. If this is not the case, then $xz \notin E$, which means x and z are comparable in a poset. This contradicts both being sinks, with the possible special case that xz is a maximal chain in the poset. That is, without loss of generality x is a sink, and z is a source. Since y is also an end vertex of a $\texttt{LexBFS}$, y is also a sink (or a source). If y is a sink then $x \| y$ and $xy \in E$, which using σ_2 implies that x is universal in G and thus adjacent to z. If y is a source then $y \| z$ and $yz \in E$, which using σ_3 implies y is universal in G, and in particular $yx \in E$. Using this in σ_2 and the fact that $z \prec_{\sigma_2} y$ implies that x is also universal in G, and thus adjacent to z.
Therefore $\operatorname{dist}(x, z) \leq 1$. □

In AT-free graphs, we get the following:

Lemma 4. Consider a sequence of $\texttt{LexBFS}^+$ orderings on an AT-free graph:

$$\sigma_1 = \texttt{LexBFS}(G, a) = a \dots x \qquad \sigma_2 = \texttt{LexBFS}^+(\sigma_1) = x \dots y \qquad \sigma_3 = \texttt{LexBFS}^+(\sigma_2) = y \dots z$$

Then $\operatorname{dist}(x, z) \leq 2$.

For succintness, let P_{ab} denote an induced ab path in the graph. If c is a vertex on P_{ab}, then we write P_{ac} to refer to the induced subpath from a to c in P_{ab}. If c is b's neighbour on P_{ab}, then we can write $P_{ab} = P_{ac} \cdot cb$ to refer to the concatenation of P_{ac} with the edge cb. We also write $|P_{ab}|$ to denote the length of this path.

Proof. Suppose for sake of contradiction that $z \neq x$ and $xz \notin E$. By Theorem 5, both (x, y) and (y, z) are dominating pairs in G. Let P_{yx} be a shortest induced yx path. Then P_{yx} is dominating, and z is either on P_{yx} or has a neighbour on P_{yx}. If the former, then since P_{yx} is induced, the subpath P_{yz} is induced and thus x misses P_{yz} unless $xz \in E$, which is a contradiction to our assumption. But then (y, z) is a dominating pair, which means P_{yz} must be dominating. Thus z has a neighbour on P_{yx}, let w be such a neighbour, and choose w as the left most vertex on $P_{yx} = y \dots w \dots x$ that is adjacent to z. Thus for all $v \in P_{yw}, zv \notin E$.

If $w = y$, then $yz \in E$. This implies that G is a clique since y is a $\texttt{LexBFS}$ end vertex and σ_3 starts at y and ends at z, which in turn implies $zx \in E$. So suppose $w \neq y$, and suppose further that $|P_{wx}| > 1$, then $P_{yw} \cdot wz$ is a dominating path that misses x. Thus $|P_{wx}| = 1$, i.e. $wx \in E$. Therefore

$$\operatorname{dist}(x, z) = 2 = |xw \cdot wz| \tag{1}$$

□

To prove the next theorem, we need the simple key observation:

Fact 2.2. *If G is AT-free and $x \in I[y, z]$, then $y \notin I[x, z]$ and $z \notin I[x, y]$ otherwise x, y, z induces an asteroidal triple.*

Theorem 7. Let G be an AT-free graph. Every admissible vertex in G has at least one admissible neighbour.

Proof. Suppose not, then there exists a vertex x admissible in G such that for all $y \in N(x)$, there exists a, b where $y \in I[a, b]$. Fix a $y \in N(x)$ and induced paths P_{ya}, P_{yb} to witness $y \in I[a, b]$. Then x is adjacent to at least one of a and b otherwise $x \in I[a, b]$. This can happen in two cases, either x lies on one of P_{ya}, P_{yb} or it does not. Suppose without loss of generality, that x is on P_{ya} as $P_{ya} = P_{yx} \cdot P_{xa}$, since $y \in N(x)$, this gives $P_{ya} = xy \cdot P_{xa}$. If $|P_{xa}| > 1$, then P_{xa} and $xy \cdot P_{yb}$ are two paths that witness $x \in I[a, b]$. This is true because b misses P_{ya} by construction and thus $bx \notin E$. Therefore $|P_{xa}| = 1$ and $a \in N(x)$. Suppose on the other hand that x does not lie on neither P_{ya} nor P_{yb}, then $xy \cdot P_{ya}$ and $xy \cdot P_{yb}$ would witness $x \in I[a, b]$, thus at least one of a, b is adjacent to x.

The idea of the proof is to show that for every non-admissible neighbour of x, we must introduce a new vertex that will eventually be a neighbour of x itself, by finiteness of $N(x)$, this cannot keep going. Suppose when we started with $y \in I[a, b]$, all three vertices $a, b, y \in N(x)$. Since $y \in I[a, b]$, by Fact 2.2 above and our assumption, a is non-admissible and $a \in I[\alpha, \beta]$ where at most one of α, β is b or y. Without loss of generality suppose $\alpha \in \{b, y\}$, then β is a new introduced vertex. Similarly for b, by assumption b is non-admissible and $b \in I[\gamma, \delta]$ where at most of the these endpoints is either a or y. Without loss of generality suppose it is $\gamma \in \{a, y\}$, then δ is a new introduced vertex. Using the same argument above, at least of β and δ must be adjacent to x, otherwise we will get $x \in I[\beta, \delta]$. We will unpack this idea case by case below.

Without loss of generality, suppose first that $a \in N(x)$ only. Then by assumption a is non-admissible and there exists two vertices c, d such that $a \in I[c, d]$. If both c, d are different from b, y then x must be adjacent to at least one of them and we thus repeat the same argument. Suppose then that one of c, d is either b or y. Notice that it cannot be both because otherwise a, b, y would induce an asteroidal triple (Fact 2.2).

(i). Suppose $c = b$. Then $y \in I[a, b]$ with P_{ya}, P_{yb} and $a \in I[b, d]$ with Q_{ab}, Q_{ad}. Notice that y must hit every ab path (Q_{ab} in particular) otherwise a, b, y induces an asteroidal triple. Notice further that $a \in I[b, d]$ implies that at least one of $b, d \in N(x)$, otherwise once again $x \in I[b, d]$. Without loss of generality, suppose $b \in N(x)$, we repeat the same argument. By assumption b is not admissible, thus there exists s, t such that $b \in [s, t]$. One of s, t must be a neighbour of x. Since G is finite, and $N(x)$ is finite, by repeating this argument, at some point, we must re-use neighbours of x that we have already introduced.

(ii). Suppose then that $c = y$. Then $y \in I[a, b]$ with P_{ya}, P_{yb} and $a \in I[y, d]$ with Q_{ay}, Q_{ad}, and both $a, y \in N(x)$. In this case, notice that d misses P_{yb}, otherwise a, y, d would induce an asteroidal triple. Similarly b misses Q_{ad}, otherwise a, b, y induces an asteroidal triple. Thus using paths $P_{xd} = xa \cdot Q_{ad}$ and $P_{xb} = xy \cdot P_{yb}$, we get that $x \in I[b, d]$ unless one of these interval endpoints is adjacent to x. We then repeat the same argument, where every interval introduces a new neighbour of x that is not admissible by assumption; $N(x)$ is finite, and thus at least one neighbour of x must be admissible. □

Corollary 2.3. *Every AT-free graph has at least 3 admissible vertices or is a clique.*

Proof. Clearly every AT-free graph has at least two admissible vertices (unless $|V| = 1$), it suffices to use two `LexBFS` sweeps on G to find them. Using Theorem 7, each one of these two admissible vertices has an admissible neighbour. Thus every G with large enough diameter has at least four admissible vertices, with the exception of AT-free graphs that have diameter two. □

This corollary is reminiscent of the classical property on chordal graphs by Dirac:

Theorem 8. [11] Every chordal graph that is not a clique has at least two non-adjacent simplicial vertices.

We next give a few properties on AT-free graphs by looking at the underlying convex geometry and antimatroid structure of this graph class. For the remainder of this work, let $G(V, E)$ be an AT-free graph. We begin with a result in [4], where Chang, Kloks, and Wang showed that the 2p-convexity is a convex geometry on AT-free graphs.

Theorem 9. [4] The 2p-convexity yields a convex geometry if and only if G is AT-free.

We independently proved that AT-free graphs admit an underlying antimatroid, however the advantage of Theorem 9 is that they presented a convex geometry that explicitly described the betweenness relation that holds for this graph class. Whereas our result just follows now from Theorem 2. Using this convex geometry on AT-free graphs, we prove a De Bruijn-Erdős type theorem for AT-free graphs.

A De Bruijn-Erdős Type Theorem for AT-free Graphs. A classical result of De Bruijn and Erdős shows that any given n points are either collinear or determine at least n distinct lines in the plane. Chen and Chvátal conjectured that a generalization of this result in all metric spaces, with appropriate definitions of lines, holds as well. We show that the Chen-Chvátal generalization holds for AT-free graphs, where lines are defined using the notion of betweenness that rises from the $2p$-convexity.

The Sylverster-Gallai theorem asserts that any set of n points is either collinear or there is a line that goes through exactly two distinct points. One of the corollaries of this result is the De Bruijn-Erdős theorem; which asserts that any n collinear points in the plane determine at least n distinct lines [13]. This in fact holds in settings where distances and angles are not needed; just the notions of lines and points on lines in ordered geometries.

Recall that betweenness is a ternary relation that relates the "placement" of a point x between two other points p and q. We write $[pxq]$ to say that x lies between p and q, and write $(E, \mathcal{B})$ to denote the set system defined on the ground set E of points, and a betweenness relation $\mathcal{B}$ defined on the elements of E. In this setting, one can define a *line* $\overline{pq}$ between any two distinct points p and q as follows:

$$\overline{pq} = \{p, q\} \cup \{x : [xpq] \in \mathcal{B} \vee [pxq] \in \mathcal{B} \vee [pqx] \in \mathcal{B}\} \tag{2}$$

Thus the definition of a line varies with every notion of betweenness. For instance, in Euclidean space, betweenness translates naturally to the Euclidean metric:

$$[pxq] \iff p, q, x \text{ are distinct points and } dist(p, x) + dist(x, q) = dist(p, q),$$

However in arbitrary spaces, metric betweenness leads to other types of lines. In fact, it can lead to families of lines with different behaviour. For instance, a line can be properly contained in another line, as shown in the example below [5]. Given

$$\begin{aligned} dist(u, v) = dist(v, x) = dist(x, y) = dist(y, z) = dist(z, u) = 1 \\ dist(u, x) = dist(v, y) = dist(x, z) = dist(y, u) = dist(z, v) = 2, \end{aligned}$$

the line $\overline{vy} = \{v, x, y\}$ is properly contained in the line $\overline{xy} = \{v, x, y, z\}$.

Chen and Chvátal conjecture in [5] the following

Conjecture 2.4. *Every metric space on $n \geq 2$ points either has at least n distinct lines or has a line that contains all the points.*

Clearly, the definition of a line varies with that of betweenness. Let's call a line *universal* if it contains all the points. And let's say that a metric space satisfies the De Bruijn-Erdős property if it either has a universal line or at least n distinct lines. Conjecture 2.4 has been proven for special cases where the notion of betweenness is well understood. In particular, it was shown that the conjecture holds for posets and chordal graphs, and other graph classes, see for instance [1, 2, 7]. For many of these cases, the notion of betweenness rises naturally from an underlying structure of the graph class or the combinatorial objects. In particular, chordal graphs admit a monophonic convexity, and it is indeed what is used to define the betweenness and the lines used to prove the conjecture in [2]. Posets admit a double shelling antimatroid, whose corresponding convex geometry is precisely captured by the betweenness used to prove the Chen-Chvátal conjecture in [1]. We discuss these two examples in more detail below.

For chordal graphs, a convex set contains all the vertices in a chordless ac path. That is all the vertices b between a and c. In the poset, every b between a and c in the poset is also in the convex set. Thus in both cases, we have the following

$$S \text{ is convex if } \quad \forall a, c \in S, b \in I(a,c) \implies b \in S$$

And to use the line notation above, we write $[abc]$ for both b is in a chordless ac path in a the chordal graph G, or b is comparable to both a, c in the poset P. Using the line definition in (1), Beaudou et al. proved in [2] that the Chen-Chvátal conjecture holds for all metric spaces induced by connected chordal graphs. This line definition makes sense for this graphs since it uses its underlying convex geometry structure. Similarly, in [1], Aboulker et al. proved that posets also satisfy the Chen-Chvátal conjecture for the betweenness that rises from the double shelling antimatroid, where a line is defined as:

$$\overline{ac} = \{a, c, \} \cup \{b : b \text{ is comparable to both } a \text{ and } c\}$$

Once again, this makes sense for posets since the notion of line rises naturally from the underlying convex geometry.

using the convex geometry and strict betweenness defined by AT-free graphs, the definition of the line in (1) reduces to

$$\begin{aligned} \overline{xy} &= \{x, y\} \cup \{z : [xzy]\} \\ &= \{x, y\} \cup \{z : z \in I_G(x, y)\} \end{aligned}$$

We give a short proof that AT-free graphs have the De Bruijn-Erdős property.

Theorem 10. A connected AT-free graph on $n \geq 2$ vertices either induces at least n distinct lines or has a universal line.

Proof. Let's call a line *trivial* if it only contains its end points, i.e., $\overline{xy} = \{x, y\}$. We begin by a simple remark: If G is AT-free graph on n vertices, then unless $n = 2$, G will never have a universal line. To see this, it suffices to notice that for a line $\overline{ab}$ to be universal, it must contain all of V. However, for all $a' \in N(a), b' \in N(b), a', b' \notin I_G(a, b)$.

We first consider an extreme case: If G has no AT-free intervals, an induced cycle on five vertices for instance, then G has roughly n^2 lines. Indeed every pair of vertices defines a distinct line. And so G has $\binom{n}{2} \approx n^2$ lines. We in fact get more than n lines if $n \geq 3$. Thus again for $n = 2$, the points are collinear.

let G be an arbitrary AT-free graph. For every vertex $v \in V$, and every neighbour $u \in N(v)$, the pair u, v defines a trivial line $\overline{uv}$. Thus every graph has m trivial distinct lines, where m is the number of edges of G. Since G is connected $m \geq n - 1$. Thus we have $l_1, \ldots, l_{n-1}$ distinct lines.

If G is a clique, then similar to the extreme case above, we get $\approx n^2$ lines. If G is not a clique, then there exists a pair of vertices a, b such that $ab \notin E$. Either $I_G(a, b) = \emptyset$ or not. If $I_G(a, b) = \emptyset$, then the line $\overline{ab}$ is trivial and is distinct from all $l_1, \ldots, l_{n-1}$ lines defined by the m edges, since $ab \notin E$. If $I_G(a, b) \neq \emptyset$, then there exists vertex c between a and b and thus $[acb]$, and $\overline{ab} \subseteq \{a, b, c\}$, which is distinct from all $l_1, \ldots, l_{n-1}$ lines.

In all cases we get at least n lines. $\square$

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