# Homework 6

*To be finished individually without resorting to external references other than those listed on course website. Due at the end of class of Thursday, March 10, 2011.*

1. *(10 points) Consider a version of the network flow problem, where there is a (possibly negative) per-unit cost* $c_{u,v}$ *associated with every arc* $(u, v)$ *in addition to the capacity bound* $b_{u,v}$*, and a target amount of flow* $d > 0$ *is given. So for every unit of flow transmitted along* $(u, v)$*, a cost of* $c_{u,v}$ *is incurred. As usual we have a source node* $s$ *and a sink node* $t$*. The goal is to find an* $s - t$ *flow of amount at least* $d$ *that minimizes the total cost, or report that such a flow does not exist.*

   (a) *Show how to use linear programming to solve this problem.*

   (b) *Show that the* $s - t$ *shortest path problem and the maximum flow problem are both special cases of this problem.*

2. *(10 points) Given a network where every arc* $(u, v)$ *is associated with a minimum flow requirement* $l_{u,v} \geq 0$ *in addition to the capacity bound* $b_{u,v}$ $(b_{u,v} \geq l_{u,v})$*. There is no source or sink node in this problem. We say a flow* $f$ *is feasible if* $l_{u,v} \leq f_{u,v} \leq b_{u,v}$ *for all arc* $(u, v)$*. Prove that there exists a feasible flow if and only if for every node set* $S$*,*

$$\sum_{u \notin S, v \in S} l_{u,v} \leq \sum_{u \in S, v \notin S} b_{u,v}.$$

   *Hint: For the if direction, try to mimic the Ford-Fulkerson analysis. Repeatedly find an infeasible arc, and try to eliminate its "infeasibility" by transmitting flow along some cycle. Either you get a feasible flow by the end, or you get stuck. If you get stuck, extract the witness set* $S$ *in a natural way similar to Ford-Fulkerson.*

3. *(10 points) Suppose you are lost in a dark sinuous tunnel of length* $L$ *for some integer* $L > 0$*, and you want to walk out of it. Your current position is some unknown* $x$ *for some integer* $0 < x < L$*, and every step you can walk one step in either direction. I.e., you can either increase or decrease* $x$ *by 1. Once your location becomes either* $0$ *or* $L$*, you are out of the tunnel. Your task is to find a deterministic strategy for walking out of the tunnel such that the number of steps you take is at most a constant times the optimal number of steps. Prove that the constant factor you get is best possible.*

4. *(extra credit) Let's play an $n$-round game of deal or no deal. You start with round 1, and at round $i$ for $i = 1, 2, \ldots, n$ if you reach that round, you are given a money offer of $v_i$, which is randomly drawn from a distribution $F_i$ over* $[0, 1]$. *Then you can either take the money offer and walk away with a winning of $v_i$ (without playing the remaining rounds), or you can refuse it, and continue to round $i + 1$ if $i < n$. Suppose you know the distributions $F_i$'s. Your goal is to show that there is a strategy such that your expected winning is at least a positive constant fraction of the expected maximum possible winning you can get assuming you know the actual values, which is* $E[\max_i v_i]$.

   - *Prove the claim for the case where all distributions are identical. Try reducing to the secretary problem.*
   - *Prove the claim for the case where the distributions are not necessarily identical.*