Homework 5

To be finished individually without resorting to external references other than ones listed on course website. Due at the end of class of Thursday, March 3, 2011.

1. *(10 points) Suppose we are given an undirected graph $G = (V, E)$, which is not necessarily bipartite. Prove that the maximum size of a matching in this graph is upper-bounded by the following quantity:*

$$\min_{U \subseteq V} \frac{1}{2}(|V| + |U| - o(G - U)),$$

where $G - U$ is the graph obtained from G by removing vertices U along with the incident edges, and $o(G - U)$ is the number of connected components of $G - U$ that have an odd number of vertices.

2. *(10 points) We say that a square 0-1 matrix is a type-k P-matrix if each of its rows contains exactly k entries that equal to one, and each of its columns contains exactly k entries that equal to one. Show that each type-k P-matrix can be written as a sum of k type-1 P-matrix.*

3. *(10 points) Given an undirected bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$, we sample the right hand side nodes of G to form a new bipartite graph $G' = (L', R', E')$ (also with $|L'| = |R'| = n$) in the following way:*

   - *Initially let $L' = L$ and $R' = E' = \emptyset$.*
   - *Repeat the following n times:*
     - *Pick a right hand side node $r \in R$ of G independently at random, and include a copy of r into G' along with the edges incident to r in G. To be specific, add a new node r' into R', and for each $l \in L$ add (l, r') into the edge set E' if and only if (l, r) is in E.*

   Now the question is to prove that the expected size of the maximum matching in G' is at least a $1 - 1/e$ times the size of the maximum matching of G, where the expectation is over the randomness involved in sampling G'.

4. *(extra credit) Given a bipartite graph $G = (L, R, E)$ with nonnegative weights $w(l, r)$ on the edges. For a subset S of L, define $f(S)$ as the maximum total weight of a matching that only matches nodes between S and R.*

- *If all the weights are one, i.e.,* $w(l,r) = 1$ *for all* $l \in L, r \in R$*, show that* $f(S)$ *is monotone in* S*, i.e.,* $f(S) \leq f(T)$ *whenever* $S \subseteq T$*, and that*

  $$f(S \cup \{i\}) - f(S) \geq f(S \cup \{i, i'\}) - f(S \cup \{i'\}),$$

  for any $S \subseteq L$ *and* $i, i' \in L$*. Intuitively, this inequality says that the less you "had", the more marginal gain in* f *you can achieve.*

- *Prove the two inequalities for arbitrary nonnegative weights. Try proof by induction.*