CS364B: Frontiers in Mechanism Design
Lecture #3: The Crawford-Knoer Auction*

Tim Roughgarden†

January 15, 2014

1 The Story So Far

Our current theme is the design of ex post incentive compatible (EPIC) ascending auctions in which sincere bidding leads to a welfare-maximizing allocation. Recall that in an EPIC auction, sincere bidding — answer all queries honestly — is an ex post Nash equilibrium (EPNE) that guarantees nonnegative utility.

Recall the current scenario of *unit demand* bidders. **Scenario #3:**

- A set U of m non-identical items.
- Each bidder i has a private valuation v_{ij} for each item j. The number n of bidders can be more or less than m.
- Each bidder i has unit demand, meaning its value for a bundle $S \subseteq U$ of items is

$$v_i(S) := \max_{j \in S} v_{ij}.$$

Recall that in this scenario, the VCG mechanism — which is always DSIC and welfare-maximizing — can be implemented in polynomial time, as computing a welfare-maximizing allocation (and also the VCG payments) reduces to a bipartite matching problem. By the Revelation Principle, this is a logical prerequisite for the type of ascending auction that we're looking for.

Natural ascending auctions terminate at (approximate) *Walrasian equilibria (WE)*. Recall that a WE is a price vector $\mathbf{q}$ and an allocation (i.e., matching) M such that:

1. Every bidder i gets a preferred good, meaning that $M(i) \in \text{argmax}\{v_u(j) - q(j)\}$.[1]

†Department of Computer Science, Stanford University, 462 Gates Building, 353 Serra Mall, Stanford, CA 94305. Email: tim@cs.stanford.edu.

[1]If $M(i) = \emptyset$, we interpret $v_i(M(i)) = q(M(i)) = 0$. Thus every bidder has the option of selecting no good and paying nothing.

2. Item j is unsold only if $q(j) = 0$.

We'll see later that our goal of designing an EPIC and welfare-maximizing ascending auction requires simulating the truthful VCG outcome — both its allocation *and* its payment — when bidders bid sincerely. This seems intimidating because VCG payments, while polynomial-time computing in scenario #3 as the different between two matching problems, do not appear simple. The key result from last lecture was a characterization of the VCG payments in scenario #3 as the smallest WE price vector.

Theorem 1.1 (Last Lecture) *In scenario #3, let* $\mathbf{v}$ *be a valuation profile. In the truthful VCG outcome* $(\mathbf{p}, M)$*, let* $p(j)$ *denote the payment of the winner of item* j*, or* $p(j)$ *if* j *was not sold. Then:*

1. $(\mathbf{p}, M)$ *is a WE;*
2. $\mathbf{p} \leq \mathbf{q}$ *component-wise for every WE price vector* $\mathbf{q}$.

We'll see in the next lecture that the first part of Theorem 1.1 crucially depends on the unit-demand assumption.

Summarizing, we've reduced our overarching goal to the design of an auction that satisfies the following three properties.

1. EPIC.
2. Sincere bidding leads to the smallest WE.
3. Simple and (pseudo)polynomial-time.

2 The Crawford-Knoer Auction

The following auction was proposed by Crawford and Knoer [2]; our description and analysis follows Demange et al. [4]. The keen reader will notice similarities to the deferred acceptance algorithm of Gale and Shapley [5]. A similar algorithm for bipartite matching was proposed independently by Berksekas [1]. Incentive issues were only considered later, by Leonard [6] and Demange and Gale [3].

Crawford-Knoer (CK) Auction:

1. Initialize the price of every item j to $q(j) = 0$.
2. Initially all bidders are unassigned.
3. while (TRUE):
   (a) Ask each bidder for a favorite item (or $\emptyset$) at the prices $q + \epsilon$, meaning an item $j \in D_i(\mathbf{q} + \epsilon) := \text{argmax}\{v_i(j) - (q(j) + \epsilon)\}$. Treat this as a "bid" for item j.[2]

[2]It is remarkable that the auction can get away with asking for one of possibly many favorite items. Several other ascending auctions require the full demand set $D_i(\mathbf{q} + \epsilon)$.

(b) If no unassigned bidder submits a bid, then halt with the current allocation and prices $\mathbf{q}$.

(c) Otherwise, pick an arbitrary unassigned bidder i that bid for item j and assign j to i.

i. If item j was previously assigned to bidder i', mark i' as unassigned and increase the price $q(j)$ by ϵ.

We first make several immediate observations about the CK auction. The bidder to bid on an item j receives it at price 0; each subsequent bid on j increments its price by ϵ. After a bidder i bids for j, it remains assigned to j until it is outbid by some other bidder — it cannot relinquish the item by its own volition. Thus, once an item has been bid on, it is forevermore assigned to some bidder. At termination, an item is assigned to the most recent bidder for it. Assuming sincere bidding — meaning all bidders i always honestly report an item from $D_i(\mathbf{q} + \epsilon)$ (or report that there none) — the CK auction will terminate in a pseudopolynomial number of iterations. If $v_{\max} = \max_{i,j} v_{ij}$, then there will be at most $v_{\max}/\epsilon$ bids on each of the m items. One could of course imagine variants of the CK auction designed to reduce the number of iterations, for example by letting the price increment ϵ grow larger as the auction proceeds.

3 Analysis of the CK Auction

We next establish the following fundamental result.

Theorem 3.1 *Up to ϵ terms, the outcome of the CK auction under sincere bidding is the VCG outcome under truthful revelation.*

From last lecture, all we need to prove is that the sincere bidding outcome of the CK auction is approximately the smallest Walrasian equilibrium. The formal statement of Theorem 3.1 is the conjunction of the next three lemmas. Lemma 3.2 argues that the outcome is an ϵ-Walrasian equilibrium, which should not be surprising given its termination condition. Lemma 3.4 is the most significant piece: the prices computed by CK auction with sincere bidding are approximately lower than every Walrasian equilibrium. It is not immediately apparent from the CK auction's description why it should find an ϵ-WE in such a parsimonious way. Lemma 3.5 proves a converse that, as an ϵ-WE, the prices computed by the CK auction cannot be much smaller than those of the smallest (exact) WE.

Lemma 3.2 *If all bidders bid sincerely, then the CK auction terminates at an ϵ-WE $(\mathbf{q}, M)$.*[3]

Recalling an exercise from last week, Lemma 3.2 immediately gives the following corollary.

[3]Recall from last lecture this means that unsold goods have price 0, and that each bidder's utility is within ϵ of $\max_{j \in U \cup \{\emptyset\}} v_i(j) - q(j)$.

Corollary 3.3 *If all bidders bid sincerely, then the CK auction terminates with an allocation that has surplus within $m\epsilon$ of the maximum possible.*

Proof of Lemma 3.2: Our discussion of the CK auction shows that an unsold item j was never bid on and hence satisfies $q(j) = 0$. So, consider a bidder i that was matched with $j \in U \cup \{\emptyset\}$. Suppose that at the time of i's most recent bid, the item prices were r. By definition, j is i's preferred option at the prices $r + \epsilon$. After i's bid, j's price was incremented by ϵ (unless $j = \emptyset$) and other prices stayed the same, so j remained i's preferred option, up to ϵ. For the remainder of the CK auction, j's price remained the same (so $q(j) = r(j) + \epsilon$) while other prices only increased (so $q(\ell) \geq r(\ell)$ for $\ell \neq j$). Thus j is i's preferred option, up to ϵ, at the conclusion of the CK auction.[4] ■

Corollary 3.3 states that the CK auction's *allocation* under sincere bidding is essentially the same as that in truthful VCG outcome. But what about the payments?

Lemma 3.4 *Fix a valuation profile* $\mathbf{v}$. *Let* $(\mathbf{q}, M)$ *denote an outcome of the CK auction at termination, under sincere bidding. Let* $(\mathbf{p}, M^*)$ *denote the outcome of the VCG mechanism under truthful revelation. Then*

$$q(j) \leq p(j) + \epsilon \cdot \min\{m, n\} \tag{1}$$

for every item j.

Proof: The proof is a little tricky. Consider an iteration of the CK auction in which, for a positive integer ℓ, a bidder i_1 is about to bid on an item j_i such that, under the current prices r,

$$r(j_1) > p(j_1) + \epsilon \cdot \ell. \tag{2}$$

Under this assumption, we'll construct sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_\ell$ of goods and sets $B_1 \subseteq B_2 \subseteq \cdots B_\ell$ with the following properties for $k = 1, 2, \ldots, \ell$:

(1) U_k has k items and B_k has $k+1$ bidders.

(2) For every item j of U_k,

$$r(j) > p(j) + \epsilon \cdot (\ell - k + 1). \tag{3}$$

(3) For every bidder i of B_k, i's most recent bid was for an item in U_k.[5]

(4) Every bidder of B_k is matched under the VCG allocation M^* to some item.

The point is that such a construction implies that, by (1) and (4), B_ℓ has $\ell + 1$ bidders that are all matched under M^*. This implies that $\ell + 1 \leq \min\{n, m\}$, and so no bidder ever bids

[4] Later we'll generalize the CK auction and its properties to *substitutes* valuation, which are essentially the preferences that enjoy similar stability as item prices ascend.

[5] For bidder i_1, we consider its current bid for item j_1 as its most recent.

in the CK auction on a good j with $r(j) - p(j) > \epsilon \cdot (\min\{n, m\} - 1)$, which proves the lemma.

We construct the U_k's and the B_k's by induction. For the base case, we set $U_1 = \{j_1\}$; (2) is satisfied. Since $r(j_1) > 0$, there is some tentative winner i_2 that bidder i_1 is about to outbid (recall ??). Define $B_1 = \{i_1, i_2\}$; (1) and (3) are satisfied. To prove property (4), note that since i_1 and i_2 are the two most recent bidders on j_1, $v_{i_1}(j_1), v_{i_2}(j_1) \geq r(j_1) > p(j_1)$. That is, with respect to the prices $\mathbf{p}$, each of i_1, i_2 has an item that would net them positive utility (namely, j_1). Since $(\mathbf{p}, M^*)$ is a WE (recall ??), neither i_1 nor i_2 can be unassigned in M^*.

For the inductive step, let U_{k-1} and B_{k-1} satisfy (1)–(4). We need to augment each by one new member, while preserving the properties (2)–(4). We begin by identifying the next item to add. By (1) and (4) of the inductive hypothesis, there is a bidder $i \in B_{k-1}$ matched to a good $j_k \notin U_{k-1}$ under M^*. By (3), bidder i's most recent bid in the CK auction was for some item $j \in U_{K-1}$. Why did i bid for j instead of j_k, especially in light of the fact that, by (2), all of the goods of U_{k-1} are relatively expensive? The only reason can be that j_k is expensive as well.

In more detail, consider the iteration of i's most recent bid, for the item $j \in B_k$. At the prices s at that time, it must have been that

$$v_i(j) - s(j) \geq v_i(j_k) - s(j_k) - \epsilon.$$

Since j's price has not changed since $(r(j) = s(j))$[6] price of j_k can only have increased since $(r(j_k) \geq s(j_k))$, the same inequality holds in the current iteration:

$$v_i(j) - r(j) \geq v_i(j_k) - r(j_k) - \epsilon.$$

On the other hand, since $(\mathbf{p}, m^*)$ is a WE,

$$v_i(j_k) - p(j_k) \geq v_i(j) - p(j) - \epsilon.$$

This flip in i's preferences at the different prices can only be explained by j_k being at least as expensive at r relative to $\mathbf{p}$ compared to j (up to ϵ):

$$r(j_k) - p(j_k) \geq r(j) - p(j) - \epsilon \geq \epsilon \cdot (\ell - k + 1), \tag{4}$$

with the last inequality following from property (2) of the inductive hypothesis (recall (3)). Thus, we can define $U_k = U_{k-1} \cup \{j_k\}$ and satisfy property (3).

The inequality (4) implies that $r(j_k) > 0$, so item j_k is currently assigned to some bidder i_{k+1}. By property (3) of the inductive hypothesis, $i_{k+1} \notin B_{k-1}$. Setting $B_k = B_{k-1} \cup \{i_{k+1}$ satisfies properties (1) and (3).

Finally, property (4) holds by the same argument as in the base case: since $r(j_k) > p(j_k)$ and $(\mathbf{p}, M^*)$ is a WE, j_k must be matched to an item in M^*. ■

[6] If $i \neq i_1$, then i is still in possession of the good j and the price is the same. Bidder i_1 is the one bidding right now, so $s = r$.

Lemma 3.4 states that, up to a small error, the prices computed by the CK algorithm are no higher than the VCG prices. We know from last time that an *exact* WE cannot be lower than the VCG prices. The next lemma extends this property to ϵ-WE, such as the one computed by the CK auction. It plays an important role in our analysis of incentives in the auction.

Lemma 3.5 *Fix a valuation profile* $\mathbf{v}$*. Let* $(\mathbf{q}, M)$ *denote an* ϵ*-WE for* $\mathbf{v}$*. Let* $(\mathbf{p}, M^*)$ *denote the outcome of the VCG mechanism under truthful revelation. Then*

$$q(j) \geq p(j) - \epsilon \cdot \min\{m, n\} \tag{5}$$

for every item j*.*

Lemma 3.5 can be proved using an inductive construction similar to that in the proof of Lemma 3.4, although the details differ. The exercises provide a step-by-step guide to the proof.

4 Incentives in the CK Auction

Section 3 shows that, *assuming sincere bidding*, the CK auction approximately simulates the VCG outcome. Does this automatically imply that sincere bidding is an (approximate) EPNE? After all, truthful revelation is a dominant strategy in the VCG mechanism that is being simulated.

The answer is subtle. Simulating the VCG outcome gives us a property that is weaker than EPIC. To explain, we can group the possible actions of a player i with a valuation v_i into three categories, from least to most devious.

1. Answer all queries honestly (with respect to v_i).
2. For some valuation $v_i' \neq v_i$, answer all queries *as if* its valuation was v_i'.
3. Answer queries in an arbitrary, possibly inconsistent, way. valuation.

Call an action *consistent* if it is realized by sincere bidding with respect to some valuation. The first two categories above are consistent actions. In indirect auctions, however, not all actions are consistent. The "crazy bidder #2" from the first lecture, used to illustrate the impossibility of a DSIC guarantee for all but the simplest iterative auction, used an inconsistent action.

If an auction $\mathcal{A}$ simulates the VCG outcome, there is an outcome-preserving bijective correspondence between profiles of consistent actions in $\mathcal{A}$ and bid profiles in the VCG mechanism. Since no bid deviations improve over direct revelation in the VCG mechanism, no deviations to consistent actions improve over sincere bidding in the auction $\mathcal{A}$.

Proposition 4.1 *Let* $\mathcal{A}$ *be an iterative auction such that the sincere bidding outcome of* $\mathcal{A}$ *is the same as the truthful revelation outcome of the VCG mechanism. For every bidder* i

and valuation profile $\mathbf{v}$*, if every player other than* i *bids sincerely, then sincere bidding is bidder* i*'s best response among consistent actions.*

The guarantee in Proposition 4.1 is weaker than EPIC, which states that sincere bidding is bidder i's best response among *all* actions, consistent or not. Indeed, the hypothesis of the proposition do not guarantee that the auction $\mathcal{A}$ is EPIC. As an exercise, we invite the reader to concoct an auction $\mathcal{A}$ for which sincere bidding leads to the VCG outcome but in which a player can have an incentive to deviate to an inconsistent action when the others are bidding sincerely.

For the special of the CK auction, however, we can prove that it possesses the stronger EPIC guarantee. The example above shows that the argument must make use of the special structure of the auction.

Theorem 4.2 *The CK auction is EPIC (up to error* $2\epsilon \cdot \min\{m, n\}$*).*

Proof: Fix valuations $\mathbf{v}$ and a bidder i. Assume that all bidders other than i bid sincerely. Proposition 4.1 assures us that deviations to consistent actions can't help bidder i. The plan is to show that, for every deviation to an inconsistent action, there is an equally (in)effective deviation to a consistent action.

Consider an arbitrary deviation by bidder i. Clearly, this deviation can only improve over sincere bidding if the auction terminates with i being allocated some item, say item j. Let $(\mathbf{q}, M)$ denote the outcome of the CK auction.

Intuitively, there is a very direct way that i could have acquired item j: bid consistently with the valuation v_i', defined as $v_i'(j) = +\infty$ (or sufficiently large) and $v_{ij'}' = 0$ for $j \neq j'$. The worry is that i figured out an inconsistent action that netted item j at a cheaper price; we proceed to show that this cannot be the case.

We claim that $(\mathbf{q}, M)$ is an ϵ-WE for the valuation profile $(v_i', \mathbf{v}_{-i})$. First, bidder i clearly gets its favorite good j with respect to the valuation v_i'. All of the other bidders get their favorite goods, up to ϵ, by the stopping condition of the CK auction (as in the proof of Lemma 3.2). Finally, in the CK auction, an item goes unsold only if it no one ever bid on it, in which case its price is 0. In particular, bidder i has no ability to relinquish an item that it is assigned to.

Let $(\mathbf{p}', M')$ be the outcome of the VCG mechanism with the bid profile $(v_i', \mathbf{v}_{-i})$; by construction of v_i', $M'(i) = j$. The claim above and Lemma 3.5 imply that $q(j) \geq p'(j) - \epsilon \cdot \min\{m, n\}$ for every j. We thus have

$$\begin{aligned} \text{utility of } i \text{ in } (\mathbf{q}, M) &\leq \text{utility of } i \text{ in VCG with profile } (v_i', \mathbf{v}_{-i}) + \epsilon \cdot \min\{m, n\} \\ &\leq \text{utility of } i \text{ in VCG with profile } (v_i, \mathbf{v}_{-i}) + \epsilon \cdot \min\{m, n\} \\ &\leq \text{utility of } i \text{ bidding sincerely} + 2\epsilon \cdot \min\{m, n\}, \end{aligned}$$

where the second inequality follows from the fact that the VCG mechanism is DSIC and the third inequality follows from Lemma 3.4. ■

References

[1] D. Bertsekas. A distributed algorithm for the assignment problem. MIT LIDS Technical Report, 1979.

[2] V. P. Crawford and E. M. Knoer. Job matching with heterogenous firms and workers. *Econometrica*, 49(2), 1981.

[3] G. Demange and D. Gale. The strategy structure of two-sided matching markets. *Econometrica*, 53(4), 1985.

[4] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *The Journal of Political Economy*, 94(4), 1986.

[5] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.

[6] H. B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, 591(3):461–479, 1983.