CS364A: Algorithmic Game Theory
Lecture #9: Beyond Quasi-Linearity*

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1 Budget Constraints

Our discussion so far has assumed that each agent has *quasi-linear utility*, meaning that it acts to maximize its valuation $v_i(\omega)$ for the chosen outcome ω minus the payment p_i that it has to make. Thus, a bidder's utility is a linear function of the payment made. We have placed no restrictions on payments, other than the minimal conditions that they are nonnegative and no more than the bid $b_i(\omega)$ agent i made for the chosen outcome.

In some important applications, payments are constrained. We first focus on *budget constraints*, which limit the amount of money that an agent can pay. Sometimes, there is little need to incorporate budget constraints. In a single-item auction, where we interpret the valuation of an agent as its maximum willingness-to-pay, its valuation is presumably bounded above by its budget. In other applications, especially where an agent might wind up buying a large number of items, budgets are crucial.

For example, every keyword auction used in practice asks a bidder for its bid-per-click (e.g., \$.25) and its daily budget (e.g., \$100). Per-item values and overall budgets model well how many people made decisions in auctions with lots of items, especially when the items are identical.

The simplest way to incorporate budgets into our existing utility model is to redefine the utility of player i with budget B_i for outcome ω and payment p_i as

$$\begin{array}{rl} v_i(\omega) - p_i & \text{if } p_i \le B_i \\ -\infty & \text{if } p_i > B_i. \end{array}$$

One can of course study smoothed version of this utility function, where there is a cost that is an increasing function of the budget violation.

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Payment constraints such as budget constraints join the two other types of constraints we've been operating under along: incentive constraints, often in the form of monotonicity; and allocation constraints, such as allocating each good to at most one agent. Surplus maximization, where payments appear neither in the objective function nor in the constraints (other than being between 0 and bidders' bids), is special. The VCG mechanism, and its precursors in Lectures 2–4, maximizes surplus "ex post," meaning as well as if all of the private data is known a priori. Maximizing revenue, where payments participate in the objective function, requires new auction formats and new measures of success (Lectures 5 and 6). The same is true of mechanism design with payment constraints, the subject of this and the next lecture.

We certainly can't maximize the surplus $\sum_{i=1}^{n} v_i(\omega)$ ex post when there are budget constraints. Consider the simple case of a single-item auction, where every bidder has a known budget of 1 and a private valuation.[1] The Vickrey auction charges the winner the second-highest bid, which might well be more its budget. Since the Vickrey auction is the unique DSIC surplus-maximizing auction (Exercise 9), surplus-maximization is impossible without violating budgets. As shown in the exercises, no DSIC auction that respects budgets can approximate the surplus well. We need new auction formats to accommodate budget constraints.

2 The Clinching Auction

The original clinching auction, by Ausubel [1], is an ascending implementation of the VCG mechanism when there are multiple identical items, analogous to the English auction for a single item. In [1], a bidder might want more than one item but is assumed to have nonincreasing marginal values for items. We discussed last lecture why ascending auctions can be more desirable than direct-revelation mechanisms. Unlike the SAA format discussed last lecture, the clinching auction in [1] is immune to demand reduction. See also Problem Set #3.

We discuss the variation of the clinching auction, due to Dobzinski et al. [3], that accommodates budget constraints. There are m identical goods, and each bidder might want many of them (like clicks in a keyword auction). Each bidder i has a private valuation v_i for each good that it gets — so if it gets k goods, its valuation for them is $k \cdot v_i$. Each bidder has a budget B_i that we assume is *public*, meaning it is known to the seller in advance.[2] The clinching auction described in this section is not DSIC when budgets are private (see the exercises).

[1] We argued that budgets are often superfluous in a single-item auction, but the point we're making here is general.

[2] We'd love to assume that budgets are private and thus also subject to misreport, but private budgets make the problem tougher, even impossible in some senses [3]. The version of the problem with public budgets is hard enough already — as shown above, surplus maximization ex post is impossible — and it guides us to some elegant and potentially useful auction formats, which of course is the whole point of the exercise.

2.1 First Cut: Using the Market-Clearing Price

We first describe an auction that is more naive than the clinching auction. One can view the clinching auction as a revised, more sophisticated version of this naive auction. We give a direct-revelation description; it will be clear that there is an ascending implementation of it.

The first auction is based on selling goods at the "market-clearing price", where supply equals demand. It's clear what the supply is (m, the number of goods). The demand of a bidder depends on the current price, with higher prices meaning less demand. Formally we define the *demand of bidder i at price p* as:

$$D_i(p) = \begin{cases} \min\left\{\lfloor \frac{B_i}{p} \rfloor, m\right\} & \text{if } p < v_i \\ 0 & \text{if } p > v_i. \end{cases}$$

To explain, recall that bidder i has value v_i for every good that it gets. If the price is above v_i it doesn't want any (i.e., $D_i(p) = 0$), while if the price is below v_i it wants as many as it can afford (i.e., $D_i(p) = \lfloor \frac{B_i}{p} \rfloor$). When $v_i = p$ the bidder does not care how many goods it gets, as long as its budget is respected, and in the auction and its analysis we can take $D_i(v_i)$ to be a convenient integer in $\{0, 1, 2, \ldots, \lfloor \frac{B_i}{p} \rfloor\}$ of our choosing.

As the price p rises, demand $D_i(p)$ goes down, from $D_i(0) = m$ to $D_i(\infty) = 0$. A demand drop can have two different forms: from an arbitrary positive integer to 0 (when p hits v_i), or by a single unit (when $\lfloor B_i/p \rfloor$ becomes one smaller).

Let p^* be the smallest price with $\sum_i D_i(p^*) = m$. Or, more generally, the smallest value such that $\lim_{p \uparrow p^*} \sum_i D_i(p) \geq m \geq \lim_{p \downarrow p^*} \sum_i D_i(p)$. Then, the auction gives $D_i(p^*)$ goods to each bidder i, each at the price p^* (defining $D_i(p^*)$'s for bidders i with $v_i = p^*$ so that all m goods are allocated).

The good news is that, by the definition of the demand $D_i(p)$, this auction respects bidders' budgets. The bad news is that it is not DSIC; it is vulnerable to demand reduction, similar to the simultaneous ascending auction format discussed last lecture.

Example 2.1 (Market-Clearing Price Is Not DSIC) Suppose there are two goods and two bidders, with $B_1 = +\infty$, $v_1 = 6$, and $B_2 = v_2 = 5$. First suppose that both bidders bid truthfully. The total demand $\sum_i D_i(p)$ is at least 3 until the price hits 5, at which point $D_1(5) = 2$ and $D_2(5) = 0$. The auction thus allocates both goods to bidder 1 at a price of 5 each, for a utility of 2. If bidder 1 falsely bids 3, however, it does better. The reason is that bidder 2's demand drops to 1 at the price $\frac{5}{2}$ (it can no longer afford both), and the auction will terminate at the price 3, at which point $D_1(3)$ will be defined as 1. Bidder 1 only gets one good, but the price is only 3, so its utility is 3, more than with truthful bidding.

We haven't studied any non-DSIC auctions since Myerson's Lemma (Lecture 3), which in some sense gives a complete solution to DSIC auction design in single-parameter settings like the present one. The allocation rule in the market-clearing price auction is monotone, as you are invited to check, so Example 2.1 shows that we got the payment rule wrong. We could apply Myerson's Lemma to this allocation rule to derive the appropriate payments to recover DSIC, but we'll want a slightly more sophisticated allocation rule, as well.

2.2 The Clinching Auction for Bidders with Budgets

We'll again give a direct-revelation description, but keep in mind that the auction admits a natural ascending implementation, and that this was the original point of the clinching auction.

Rather than sell all the goods in one shot, we will sell them piecemeal, at different prices. In addition to the current price p, the auction keeps track of the current supply s (initially m) and the residual budget $\hat{B}_i$ (initially B_i) of each bidder i. The demand $\hat{D}_i(p)$ of bidder i at price p is defined with respect to the residual budget and supply, as $\min\{\lfloor\frac{\hat{B}_i}{p}\rfloor, s\}$ if $p < v_i$ and as 0 if $p > v_i$.

Clinching Auction for Budgeted Bidders

- Initialize $p = 0$, $s = m$.
- While $s > 0$:
  - Increase p until there is a bidder i such that $\underbrace{s - \sum_{j \neq i} \hat{D}_j(p)}_{:=k} > 0$.
  - Give k goods to bidder i at price p (theses good are "clinched").
  - Decrease s by k.
  - Decrease $\hat{B}_i$ by $p \cdot k$.

Observe that different goods are sold at different prices, with selling prices increasing over the course of the auction. Observe also that budgets are respected — equivalently, the number of goods k clinched by a bidder i is at most its current demand $\hat{D}_i(p)$.[3]

Example 2.2 Let's return to the setting of Example 2.1 — two goods and two bidders, with $B_1 = +\infty$, $v_1 = 6$, and $B_2 = v_2 = 5$. Suppose both bidders bid truthfully. In Example 2.1, bidder 1 was awarded both goods at a price of 5. Here, because the demand $D_2(p)$ of the second bidder drops to 1 once $p = \frac{5}{2}$, bidder 1 clinches one good at a price of $\frac{5}{2}$. The second good is sold to bidder 1 at price 5, as before. Thus bidder 1 has utility $\frac{9}{2}$ when it bids truthfully in the clinching auction. As we'll see, no false bid could be better.

Theorem 2.3 *The clinching auction for bidders with public budgets is DSIC.*

Proof: We could proceed by verifying that the allocation rule is monotone and the payments conform to Myerson's payment formula, but it's easier to just verify the DSIC condition directly. So, fix a bidder i and bids $\mathbf{b}_{-i}$ by the others. Since bidder i's budget is public, it cannot affect the term $\lfloor \hat{B}_i/p \rfloor$ of its demand function $\hat{D}_i(p)$. It can only affect the time at which it is kicked out of the auction (meaning $\hat{D}_i(p) = 0$ forevermore), which is precisely

[3]If not, then $\sum_j \hat{D}_j(p) < s$. But the auction maintains the invariant that the sum of the current demands is at least the current supply.

when the price p reaches its bid b_i. Note that every good clinched by bidder i when $p < v_i$ contributes positively to the bidder's utility, while every good clinched when $p > v_i$ contributes negatively.

First compare the utility earned by bid $b_i < v_i$ to that earned by a truthful bid. Imagine running the clinching auction twice in parallel, once when i bids b_i and once when i bids v_i. By induction on the number of iterations, the execution of the clinching auction will be identical in the two scenarios as the price ascends from 0 to b_i. Thus, by bidding b_i, the bidder can only lose out on goods that it otherwise would have cliched (for nonnegative utility) in the price interval $[b_i, v_i]$.

Similarly, if i bids $b_i > v_i$, all that changes is that the bidder might acquire some additional goods for nonpositive utility in the price interval $[v_i, b_i]$. Thus, no false bid nets i more utility than a truthful one. ■

If budgets are private and the clinching auction is run with reported budgets instead, then it is no longer DSIC (see the exercises).

Taken alone, Theorem 2.3 is not compelling. There are other simple budget-respecting DSIC auctions, such as giving away all the goods to random bidders for free. We would like to additionally say that the clinching auction computes a "good" allocation, such as one with surplus close to the maximum possible (subject to budget constraints). The original clinching auction [1], without budgets, implements the VCG outcome and hence is surplus-maximizing. As we've seen, no budget-respecting mechanism can have surplus close to that of the VCG mechanism (which need not respect budgets).

Researchers have explored at least three approaches to justifying the clinching auction with budgets on surplus grounds. None are fully satisfying. While there is strong belief that the clinching auction is "the right solution," researchers are struggling to formulate a model to make this intuition precise.

The key challenge is to identify a good benchmark to compare to the performance of the clinching auction. Dobzinski et al. [3] study Pareto optimality rather than an objective function. An allocation is *Pareto optimal* if and only if there's no way to reassign goods and payments to make some agent (a bidder or the seller) better off without making another worse off, where the seller's utility is its revenue. The good news is that Pareto optimality strongly advocates for the clinching auction — it is the unique deterministic DSIC auction that always computes a Pareto optimal allocation. The bad news is that Pareto optimality is not always necessary or sufficient for an auction to be desirable. For example, Bayesian-optimal mechanisms, discussed below, need not be Pareto optimal.

The second approach is to posit a distribution over bidders' valuations and solve for the DSIC mechanism that maximizes expected surplus subject to the given budget constraints (cf., Lecture 5). With this average-case approach, there is an unambiguous notion of "optimal" auctions — those with the highest expected surplus. It is also interesting to prove "simple near-optimal" and "prior-independent" approximations in this setting, along the lines of the results in Lecture 6. Progress in these directions have been slow but steady [6]. Common budgets are currently better understood than general budgets, and in this special case the clinching auction is provably near-optimal [2].

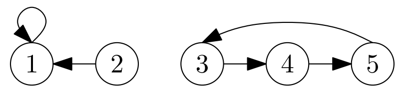

Figure 1: An iteration of the Top Trading Cycle Algorithm (TTCA) with two directed cycles.

A third approach is to modify the surplus objective function to take budgets into account. The most common proposal is to replace $\sum_i v_i x_i$ by $\sum_i \min\{B_i, v_i x_i\}$. The good news is that the clinching auction is provably near-optimal with respect to this objective function [4]. The bad news is that this modified objective does not make much sense in some settings; see the exercises and [5, §3.10].

3 Mechanism Design without Money

There are a number of important applications where there are significant incentive issues but where money is infeasible or illegal. This is equivalent to all agents having a budget of zero. Mechanism design without money is relevant for designing and understanding methods for voting, organ donation, school choice, and labor markets. The designer's hands are tied without money — even tighter than with budget constraints. There is certainly no Vickrey auction, for example. Despite this, and strong impossibility results in general settings, some of mechanism design's greatest hits are motivated by applications without money.

Shapley and Scarf [7] defined the following *house allocation problem*. There are n agents, and each initially owns one house. Each agent has a total ordering over the n houses, and need not prefer their own over the others. The question is: how to sensibly reallocate the houses to make the agents better off? Consider the following *Top Trading Cycle Algorithm (TTCA)*, credited to Gale in [7].

- While agents remain:
  - Each remaining agent points to its favorite remaining house. This induces a directed graph G on the remaining agents in which every vertex has out-degree 1 (Figure 1).
  - The graph G has at least one directed cycle.[4] Self-loops count as directed cycles.
  - Reallocate as suggested by the directed cycles, with each agent on a directed cycle C giving its house to the agent that points to it, that is, to its predecessor on C.
  - Delete the agents and the houses that were reallocated in the previous step.

Observe that the TTCA terminates with each agent possessing exactly one house. As a sanity check for its reasonableness, observe that every agent is only made better off by the algorithm. To see why, note that the algorithm maintains the invariant that the remaining agents still own their original houses. Thus, every iteration, an agent points either to its

[4]Keep following outgoing arcs; eventually, a vertex will be repeated, exposing a directed cycle.

own house or to a house that it likes better. Finally, when an agent is deleted, it receives the house that it had been pointing to.

When agents' preferences are privately known, we can apply the TTCA to agents' reported preferences in a direct-revelation mechanism. There is no incentive for agents to misreport their preferences.

Theorem 3.1 *The TTCA induces a DSIC mechanism.*

Proof: Let N_j denote the agents allocated in the jth iteration of the TTCA when all agents report truthfully. Each agent of N_1 gets its first choice and hence has no incentive to misreport. An agent i of N_2 is not pointed to by any agent of N_1 in the first iteration — otherwise, i would belong to N_1 rather than N_2. Thus, no misreport by i nets a house originally owned by an agent in N_1. Since i gets its first choice outside of the houses owned by N_1, it has no incentive to misreport. In general, an agent i of N_j is never pointed to in the first $j-1$ iterations of the TTCA by any agents in $N_1 \cup \cdots \cup N_{j-1}$. Thus, whatever it reports, i will not receive a house owned by an agent in $N_1 \cup \cdots \cup N_{j-1}$. Since the TTCA gives i its favorite house outside this set, it has no incentive to misreport. ■

As with the clinching auction, Theorem 3.1 by itself is not impressive — the mechanism in which every agent keeps its initial house is also DSIC. To argue that the TTCA is in some sense optimal, we introduce the notion of a *core allocation* — an allocation such that no coalition of agents can make all of its members better off via internal reallocations.

Theorem 3.2 *For every house allocation problem, the allocation computed by the TTCA is the unique core allocation.*

Proof: To prove the computed allocation is a core allocation, consider an arbitrary subset S of agents. Define N_j as in the proof of Theorem 3.1. Let ℓ be the first iteration in which $N_\ell \cap S \neq \emptyset$, with agent $i \in S$ receiving its house in the ℓth iteration of TTCA. TTCA gives agent i its favorite house outside of those owned by $N_1, \ldots, N_{\ell-1}$. Since no agents of S belong to $N_1, \ldots, N_{\ell-1}$, no reallocation of houses among agents of S can make i strictly better off.

We now prove uniqueness. In the TTCA allocation, all agents of N_1 receive their first choice. This must equally be true in any core allocation — in an allocation without this property, the agents of N_1 that didn't get their first choice form a coalition for which internal reallocation can make everyone strictly better off. Similarly, in the TTCA allocation, all agents of N_2 receive their first choice outside of N_1. Given that every core allocation agrees with the TTCA allocation for the agents of N_1, such allocations must also agree for the agents of N_2 — otherwise, the agents of N_2 that fail to get their first choice outside N_1 can all improve via an internal reallocation. Continuing inductively, we find that the TTCA allocation is the unique core allocation. ■

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