Coefficients of the n-fold Theta Function and Weyl Group Multiple Dirichlet Series

Benjamin Brubaker, Daniel Bump, Solomon Friedberg, Jeffrey Hoffstein

Dedicated to Professor Samuel J. Patterson in honor of his sixtieth birthday

May 10, 2010

Abstract

We establish a link between certain Whittaker coefficients of the generalized metaplectic theta functions, first studied by Kazhdan and Patterson [14], and the coefficients of the stable Weyl group multiple Dirichlet series defined in [3]. The generalized theta functions are the residues of Eisenstein series on a metaplectic n-fold cover of the general linear group. For n sufficiently large, we consider *different* Whittaker coefficients for such a theta function which lie in the orbit of Hecke operators at a given prime p. These are shown to be equal (up to an explicit constant) to the p-power supported coefficients of a Weyl group multiple Dirichlet series (MDS). These MDS coefficients are described in terms of the underlying root system; they have also recently been identified as the values of a p-adic Whittaker function attached to an unramified principal series representation on the metaplectic cover of the general linear group.

1 Introduction

This paper links the coefficients of two different Dirichlet series in several complex variables that arise in the study of automorphic forms on the metaplectic group. We begin with a brief discussion of the metaplectic group. Let F be a number field containing the group μ_{2n} of $2n$-th roots of unity, and let F_v denote the completion of F at a place v. Let $\widetilde{G}_v$ denote the n-fold metaplectic cover of $\mathrm{GL}_{r+1}(F_v)$. Recall that $\widetilde{G}_v$ is a central extension of $\mathrm{GL}_{r+1}(F_v)$ by μ_n:

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G}_v \longrightarrow \mathrm{GL}_{r+1}(F_v) \longrightarrow 1.$$

This group is described by a 2-cocycle whose definition involves the n-th power Hilbert symbol. See Matsumoto [16] or Kazhdan and Patterson [14] for this construction. The group $\widetilde{G}_v$ is generally not the F_v-points of an algebraic group. One may then take a suitable restricted direct product to define a global metaplectic cover $\widetilde{G}$ over $\mathrm{GL}_{r+1}(\mathbb{A}_F)$, the adelic points of the group. (The assumption that F contains μ_{2n} rather than μ_n is not necessary, but greatly simplifies the description of the cocycle and resulting formulas.)

Generalized theta series were introduced on the metaplectic covers of GL_2 by Kubota, and for GL_n in the visionary paper of Kazhdan and Patterson [14]. These remarkable automorphic forms are residues of the minimal parabolic Eisenstein series on the global metaplectic cover. They generalize classical theta functions of Jacobi and Siegel which were shown by Weil to live on the metaplectic double covers of GL_2 and symplectic groups.

After Kubota introduced generalized theta series on the higher metaplectic covers of GL_2, Patterson and Heath-Brown [18] exploited the fact that when $n = 3$ their Fourier (Whittaker) coefficients are Gauss sums in order to settle the Kummer conjecture. Yet it was found by Suzuki [19] that one could not so readily determine the coefficients of the theta function on the 4-fold cover of GL_2. See Eckhardt and Patterson [10] for further discussion of this case. The difficulty in determining these coefficients is linked to the non-uniqueness of Whittaker models (cf. Deligne [9]).

Thus determining the Whittaker coefficients of generalized theta series was recognized as a fundamental question. The non-degenerate Whittaker coefficients on the n-fold cover of GL_{r+1} are non-zero only if $n \geq r + 1$ ([14]). Due to non-uniqueness of Whittaker models their complete description is unavailable. Though they are thus mysterious, the partial information that is available is interesting indeed. They satisfy a periodicity property modulo n-th powers, which is a generalization of the periodicity of the coefficients of the classical Jacobi theta function modulo squares. Moreover Kazhdan and Patterson found an action of the Weyl group on the coefficients modulo this periodicity in which each simple reflection adds or deletes a Gauss sum. This is an elegant formulation of the information that is available from Hecke theory. The non-uniqueness of Whittaker models when $n > r + 2$ is a consequence of the fact that there is more than one free orbit in this Weyl group action. We review the definition of the generalized theta functions and expand on this discussion in Section 3.

More recently, Brubaker, Bump and Friedberg [6] have given an explicit description of the Whittaker coefficients of Borel Eisenstein series on the n-fold metaplectic cover of GL_{r+1}. In particular, they showed that the first nondegenerate Whittaker coefficient is a Dirichlet series in r complex variables (a “multiple Dirichlet series”) that is roughly of the form

$$\sum_{c_1,\dots,c_r} H(c_1,\dots,c_r)|c_1|^{-2s_1}\dots|c_r|^{-2s_r}. \tag{1}$$

Here the sum runs over r-tuples of S-integers $\mathfrak{o}_S \subset F$ for a finite set of bad primes S and the coefficients H are sums of products of Gauss sums built with n-th power residue symbols. The general expression for the coefficients H is best given in the language of crystal graphs, but this full description will not be needed here. Indeed, we will restrict our attention to cases where the degree of the cover n is at least $r+1$ (which is "stable" in the vocabulary of [3]), in which case the description of the coefficients simplifies considerably. In this situation, the coefficients supported at powers of a given prime p are in one-to-one correspondence with the Weyl group S_{r+1} of GL_{r+1}, and have a description in terms of the underlying root system [3]. Though we have described the series (1) in terms of global objects (Eisenstein series), let us also mention that the p-power supported terms are known to match the p-adic Whittaker function attached to the spherical vector for the associated principal series representation used to construct the Eisenstein series. This follows from combining the work of McNamara [17] with that of Brubaker, Bump and Friedberg [2, 6], or by combining [2, 17] and an unfolding argument of Friedberg and McNamara [11]. The precise definition of the coefficients H in the "stable" case will be reviewed in Section 2.

This paper establishes a link between *some* of the Whittaker coefficients of generalized theta functions and the coefficients of a stable Weyl group multiple Dirichlet series. Let us explain which coefficients are linked. We will show that, for $n > r+1$ fixed, the coefficients at p determined by Hecke theory are in one-to-one correspondence with the coefficients at p of the series (1). This is accomplished by comparing the two Weyl group actions – one on the Whittaker coefficients of generalized theta series found by Kazhdan and Patterson, and another on the permutahedron supporting the stable multiple Dirichlet series coefficients. We know of no *a priori* reason for this link. On the one side, we have *different* Whittaker coefficients attached to a *residue* of an Eisenstein series. On the other, we have multiple Dirichlet series coefficients that contribute to the representation of a *single* Whittaker coefficient of the Eisenstein series itself. (More precisely, these contribute to the first non-degenerate coefficient.) For $n = r+1$ there is also a link, but this time to a multiple Dirichlet series coefficient attached to an Eisenstein series on the n-fold cover of GL_r, rather than on the n-fold cover of GL_{r+1}. Both comparison theorems for $n > r+1$ (Theorem 2) and $n = r+1$ (Theorem 3) are stated and proved in Section 4 of the paper. These Theorems sharpen and extend the work of Kazhdan and Patterson (see [14], Theorems I.4.2 and II.2.3) on this connection.

This work was supported by NSF FRG grants DMS-0652529, DMS-0652817, DMS-0652609, and DMS-0652312, and by NSA grant H98230-10-1-0183.

2 Weyl group multiple Dirichlet series

In [3], Brubaker, Bump, and Friedberg defined a *Weyl group multiple Dirichlet series* for any reduced root system and for n sufficiently large (depending on the root system). The requirement that n be sufficiently large is called stability, as the coefficients of the Dirichlet series are uniformly described Lie-theoretically for all such n. In this paper we will be concerned with root systems of type A_r and in this case, the stability condition is satisfied if $n \geq r$.

As above, let F be a number field containing the group μ_{2n} of $2n$-th roots of unity. Let S be a finite set of places of F containing the archimedean ones and those ramified over $\mathbb{Q}$ and that is large enough that the ring $\mathfrak{o}_S$ of S-integers in F is a principal ideal domain.

The multiple Dirichlet series coefficients are built from Gauss sums g_t, whose definition we now give. Let ψ be an additive character of $F_S = \prod_{v \in S} F_v$ that is trivial on $\mathfrak{o}_S$ but no larger fractional ideal. If $m, c \in \mathfrak{o}_S$ with $c \neq 0$ and if $t \geq 1$ is a rational integer, let

$$g_t(m,c) = \sum_{a \bmod^{\times} c} \left(\frac{a}{c}\right)^t \psi\left(\frac{am}{c}\right), \tag{2}$$

where $\left(\frac{a}{c}\right)$ is the n-th power residue symbol and the sum is over a modulo c with $(a,c) = 1$ in $\mathfrak{o}_S$. For convenience, we let $g(m,c) = g_1(m,c)$. Let p be a fixed prime element of $\mathfrak{o}_S$, and q be the cardinality of $\mathfrak{o}_S/p\mathfrak{o}_S$. For brevity, we may sometimes write $g_t = g_t(1,p)$.

The multiple Dirichlet series of type A_r defined in [3] has the form

$$Z_\Psi(s_1, \cdots, s_r) = \sum H\Psi(c_1, \cdots, c_r)\mathbb{N}c_1^{-2s_1} \cdots \mathbb{N}c_r^{-2s_r} \tag{3}$$

where the sum is over nonzero ideals c_i of $\mathfrak{o}_S$. Here H and Ψ are defined when c_i are nonzero elements of $\mathfrak{o}_S$, but their product is well-defined over ideals, since H and Ψ behave in a coordinated way when c_i is multiplied by a unit. Thus the sum is essentially over ideals $c_i\mathfrak{o}_S$. However we will want to consider H independently of Ψ, so for each prime $\mathfrak{p}$ of $\mathfrak{o}_S$ we fix a generator p of $\mathfrak{p}$, and only consider c_i which are products of powers of these fixed p's.

The function Ψ is chosen from a finite-dimensional vector space that is well-understood and defined in [3] or [2], and we will not discuss it further here. The function H contains the key arithmetic information. It has a twisted multiplicativity, so that while Z_Ψ is not an Euler product, the specification of its coefficients is reduced to the case where the c_i are powers of the same prime p. See [1], [3] or [4] for further details.

To describe $H(p^{k_1}, \ldots, p^{k_r})$, let Φ be the roots of A_r with Φ^+ (resp. Φ^-) the positive (resp. negative) roots. The Weyl group W acts on Φ. Let

$$\Phi(w) = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}.$$

Also, let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the Weyl vector and let $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$ denote the set of simple roots. Then, as described in [3], we have:

- $H(p^{k_1}, \ldots, p^{k_r}) \neq 0$ if and only if $\rho - w\rho = \sum_{i=1}^r k_i\alpha_i$ for some $w \in W$.
- If $\rho - w\rho = \sum_{i=1}^r k_i\alpha_i$, then

  $$H(p^{k_1}, \ldots, p^{k_r}) = \prod_{\alpha\in\Phi(w)} g(p^{d(\alpha)-1}, p^{d(\alpha)}) \tag{4}$$

  with $d(\sum_{i=1}^r k_i\alpha_i) = \sum_{i=1}^r k_i$.

Thus in the stable case the Weyl group multiple Dirichlet series of type A_r has exactly $(r+1)!$ nonzero coefficients at each prime p. For motivation, more details, and generalizations to the case of smaller n (where there are additional nonzero coefficients), see [1]—[6].

3 Theta functions

As in the introduction, $\widetilde{G}$ denotes the n-fold metaplectic cover of $\mathrm{GL}_{r+1}(\mathbb{A}_F)$. Let $\theta_r^{(n)}$ denote the theta function on $\widetilde{G}$. This function is the normalized K-fixed vector in the space spanned by the residues at the rightmost poles of the minimal parabolic Eisenstein series on $\widetilde{G}$. Here K denotes a suitable compact open subgroup. We will be concerned with the Whittaker coefficients of this vector, when they exist. By Hecke theory, these are determined by the values of these coefficients at prime power indices, or equivalently by the values of the local Whittaker functions for the exceptional theta representations $\Theta_r^{(n)}$, in the sense of Kazhdan and Patterson [14]. We now pass to a fixed completion of F at a good finite prime. In Section I.3 of [14], these authors have shown:

1. The representation $\Theta_r^{(n)}$ has a unique Whittaker model if and only if $n = r + 1$ or $n = r + 2$.

2. The representation $\Theta_r^{(n)}$ does not have a Whittaker model if $n \leq r$.

3. The representation $\Theta_r^{(n)}$ has a finite number of independent nonzero Whittaker models if $n > r + 2$.

4. When the Whittaker model for $\Theta_r^{(n)}$ is nonzero, it is completely determined by the values of the associated Whittaker function on diagonal matrices of the form
$$\begin{pmatrix} \varpi^{f_1} & & & \\ & \varpi^{f_2} & & \\ & & \cdots & \\ & & & \varpi^{f_{r+1}} \end{pmatrix}$$
with $0 \leq f_i - f_{i+1} \leq n - 1$ for $1 \leq i \leq r$.

The reason that this last holds is that the remaining values are determined by Kazhdan and Patterson's Periodicity Theorem. This states that shifting one of the $f_i - f_{i+1}$ by a multiple of n multiplies the Whittaker value by a specific power of q.

Suppose $n \geq r + 1$. Fix a prime element p of $\mathfrak{o}_S$. Let $\tau_{n,r}(k_1, \ldots, k_r)$ be the $(p^{k_1}, \ldots, p^{k_r})$-th Whittaker coefficient of $\theta_r^{(n)}$. This coefficient is obtained by integrating against the character

$$\psi_U(u) = \psi\left(\sum_{i=1}^{r} p^{k_i} u_{i,i+1}\right)$$

of the subgroup U of upper triangular unipotents of GL_{r+1}, which is embedded in $\widetilde{G}$ via the trivial section.

Kazhdan and Patterson observed that Hecke theory may be used to compute all these Whittaker coefficients in the unique model case, and a subset of the coefficients in general. (See also Bump and Hoffstein [8] and Hoffstein [12].) We shall now review their description.

Let W denote the Weyl group for the root system A_r, isomorphic to the symmetric group S_{r+1}. In Section I.3 of [14], Kazhdan and Patterson define an action of W on the weight lattice (identified with $\mathbb{Z}^{r+1}$) by the formula

$$w[\mathbf{f}] = w(\mathbf{f} - \boldsymbol{\rho}) + \boldsymbol{\rho}$$

Element σ of S_3	$\sigma((0,0))$
e	$(0,0)$
σ_1	$(n-2,1)$
σ_2	$(1,n-2)$
$\sigma_1\sigma_2$	$(n-3,0)$
$\sigma_2\sigma_1$	$(0,n-3)$
$\sigma_1\sigma_2\sigma_1$	$(n-2,n-2)$

Table 1: The orbit of $(0,0)$ under S_3.

where $\mathbf{f} = (f_1, \ldots, f_{r+1})$, the Weyl vector $\boldsymbol{\rho} = (r, r-1, \ldots, 0)$, and

$$w(\mathbf{f}) = (f_{w^{-1}(1)}, \ldots, f_{w^{-1}(r+1)}).$$

This action of W on $\mathbb{Z}^{r+1}$ may then be projected down to $(\mathbb{Z}/n\mathbb{Z})^{r+1}$.

Because we prefer to use coordinates on the root lattice, we will reformulate this action in the language of the previous section. It suffices to define it for simple reflections σ_i which generate W. Let $\mathbf{K}_r = \{\mathbf{k} = (k_1, \ldots, k_r) \mid 0 \leq k_j < n \text{ for all } j\}$. Then the action of σ_i for any $i = 1, \ldots, r$ on multi-indices $\mathbf{k} \in \mathbf{K}_r$ is given by $\sigma_i(\mathbf{k}) = \mathbf{m}$ with

$$\begin{aligned} m_{i-1} &= \begin{cases} 1 + k_i + k_{i-1} & \text{if } 1 + k_i + k_{i-1} < n \\ 1 + k_i + k_{i-1} - n & \text{if } 1 + k_i + k_{i-1} \geq n \end{cases} \\ m_i &= \begin{cases} n - 2 - k_i & \text{if } k_i < n-1 \\ 2n - 2 - k_i & \text{if } k_i = n-1 \end{cases} \\ m_{i+1} &= \begin{cases} 1 + k_i + k_{i+1} & \text{if } 1 + k_i + k_{i+1} < n \\ 1 + k_i + k_{i+1} - n & \text{if } 1 + k_i + k_{i+1} \geq n \end{cases} \\ m_j &= k_j \qquad \text{if } j \neq i-1, i, i+1. \end{aligned} \tag{5}$$

In these formulas, we take $k_0 = k_{r+1} = 0$. It is a simple exercise to verify that this matches the action on the weight lattice described above.

To illustrate, the orbit of the origin when $r = 2$ and $n \geq 3$ is given in Table 1, and the orbit of the origin when $r = 3$ and $n \geq 4$ is given in Table 2. We shall also show that the stabilizer of the origin is trivial for $n > r + 1$. However, this fails for $n = r + 1$, as one sees immediately in these two examples.

Element σ of S_4	$\sigma((0,0,0))$	Element σ of S_4	$\sigma((0,0,0))$
e	$(0,0,0)$	$\sigma_3\sigma_1\sigma_2$	$(n-3,2,n-3)$
σ_1	$(n-2,1,0)$	$\sigma_2\sigma_3\sigma_2$	$(2,n-2,n-2)$
σ_2	$(1,n-2,1)$	$\sigma_1\sigma_2\sigma_3$	$(n-4,0,0)$
σ_3	$(0,1,n-2)$	$\sigma_1\sigma_3\sigma_2\sigma_1$	$(n-2,1,n-4)$
$\sigma_1\sigma_2$	$(n-3,0,1)$	$\sigma_2\sigma_3\sigma_2\sigma_1$	$(1,n-2,n-3)$
$\sigma_2\sigma_1$	$(0,n-3,2)$	$\sigma_1\sigma_2\sigma_3\sigma_1$	$(n-3,n-2,1)$
$\sigma_1\sigma_3$	$(n-2,2,n-2)$	$\sigma_2\sigma_3\sigma_1\sigma_2$	$(0,n-4,0)$
$\sigma_2\sigma_3$	$(2,n-3,0)$	$\sigma_1\sigma_2\sigma_3\sigma_2$	$(n-4,1,n-2)$
$\sigma_3\sigma_2$	$(1,0,n-3)$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$	$(0,n-3,n-2)$
$\sigma_1\sigma_2\sigma_1$	$(n-2,n-2,2)$	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$	$(n-3,0,n-3)$
$\sigma_3\sigma_2\sigma_1$	$(0,0,n-4)$	$\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$	$(n-2,n-3,0)$
$\sigma_2\sigma_3\sigma_1$	$(1,n-4,1)$	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$	$(n-2,n-2,n-2)$

Table 2: The orbit of $(0,0,0)$ under S_4.

The action above is essentially that corresponding to the action of the Hecke operators. Since $\theta_r^{(n)}$ is an eigenfunction of these operators, one can deduce the following relation.

Theorem 1 *Suppose that* $0 \le k_j < n$ *for* $1 \le j \le r$, $k_i \not\equiv -1 \bmod n$, *and* $\sigma_i(\mathbf{k}) = \mathbf{m}$. *Then*

$$\tau_{n,r}(\mathbf{m}) = q^{i-r/2-1+\delta(i,r,\mathbf{k})} g_{1+k_i} \tau_{n,r}(\mathbf{k}),$$

where

$$\begin{aligned}\delta(i,r,\mathbf{k}) &= \begin{cases} -(i-1)(r-i+2)/2 & \textit{if } 1+k_i+k_{i-1} \ge n \\ 0 & \textit{otherwise} \end{cases} \\ &\quad + \begin{cases} (i+1)(r-i)/2 & \textit{if } 1+k_i+k_{i+1} < n \\ 0 & \textit{otherwise.} \end{cases}\end{aligned}$$

Here we have used the Gauss sum g_t as defined in (2). The result follows from Proposition 5.3 of [12] and the periodicity property of the Fourier coefficients $\tau_{n,r}$, given in Proposition 5.1 there. (Note that $G_{1+c_i}(m_i,p)$ in [12], Proposition 5.3, is normalized to have absolute value 1, while g_{1+k_i} has absolute value $q^{1/2}$.) See also [8], and Corollary I.3.4 of [14].

4 A link between theta coefficients and Weyl group multiple Dirichlet series

To give a link between the Whittaker coefficients of the generalized theta functions that are determined by Hecke theory and the Weyl group multiple Dirichlet series, we begin by linking the action of (5) to roots. Suppose that $n \geq r+1$.

Proposition 1 *Let $w \in W$, and suppose that $w((0,\ldots,0)) = \mathbf{k}$. Then $1 + k_i \equiv d(w^{-1}\alpha_i) \bmod n$ for each i, $1 \leq i \leq r$.*

Proof We prove the Proposition by induction on $\ell(w)$, the length of w as a reduced word composed of simple reflections σ_i. The case $w = 1$ is clear. Suppose that $w((0,\ldots,0)) = \mathbf{k}$ and $1+k_i \equiv d(w^{-1}\alpha_i) \bmod n$ for each i. Choose $\sigma_j \in W$ such that $\ell(\sigma_j w) = \ell(w)+1$. If $\sigma_j w((0,\ldots,0)) = \mathbf{m}$, then $\mathbf{m} = \sigma_j(\mathbf{k})$, so by (5)

$$m_i \equiv \begin{cases} -2-k_i & j=i \\ 1+k_i+k_j & j=i+1 \text{ or } j=i-1 \\ k_i & \text{otherwise} \end{cases} \tag{6}$$

modulo n. On the other hand, we have

$$\begin{aligned} (\sigma_j w)^{-1}(\alpha_i) &= w^{-1}(\sigma_j(\alpha_i)) \\ &= \begin{cases} w^{-1}(-\alpha_i) & j=i \\ w^{-1}(\alpha_i+\alpha_j) & j=i+1 \text{ or } j=i-1 \\ w^{-1}(\alpha_i) & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$d((\sigma_j w)^{-1}(\alpha_i)) = \begin{cases} -d(w^{-1}(\alpha_i)) & j=i \\ d(w^{-1}(\alpha_i)) + d(w^{-1}(\alpha_j)) & j=i+1 \text{ or } j=i-1 \\ d(w^{-1}(\alpha_i)) & \text{otherwise.} \end{cases}$$

Using the inductive hypothesis, we see that modulo n

$$\begin{aligned} d((\sigma_j w)^{-1}(\alpha_i)) &\equiv \begin{cases} -(1+k_i) & j=i \\ 2+k_i+k_j & j=i+1 \text{ or } j=i-1 \\ 1+k_i & \text{otherwise} \end{cases} \\ &= \begin{cases} 1+(-2-k_i) & j=i \\ 1+(1+k_i+k_j) & j=i+1 \text{ or } j=i-1 \\ 1+k_i & \text{otherwise.} \end{cases} \end{aligned}$$

Comparing this to (6), we see that the Proposition holds. □

Corollary 1 *Let $w \in W$, and suppose that $w((0,\ldots,0)) = \mathbf{k}$. Then for all i, $1 \le i \le r$, $k_i \not\equiv -1 \bmod n$.*

Proof Since $w^{-1}\alpha_i$ is a root, we have $d(w^{-1}\alpha_i) \neq 0$. Moreover, from the explicit description of the roots of type A_r, for any root $\beta \in \Phi$, we have $-r \le d(\beta) \le r$. If $k_i \equiv -1 \bmod n$, Proposition 1 would imply that $d(w^{-1}\alpha_i) \equiv 0 \bmod n$, which is impossible as $n \ge r+1$ and $d(w^{-1}\alpha_i) \neq 0$. □

Corollary 2 *Suppose that $n > r+1$. Then the stabilizer in W of $(0,\ldots,0)$ is trivial. Thus the orbit of the origin has cardinality $(r+1)!$, and every point in the orbit may be described uniquely as $w((0,\ldots,0))$ for some $w \in W$.*

Proof Let $\sigma \in W$ and suppose that $\sigma((0,\ldots,0)) = (0,\ldots,0)$. By Proposition 1, $d(\sigma^{-1}\alpha_i) \equiv 1 \bmod n$ for all i, $1 \le i \le r$. But as noted above, $-r \le d(\sigma^{-1}\alpha_i) \le r$. Since $n > r+1$, the congruence can only hold if $d(\sigma^{-1}\alpha_i) = 1$ for all i, $1 \le i \le r$. Thus $\sigma^{-1}(\alpha_i) \in \Phi^+$ for all i. This implies that $\sigma^{-1}(\Phi^+) \subset \Phi^+$, which is true only if σ is the identity element. □

Note that Corollary 2 does not remain valid if $n = r+1$; it is possible that there exists an i for which $d(\sigma^{-1}\alpha_i)$ is $-r$ and not 1. This occurs, for example, when $r = 2$, $\sigma = \sigma_1\sigma_2$, and $i = 1$. More generally, see Lemma 1 below.

We may now establish a link between the Whittaker coefficients of the generalized theta function that are determined by Hecke theory and the Weyl group multiple Dirichlet series.

Theorem 2 *Suppose that $n > r+1$. Let $w \in W$, and set $w((0,\ldots,0)) = \mathbf{k}$, $\rho - w\rho = \sum f_i\alpha_i$. Then*

$$\tau_{n,r}(\mathbf{k}) = q^{\eta(w,n,r,\mathbf{k})}\, H(p^{f_1},\ldots,p^{f_r}),$$

where the function $\eta(w,n,r,\mathbf{k})$ is described in (9) below.

Remark 1 We should emphasize that in general $k_i \neq f_i$. For example, on A_2 we have $\sigma_1((0,0)) = (n-2,1)$ while $\rho - \sigma_1(\rho) = \alpha_1$, so $(f_1,f_2) = (1,0)$. Coincidentally, on A_2 with $n = 4$ (a unique model case), one obtains the same 6 integer lattice points in terms of (k_1,k_2) and (f_1,f_2), but this phenomenon does not persist to higher rank.

Proof We prove this by induction on the length of w. If w is the identity the result is clear (with $\eta(e,n,r,\mathbf{k})=0$). Suppose that the result is proved for w and that $\ell(\sigma_i w)=\ell(w)+1$. Let $\sigma_i(\mathbf{k})=\mathbf{m}$. By Corollary 1, the hypothesis of Theorem 1 is satisfied. Thus by this result, we have

$$\tau_{n,r}(\mathbf{m})=q^{i-r/2-1+\delta(i,r,\mathbf{k})}g_{1+k_i}\tau_{n,r}(\mathbf{k}).$$

By Proposition 1, $g_{1+k_i}=g_{d(w^{-1}\alpha_i)}$. Moreover, under the assumption that $\ell(\sigma_i w)=\ell(w)+1$, it follows that $w^{-1}\alpha_i\in\Phi^+$, so $d(w^{-1}\alpha_i)>0$. (See, for example, Bump [7], Propositions 21.2 and 21.10.). Thus by elementary properties of Gauss sums,

$$g_{d(w^{-1}\alpha_i)}=q^{1-d(w^{-1}\alpha_i)}g(p^{d(w^{-1}\alpha_i)-1},p^{d(w^{-1}\alpha_i)}).$$

So we arrive at the formula

$$\tau_{n,r}(\mathbf{m})=q^{i-r/2+\delta(i,r,\mathbf{k})-d(w^{-1}\alpha_i)}g(p^{d(w^{-1}\alpha_i)-1},p^{d(w^{-1}\alpha_i)})\tau_{n,r}(\mathbf{k}). \tag{7}$$

On the other hand, it is well-known (see, for example, Bump [7], Proposition 21.10) that

$$\Phi(\sigma_i w)=\Phi(w)\cup\{w^{-1}\alpha_i\}.$$

Thus (4) implies that

$$H(p^{m_1},\dots,p^{m_r})=g(p^{d(w^{-1}\alpha_i)-1},p^{d(w^{-1}\alpha_i})\,H(p^{k_1},\dots,p^{k_r}). \tag{8}$$

Comparing (7) and (8), the Theorem follows.

To give the precise power of q, suppose that $w=\sigma_{j_c}\sigma_{j_{c-1}}\dots\sigma_{j_1}$ is a reduced word for w, so $c=\ell(w)$. Let $\mathbf{k}^{(0)}=(0,\dots,0)$ and $\sigma_{j_i}(\mathbf{k}^{(i-1)})=\mathbf{k}^{(i)}$, $1\le i\le c$. Also let $\tau_1=1$ and $\tau_t=\sigma_{j_{t-1}}\sigma_{j_{t-2}}\dots\sigma_{j_1}$ for $1<t\le c$. Then applying (7) repeatedly, we find that

$$q^{\eta(w,n,r,\mathbf{k})}=q^{-r\ell(w)/2}\prod_{t=1}^{\ell(w)}q^{j_t+\delta(j_t,r,\mathbf{k}^{(t-1)})-d(\tau_t^{-1}\alpha_{j_t})}. \tag{9}$$

□

Next we turn to the case $n=r+1$. This equality implies that the Whittaker model of the theta representation is unique (see Kazhdan-Patterson [14], Corollary I.3.6 for the local uniqueness and Theorem II.2.5 for its global realization). To describe the corresponding Whittaker coefficients in terms of multiple Dirichlet series, we first describe the orbit of the origin under W. As noted above, the stabilizer of the origin is non-trivial. Indeed, we have

Lemma 1 *Suppose $n = r + 1$. Then $\sigma_1\sigma_2\cdots\sigma_r((0,\ldots,0)) = (0,\ldots,0)$.*

Proof The proof is a straightforward calculation, left to the reader. $\square$

Since the stabilizer of the origin is nontrivial, let us restrict the action of W on r-tuples to the subgroup generated by the transpositions σ_i, $1 \le i < r$. We will denote this group $\mathfrak{S}_r$; note that $\mathfrak{S}_r$ is isomorphic to the symmetric group S_r, but the action of $\mathfrak{S}_r$ on r-tuples is *not* the standard permutation action.

Lemma 2 *Suppose $n = r + 1$. Then the stabilizer in $\mathfrak{S}_r$ of $(0,\ldots,0)$ is trivial.*

Proof In this proof (and in the proof of Theorem 3 below), we write W_r instead of W for the Weyl group of type A_r. W_r acts on $\mathbf{K}_r$ by the action given in (5). Observe that under the projection π from $\mathbf{K}_r$ to $\mathbf{K}_{r-1}$ obtained by forgetting the last coordinate, the action of $\mathfrak{S}_r$ on $\mathbf{K}_r$ restricts to the action of the Weyl group W_{r-1} on $\mathbf{K}_{r-1}$. Indeed, this is true since the actions on the first $r-1$ entries are the same; note that changing the r-th entry of an element of $\mathbf{K}_r$ does not affect its image under $\pi \circ \sigma_i$ for $\sigma_i \in \mathfrak{S}_r$. Then the Lemma follows at once from Corollary 2, which applies as $n > (r-1)+1$. $\square$

Combining these, we may describe the orbit of the origin.

Proposition 2 *Suppose $n = r + 1$. Then the stabilizer in W of the origin has order $r+1$ and is the group generated by the element $\sigma_1\sigma_2\cdots\sigma_r$. The orbit of the origin under W has order $r!$, and every point in the orbit may be described uniquely as $w((0,\ldots,0))$ for some $w \in \mathfrak{S}_r$.*

Proof Since $\sigma_1\sigma_2\cdots\sigma_r$ has order $r+1$, the stabilizer $W^{(0,\ldots,0)}$ of the origin in W has order at least $r+1$. Hence $[W : W^{(0,\ldots,0)}] \le r!$. But by Lemma 2, the image of W has order at least $r!$. Since the cardinality of this image is exactly $[W : W^{(0,\ldots,0)}]$, equality must obtain, and the Proposition follows. $\square$

Finally, we give the analogue of Theorem 2 when $n = r + 1$. The link is once again between theta Whittaker coefficients and stable Weyl group multiple Dirichlet series coefficients, but this time the latter are of type A_{r-1} rather than type A_r.

Theorem 3 *Suppose that $n = r + 1$. Let $w \in \mathfrak{S}_r$, and set $w((0,\ldots,0)) = \mathbf{k}$, $\rho - w\rho = \sum f_i\alpha_i$. Then*

$$\tau_{r+1,r}(\mathbf{k}) = q^{\eta(w,r+1,r,\mathbf{k})}\, H(p^{f_1},\ldots,p^{f_{r-1}}),$$

where the coefficient H is the coefficient of the type A_{r-1} multiple Dirichlet series, and the function $\eta(w,r+1,r,\mathbf{k})$ is given by (9) above.

Remark 2 Note that since $n > r-1$, the coefficients H are stable, and account for the full set of nonzero Weyl group multiple Dirichlet series coefficients for A_{r-1}. See [3]. Also, if $w \in \mathfrak{S}_r$ and $\rho - w\rho = \sum_{i=1}^r f_i\alpha_i$, then necessarily $f_r = 0$, so the restriction to $(r-1)$-tuples in the right-hand side of the Theorem is natural. In addition, one can check that

$$\eta(\sigma_1\sigma_2\cdots\sigma_r, r+1, r, (0,\ldots,0)) = 0,$$

and that one can use any $w \in W$ to reach $\mathbf{k}$ in the orbit of $(0,\cdots,0)$ in order to compute the coefficient $\tau_{r+1,r}(\mathbf{k})$. (Doing so one obtains each coefficient $r+1$ times.)

Proof The Weyl group of type A_r, W_r, acts on its root system Φ and on $\mathbf{K}_r$. These actions each restrict: the subgroup $\mathfrak{S}_r$ acts on

$$\Phi_{r-1} = \{\alpha \in \Phi \mid \alpha = \sum_{i=1}^{r-1} m_i\alpha_i \text{ for some } m_i \in \mathbb{Z}\},$$

and, as noted in the proof of Lemma 2 above, it also acts on $\mathbf{K}_{r-1}$. These actions are each compatible with the isomorphism $\mathfrak{S}_r \cong W_{r-1}$. Thus, we may follow the argument given in the proof of Theorem 2. However, in that case we obtain the $r!$ stable coefficients of the type A_{r-1} Weyl group multiple Dirichlet series. (Note that these coefficients are a subset of the $(r+1)!$ stable coefficients of type A_r.) $\square$

In concluding, we note that one can ask if theorems that are similar to Theorems 2 and 3 hold for metaplectic covers of the adelic points other reductive groups. The theory of theta functions, that is, residues of Eisenstein series on metaplectic covers, is not yet fully established when the underlying group in question is not a general linear group. However, we do expect that it can be developed using methods similar to those of [14], and that the link between the stable Weyl group multiple Dirichlet series and the Whittaker coefficients determined by Hecke theory persists. Indeed, Brubaker and Friedberg have carried out computations of Hecke operators on the four and five-fold covers of $GSp(4)$, following the approach of T. Goetze [13]. Under reasonable hypotheses about the periodicity relation (which should vary depending on root lengths for simple roots) for theta coefficients for those groups, such a link once again holds in those cases.

References

[1] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series I. In: *Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory*, Proc. Sympos. Pure Math., 75. Amer. Math. Soc., Providence, R.I., 2006, pp. 91-114.

[2] B. Brubaker, D. Bump, and S. Friedberg. Twisted Weyl group multiple Dirichlet series: the stable case. In: *Eisenstein series and applications (Gan, Kudla and Tschinkel, eds.)*. Birkhäuser Progress in Mathematics 258, 2008, pp. 1-26.

[3] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series. II. The stable case. *Invent. Math.*, 165(2):325–355, 2006.

[4] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r. *Annals of Mathematics*, 166:293–316, 2007.

[5] B. Brubaker, D. Bump, and S. Friedberg. Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory. To appear as a volume in Annals of Mathematics Studies, Princeton Univ. Press.

[6] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Annals of Mathematics*, to appear.

[7] D. Bump. Lie Groups, Graduate Texts in Mathematics 225. Springer-Verlag, New York, 2004.

[8] D. Bump and J. Hoffstein. On Shimuras correspondence. *Duke Math. J.*, 55(3):661–691, 1987.

[9] P. Deligne. Sommes de Gauss cubiques et revêtements de SL(2) [d'après S. J. Patterson]. *Sém. Bourbaki (1978/79), Exposé No. 539.* In: Springer Lecture Notes 770, 1980, pp. 244-277.

[10] C. Eckhardt and S. J. Patterson. On the Fourier coefficients of biquadratic theta series. *Proc. London Math. Soc. (3)*, 64(2):225–264, 1992.

[11] S. Friedberg and P. McNamara, in preparation.

[12] D. R. Heath-Brown, and S. J. Patterson, The distribution of Kummer sums at prime arguments. J. Reine Angew. Math. 310 (1979), 111–130.

[13] J. Hoffstein. Eisenstein series and theta functions on the metaplectic group. In: *Theta functions: from the classical to the modern (M. Ram Murty, ed.).* CRM Proc. Lect. Notes 1, 1993, pp. 65-104.

[14] T. Goetze. Euler products associated to metaplectic automorphic forms on the 3-fold cover of $GSp(4)$. *Trans. A.M.S.*, 350(3):975–1011, 1998.

[15] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[16] T. Kubota, T. Some results concerning reciprocity law and real analytic automorphic functions. 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), pp. 382–395. Amer. Math. Soc., Providence, R.I., 1971.

[17] H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup.*, (4)2:1–62, 1969.

[18] P. McNamara. Metaplectic Whittaker functions and crystal bases. *Duke Math. J.*, to appear.

[19] T. Suzuki, Some results on the coefficients of the biquadratic theta series. J. Reine Angew. Math. 340 (1983), 70–117.

Dept. of Mathematics, MIT, Cambridge, MA 02139-4307
brubaker@math.mit.edu
Dept. of Mathematics, Stanford University, Stanford, CA 94305-2125
bump@math.stanford.edu
Dept. of Mathematics, Boston College, Chestnut Hill, MA 02467-3806
friedber@bc.edu
Dept. of Mathematics, Brown University, Providence, RI 02912-9032
jhoff@math.brown.edu