Lifting Automorphic Representations on the Double Covers of Orthogonal Groups

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Abstract

Suppose that G and H are connected reductive groups over a number field F and that an L-homomorphism $\rho : {}^LG \longrightarrow {}^LH$ is given. The Langlands functoriality conjecture predicts the existence of a map from the automorphic representations of $G(\mathbb{A})$ to those of $H(\mathbb{A})$. If the adelic points of the algebraic groups G, H are replaced by their metaplectic covers, one may hope to specify an analogue of the L-group (depending on the cover), and then one may hope to construct an analogous correspondence. In this paper we construct such a correspondence for the double cover of the split special orthogonal groups, raising the genuine automorphic representations of $\widetilde{\mathrm{SO}}_{2k}(\mathbb{A})$ to those of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. To do so we use as integral kernel the theta representation on odd orthogonal groups constructed by the authors in a previous paper [3]. In contrast to the classical theta correspondence, this representation is not minimal in the sense of corresponding to a minimal coadjoint orbit, but it does enjoy a smallness property in the sense that most conjugacy classes of Fourier coefficients vanish.

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1 Introduction

Let G and H be reductive groups and let $\mathbb{A}$ be the ring of adeles of a given global field. Given an L-homomorphism $\rho :^L G \longrightarrow^L H$, the Langlands correspondence predicts the existence of a map from the automorphic representations of $G(\mathbb{A})$ to those of $H(\mathbb{A})$. In the case that adelic points of the algebraic groups G, H are replaced by their covers, the results of Savin [13] suggest that one may specify an analogue of the L-group (depending on the cover), and then one may still expect the existence of a correspondence of automorphic representations. A first example is the Shimura correspondence.

It is not expected that the principle of functoriality works perfectly in such a context. For example, we know that the metaplectic double cover the genuine Iwahori-Hecke algebra of $\widetilde{\mathrm{SL}}_2$ is isomorphic to the Iwahori-Hecke algebra of PGL_2. Thus if the L-group formalism is extended to this context, their L-groups should be the same. This does not mean that the Shimura correspondence is a perfect bijection between automorphic representations of the two groups, since Waldspurger [15] proved that an automorphic representation π of PGL_2 is a Shimura lift if and only if $L(\frac{1}{2}, \pi) \neq 0$.

Moreover, a proper generalization of the principle of functoriality to metaplectic groups will require at least a discussion of quasisplit forms. As far as we know this has not been done. The results of Savin [13] are for split forms.

With these caveats, it may be useful to tentatively define an L-group for metaplectic groups. Let G be a reductive algebraic group defined over a ground field F containing sufficiently many roots of unity, and let $\tilde{G}^{(n)}$ denote a corresponding metaplectic n-fold cover. We would like to define

${}^L\tilde{G}^{(n)}$ to be a complex analytic group such that (if F is p-adic) the semisimple conjugacy classes of ${}^L\tilde{G}^{(n)}$ parametrize the irreducible representations of $\tilde{G}^{(n)}(F)$ that are spherical. (We are considering the connected L-group only in this assertion.) One would then have, when SO_m denotes a split orthogonal group:

$$ {}^L\widetilde{\mathrm{SO}}_{2k+1}^{(n)} \cong \begin{cases} \mathrm{Sp}_{2k}(\mathbb{C}) & \text{if } n \text{ is odd;} \\ \mathrm{SO}_{2k+1}(\mathbb{C}) & \text{if } n \text{ is even,} \end{cases} $$

while ${}^L\widetilde{\mathrm{SO}}_{2k}^{(n)} \cong \mathrm{SO}_{2k}(\mathbb{C})$ regardless of the parity of n.

From this point on, SO_k will always denote a split orthogonal group and $\widetilde{\mathrm{SO}}_k$ will denote its metaplectic double cover, whose definition is given in [3] and reviewed briefly in Section 1. We note that the existence of this cover requires that the ground field contain the fourth roots of unity. Matsumoto proved that one could construct a metaplectic n-fold cover of split semisimple simply-connected groups, but if the group is not simply connected – as in the case of orthogonal groups – then more roots of unity may be required.

Savin's results suggest that the L-group of $\widetilde{\mathrm{SO}}_k$ is just $\mathrm{SO}_k(\mathbb{C})$, and correponding to the inclusion of $\mathrm{SO}_k(\mathbb{C})$ in $\mathrm{SO}_{k+1}(\mathbb{C})$ one should be able to construct "functorial" liftings from genuine automorphic representations of $\widetilde{\mathrm{SO}}_k$ to $\widetilde{\mathrm{SO}}_{k+1}$. In this paper we construct such a map by means of a theta integral, and verify in a weak sense that it is functorial. More precisely, at any place where the representation of $\widetilde{\mathrm{SO}}_k$ is unramified, if the induction data are in general position, then we show that the lifted representation agrees with the functorial lift.

The classical theta correspondence is obtained by using as integral kernel the theta function on the symplectic group obtained from the Weil representation. The corresponding representation is minimal in the sense of being attached to a minimal coadjoint orbit. Though, as was shown by Vogan [14], there is in fact no minimal representation on odd orthogonal groups beyond SO_7, the authors in [3] established the existence of a representation which, though not minimal, was small, in the sense that most conjugacy classes of Fourier coefficients vanished (see Proposition 2). Globally this space was obtained as the residues of certain metaplectic Eisenstein series. In this paper we use the functions of this theta representation as the kernels for a family of theta lifts. We show that this construction enjoys many of the same properties as the classical theta lift. In particular, in Section 3 we show that this theta lift satisfies a tower property, so that the first nonzero theta lift is cuspidal. In Section 4 we study the nonvanishing of the lift, and show that a genuine cuspidal automorphic representation on $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ must lift nontrivially to $\widetilde{\mathrm{SO}}_{8k}(\mathbb{A})$. In Section 5 we refine these results for generic representations, and we compute the Whittaker model of the lift.

Finally, in Section 6 we study the unramified correspondence, computing the Langlands parameters of the lift from $\widetilde{\mathrm{SO}}_{2k}$ to $\widetilde{\mathrm{SO}}_{2k+1}$, effectively showing that it is functorial. We analyze quotients of the restriction of the theta representation of $\widetilde{\mathrm{SO}}_{4k+1}$ to $\widetilde{\mathrm{SO}}_{2k} \times \widetilde{\mathrm{SO}}_{2k+1}$. The general flavor of this result is similar to Kudla [11], in which the irreducible quotients of the restriction of the usual Weil representation to a dual reductive pair are studied.

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2 Preliminaries

We start by fixing some notations. Let SO_l denote the split special orthogonal group on an l dimensional space. All orthogonal groups in this paper will be represented with respect to the $l \times l$ matrix

$$J_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} .$$

The maximal unipotent subgroup of SO_l contains n simple roots, where $n = [l/2]$. Let $e_{i,j}$ denote the $l \times l$ matrix with one in the (i,j)-th entry and zero elsewhere. Let α_i $(1 \leqslant i \leqslant n)$ denote the simple roots in the usual order with respect to the standard Borel subgroup of upper triangular matrices. The corresponding one-parameter subgroups are $r \longmapsto x_{\alpha_i}(r)$ where

$$x_{\alpha_i}(r) = \exp(r(e_{i,i+1} - e_{l-i,l-i+1}))$$

if $l = 2n+1$, and

$$x_{\alpha_i}(r) = \begin{cases} \exp(r(e_{i,i+1} - e_{l-i,l-i+1})) & \text{if } 1 \leqslant i < n \\ \exp(r(e_{n-1,n+1} - e_{n,n+2}) & \text{if } i = n \end{cases}$$

if $l = 2n$. We shall denote by w_i the simple reflection corresponding to the simple root α_i.

We shall always assume that the ground field F (which may be local or global) contains four distinct fourth roots of unity. If F is global, let $\mathbb{A}$ denote its adele ring. Let $\widetilde{\mathrm{SO}}_l(F)$ (if F is local) or $\widetilde{\mathrm{SO}}_l(\mathbb{A})$ (if F is global) denote the metaplectic double cover, which is defined and studied in [3]. We recall that although $\widetilde{\mathrm{SO}}_l$ is actually a *double* cover it contains a central subgroup μ_4 of order four which we identify with the fourth roots of unity. We recall from [3] that a representation ρ of any subgroup of $\widetilde{\mathrm{SO}}_{2n+1}(F)$ which contains the embedded group μ_4 of $\widetilde{\mathrm{SO}}_{2n+1}(F)$ is called *genuine* if $\rho(\varepsilon g) = \varepsilon\rho(g)$, where we have fixed an injection $\mu_4 \longrightarrow \mathbb{C}^\times$, and by abuse of notation identify ε with its image in $\mathbb{C}^\times$. Most representations which we will consider are genuine.

For any two natural numbers $2k+1$ and $2m$ we embed the orthogonal groups SO_{2k+1} and SO_{2m} in $\mathrm{SO}_{2k+2m+1}$ as follows:

$$(h,g) \hookrightarrow \begin{pmatrix} a & 0 & b \\ 0 & g & 0 \\ c & 0 & d \end{pmatrix}, \qquad g \in \mathrm{SO}_{2k+1}, h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_{2m}\,. \tag{1}$$

Let π denote an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. If $\theta_{2k+2m+1}$ is any genuine automorphic representation on $\widetilde{\mathrm{SO}}_{2k+2m+1}$ we consider the functions

$$\tilde{f}(h) = \int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(h,g)}\, dg\ . \tag{2}$$

Here $\varphi_\pi(g)$ denotes a general vector in the space of π and $\theta_{2k+2m+1}(r)$ denotes a general function in the space of $\theta_{2k+2m+1}$. We are writing $\mathrm{SO}_{2k+1}(\mathbb{A})$ instead of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ because the product of φ_π and $\bar{\theta}_{2k+2m+1}$ is not genuine. This integral defines a mapping from the irreducible cuspidal genuine automorphic representations on the group $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ to the genuine automorphic representations on $\widetilde{\mathrm{SO}}_{2m}(\mathbb{A})$. We shall denote the image representation by $\theta_{2k+2m+1}(\pi)$.

In a similar way one can construct a mapping from the irreducible cuspidal genuine automorphic representations on $\widetilde{\mathrm{SO}}_{2m}(\mathbb{A})$ to the genuine automorphic representations on $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$.

In [3] we introduced and studied the properties of what we refer to as the theta representation on $\widetilde{\mathrm{SO}}_{2k+2m+1}$. This is an automorphic representation obtained as a residue of an Eisenstein series which is small in a certain sense. In that paper we denoted this representation by θ. Since we will vary the number m we henceforth write $\theta_{2k+2m+1}$ for θ. Fixing the number k and letting m vary, the integral (2) defines a "tower" of liftings. In the next Sections we will study the properties of this tower.

We now recall two of the main results in [3] which we will need for our computations. In Proposition 1 below, some notations are as in [3]. In particular, $\widetilde{\mathrm{GL}}_r(F)$ is the cover induced on $\mathrm{GL}_r(F)$ from the cover $\widetilde{\mathrm{SO}}_{2k+2m+1}(F)$ by its inclusion as the Levi factor of $\mathrm{SO}_{2r+1}(F)$ in the standard Siegel parabolic subgroup. It is a metaplectic double cover in the sense of Kazhdan and Patterson [10], and the representation Θ which appears in Proposition 1 is an exceptional representation of $\widetilde{\mathrm{GL}}_r(F)$ in the sense of Kazhdan and Patterson [10]. We refer to the discussion in [3], page 1370 for the precise descriptions of $\widetilde{\mathrm{GL}}_r(F)$ and its representation Θ.

Proposition 1 *Let F be a nonarchimedean local field, and let $\theta_{2k+2m+1}$ be the local theta representation of $\widetilde{\mathrm{SO}}_{2k+2m+1}(F)$. Let $P_r = (\mathrm{GL}_r \times \mathrm{SO}_{2(k+m-r)+1})U$ be a maximal parabolic subgroup of $\mathrm{SO}_{2k+2m+1}$. Then as a representation of $\widetilde{\mathrm{GL}}_r(F) \times \widetilde{\mathrm{SO}}_{2(k+m-r)}(F)$, the Jacquet module with respect to U is isomomorphic to $\Theta \otimes \theta_{2(k+m-r)+1}$, where Θ is a theta representation of $\widetilde{\mathrm{GL}}_r(F)$.*

This is Theorem 2.3 of [3]. A global statement should be true: on the adele group it should be true that as a function of (h_1, h_2) the integral

$$\int_{U(F)\backslash U(\mathbb{A})} \theta_{2k+2m+1}(u(h_1, h_2))\, du$$

is in the space of the automorphic representation $\Theta \otimes \theta_{2(k+m-r)+1}$ where Θ is the theta function on the double cover of GL_r. This statement is Conjecture 3.3 of [3], and it is proved there if $r = 1$. The local statement is sufficient for our purposes. The most important property for us of Θ is that it does not have a Whittaker model if $r \geqslant 3$.

The uniportent conjugacy classes of SO_{2n+1} are parametrized by partitions of $2n+1$ in which each even part occurs an even number of times. By abuse of notation we will identify a unipotent class with the corresponding partition. See [3], Section 4 and Collingwood and McGovern [5] for this parametrization, and for the partial order on the classes.

In [3] Section 4, a connection between unipotent conjugacy classes and Fourier coefficients is explained. Given a unipotent class, a set of Fourier coefficients is defined by (4.5) of [3]. The description of V_2^n in that formula is somewhat lengthy so we assume familiarity with [3] regarding this point.

Let $\mathcal{O}(\theta_{2k+2m+1})$ denote $(2^{2n}1)$ if $k+m=2n$ and let $\mathcal{O}(\theta_{2k+2m+1})=(2^{2n}3)$ if $k+m=2n+1$. The vanishing properties of the Fourier coefficients of the theta representation are described as follows.

Proposition 2 ([3], Theorem 4.2 (i)) *If $\mathcal{O}$ is any unipotent conjugacy class which is greater than or not comparable to $\mathcal{O}(\theta_{2k+2m+1})$ in the partial order, then all Fourier coefficients of $\theta_{2k+2m+1}$ with respect to $\mathcal{O}$ are zero.*

We will also need a couple of local consequences of the smallness of the θ representations. For the remainder of the section, F will be a nonarchimedean local field. Let $U=U_{2k+1}$ denote the unipotent radical of the standard parabolic subgroup of SO_{2k+1} with Levi factor $\mathrm{GL}_1\times\mathrm{SO}_{2k-1}$. By abuse of notation we will write U for $U(F)$ in the remainder of this section. If $r\in F^{2k-1}$, then writing a typical element of U_{2k+1} as

$$u_{2k+1}=\begin{pmatrix}1 & u & *\\ & I_{2k-1} & *\\ & & 1\end{pmatrix},\qquad u\in F^{2k-1},$$

every character of U_{2k+1} has the form $\psi_r(u)=\psi(\langle r,u\rangle)$ where if $r\in F^{2k-1}$, $\langle r,u\rangle$ denotes the inner product of r with the vector u, with respect to the split quadratic form having the matrix $\begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

Proposition 3 *Let r be a vector of nonzero length in F^{2k-1}. Let $U=U_{2k+1}$. Then the twisted Jacquet module of θ_{2k+1} with respect to the character ψ_r of U vanishes.*

Proof This is similar to Theorem 2.6 of [3], except that if the length of r is not a square, the stabilizer of ψ is not the split SO_{2k-2}, but the quasisplit one. The arguments of [3] must be repeated for this group. We omit the details, which are long but similar to [3]. $\square$

Proposition 4 *Let $\theta=\theta_{2k+1}$, where $k\geqslant 3$. Let $U=U_{2k+1}$ and let ψ_U be the character of U defined by $\psi_U(u)=\psi(u_{12})$. Let V be the unipotent radical of the parabolic subgroup with Levi factor $\mathrm{GL}(2)\times\mathrm{SO}_{2k-3}$. Then the twisted Jacquet module θ_{U,ψ_U} is a quotient of θ_V. In other words, the kernel of the natural map $\theta\longrightarrow\theta_{U,\psi_U}$ contains the kernel of $\theta\longrightarrow\theta_V$.*

Proof We embed $U_{2k-1} \longrightarrow \mathrm{SO}_{2k-1} \longrightarrow \mathrm{SO}_{2k+1}$ with $\mathrm{GL}_1 \times \mathrm{SO}_{2k-1}$ being the Levi factor of the standard parabolic subgroup having U_{2k+1} as its unipotent radical. Thus $V \subset U_{2k+1}U_{2k-1}$ and what we must show is that U_{2k-1} acts trivially on θ_{U,ψ_U}. If not, then there is a nontrivial Jacquet module with respect to some nontrivial character ψ_r of U_{2k-1}, where r is a vector in F^{2k-3}.

So assume that $r \neq 0$ and the Jacquet module $\theta_{U_{2k+1}U_{2k-1},\psi_U\psi_r} \neq 0$. There are two cases. First, suppose that r has nonzero length. Then we may conjugate $U_{2k+1}U_{2k-1}$ by the Weyl element w_2 which is the simple reflection interchanging the first two rows of $U_{2k+1}U_{2k-1}$. We disregard everything but the first row. We see that θ has a nonzero Jacquet module with respect to the following unipotent subgroup and character:

$$\begin{pmatrix} 1 & 0 & u & * & * \\ & 1 & 0 & 0 & * \\ & & I_{2k-3} & 0 & * \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \longmapsto \psi(\langle r,u\rangle).$$

Now for some $a \in F$ there will be a nonzero Jacquet module for U_{2k+1} with the character

$$\begin{pmatrix} 1 & x & u & * & * \\ & 1 & 0 & 0 & * \\ & & I_{2k-3} & 0 & * \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \longmapsto \psi(ax)\psi(\langle r,u\rangle).$$

This is the character parametrized by the vector $(a, r, 0) \in F^{2k+1}$ and since $\langle r, r\rangle \neq 0$, no matter what a is the length of this vector is nonzero, and we now have a contradiction to Proposition 3.

Therefore we must have $\langle r, r\rangle = 0$. Using GL_1 and the middle SO_{2k-3}, we may move the character and assume that $r = (1, 0, \cdots, 0)$, and we now have a zero twisted Jacquet module with respect to the character $\psi_U\psi_r$ of

$U_{2k+1}U_{2k-1}$. This is the character

$$\begin{pmatrix} 1 & u_{12} & * & * & \cdots & * & * \\ & 1 & u_{23} & * & \cdots & * & * \\ & & 1 & 0 & 0 & \vdots & \vdots \\ & & & I_{2k-5} & 0 & * & * \\ & & & & 1 & u_{12} & * \\ & & & & & 1 & u_{23} \\ & & & & & & 1 \end{pmatrix} \longmapsto \psi(u_{12}+u_{23}).$$

Now we take the Jacquet module with respect to all characters of U_{2k-3}. Some Jacquet module must be nontrivial. It cannot be with respect to the trivial character, since then the character $\psi_U\psi_r$ would be trivial on the unipotent radical of the standard parabolic subgroup with Levi factor $\mathrm{GL}(3)\times\mathrm{SO}_{2k-5}$, which affords the theta representation of $\widetilde{\mathrm{GL}}(3)$ by Proposition 1. This $\psi_U\psi_r$ would then induce a Whittaker model on the theta representation, but this representation has no Whittaker model. Therefore the character of U_{2k-3} must be nonzero. Writing it as $\psi_{r'}$ where $r'\in F^{2k-5}$, if r' has nonzero length we may argue as we did previously, using a Weyl group element to move it to the first row. We then obtain a nonzero Jacquet module with respect to the following unipotent subgroup and character:

$$\begin{pmatrix} 1 & 0 & u & * & * \\ & I_2 & 0 & 0 & * \\ & & I_{2k-5} & 0 & * \\ & & & I_2 & 0 \\ & & & & 1 \end{pmatrix} \longmapsto \psi(\langle r',u\rangle).$$

The argument is as before; for some $a,b\in F$ there will be a nonzero Jacquet module for U_{2k+1} with the character

$$\begin{pmatrix} 1 & x & u & * & * \\ & I_2 & 0 & 0 & * \\ & & I_{2k-5} & 0 & * \\ & & & I_2 & x \\ & & & & 1 \end{pmatrix} \longmapsto \psi(ax_1+bx_2)\psi(\langle r',u\rangle), \qquad x=(x_1,x_2),$$

but no matter what a and b are we get a contradiction to Proposition 3. Thus r' has length zero, and as before we may move it to the $3,4$ position.

Proceeding in this way, we eventually obtain a nonzero Jacquet functor for the Gelfand-Graev character of the maximal unipotent radical of θ, a contradiction since it has no Whittaker model. $\square$

3 The Cuspidality Tower

In this Section we will study the cuspidality property of the tower of lifting introduced in (2). We will prove

Theorem 1 *Let π be a cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose the lift $\theta_{2k+2m-1}(\pi)$ is zero. Then the lift $\theta_{2k+2m+1}(\pi)$ is a cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2m}(\mathbb{A})$.*

Let $U_{i,2k+2m+1}$ denote the unipotent subgroup of $\mathrm{SO}_{2k+2m+1}$ consisting of all matrices of the form

$$U_{i,2k+2m+1} = \left\{ \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ & 1 & u & * & 0 \\ & & I_{2(k+m-i)+1} & * & 0 \\ & & & 1 & 0 \\ & & & & I_{i-1} \end{pmatrix} \right\} \tag{3}$$

where $*$ denotes whatever is needed to guarantee that the matrix is orthogonal. It is clear that $U_{i,2k+2m+1}$ is an abelian group. Given an additive character ψ of the group $F\backslash\mathbb{A}$ define a character ψ_1 of $U_{1,2k+2m+1}$ as follows. If $u = (u_{i,j}) \in U_{1,2k+2m+1}$ then set $\psi_1(u) = \psi(u_{1,2})$. We start with

Lemma 1 *The function*

$$f(z) = \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \theta_{2k+2m+1}(uz)\,\psi_1(u)\,du$$

is left-invariant under the adelic points of $U_{2,2k+2m+1}$. In other words, $f(z) = f(vz)$ for all $v \in U_{2,2k+2m+1}(\mathbb{A})$.

Proof We expand $f(z)$ along the group $U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})$. The group $\mathrm{SO}_{2k+2m-3}(F)$ which is embedded in $\mathrm{SO}_{2k+2m+1}(F)$ as in (1) acts on the characters of $U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})$ with three types of orbits. First we have the orbits whose stabilizers are given by a quasi-split even orthogonal group $\mathrm{SO}_{2k+2m-4}$. The contributions to $f(z)$ from these orbits are integrals of the form

$$\int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})}\int_{U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})} \theta_{2k+2m+1}(uvz)\,\psi_1(u)\,\psi_2(v)\,dv\,du. \tag{4}$$

Here $\psi_2(v)=\psi(v_{2,k+m}+av_{2,k+m+2})$ where v is parametrized as in (3) and where $a\in F^\times$. However this Fourier coefficient corresponds to the unipotent class $\mathcal{O}=(51^{2k+2m-4})$ and hence by Proposition 2 this integral is zero.

Next, in the Fourier expansion of $f(z)$ along $U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})$ we consider the contribution from the nonzero isotropic vectors. In other words we have the contribution from

$$\int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})}\int_{U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})} \theta_{2k+2m+1}(uvz)\,\psi_1(u)\,\tilde{\psi}_2(v)\,dvdu \tag{5}$$

where $\tilde{\psi}_2(v)=\psi(v_{2,3})$. Now we continue by expanding this integral along

$$U_{3,2k+2m+1}(F)\backslash U_{3,2k+2m+1}(\mathbb{A}).$$

As in (4) one sees that the contribution coming from the big orbit is zero. We claim that the constant term in this case is also zero. In other words we claim that

$$\int \theta_{2k+2m+1}(uvrz)\psi_1(u)\tilde{\psi}_2(v)\,dv\,du\,dr=0$$

for all choices of data. Here r is integrated over $U_{3,2k+2m+1}(F)\backslash U_{3,2k+2m+1}(\mathbb{A})$ and u and v are integrated as before. To see that this integral is zero, notice that

$$L=U_{1,2k+2m+1}U_{2,2k+2m+1}U_{3,2k+2m+1}$$

is the unipotent radical of the parabolic subgroup of $\mathrm{SO}_{2k+2m+1}$ whose Levi part is $\mathrm{GL}_1^3\times\mathrm{SO}_{2k+2m-5}$. Hence we can write the above integral as

$$\int_{L(F)\backslash L(\mathbb{A})}\theta_{2k+2m+1}(lz)\,\psi_L(l)\,dl$$

where if $l=(l_{i,j})\in L$ then $\psi_L(l)=\psi(l_{1,2}+l_{2,3})$. This integral is a Whittaker coefficient of the constant term with respect to a maximal parabolic subgroup with Levi factor $\mathrm{GL}_3 \times \mathrm{SO}_{2k+2m-5}$. At any nonarchimedean place, this integral factors through the corresponding Jacquet module, which has no Whittaker model by Proposition 1, and so this integral is zero.

Thus in (5) we are left with the contribution which comes from the nonzero isotropic vectors. In other words, (5) is a sum of integrals of the type

$$\int \theta_{2k+2m+1}(uvrz)\, \psi_1(u)\, \tilde{\psi}_2(v)\, \tilde{\psi}_3(r)\, dv\, du\, dr$$

where $\tilde{\psi}_3(r) = \psi(r_{3,4})$. Continue by induction. We eventually obtain either the Whittaker coefficient of the maximal unipotent radical of $\mathrm{SO}_{2k+2m+1}$, which is zero by Proposition 2, or we get a Whittaker coefficient on the double cover of GL_{k+m}, and since $k+m>2$ this vanishes by applying Proposition 1 at any nonarchimedean place. Hence the above integral is zero and so is the integral (5). This shows that the contribution to Fourier expansion of $f(z)$ which comes from the nonzero isotropic vectors is also zero. Thus we are left with the constant term. But this just means that $f(z) = f(vz)$ for all $v \in U_{2,2k+2m+1}(\mathbb{A})$. $\square$

We may extend this Lemma as follows. Let $R_{2j-1} = \prod_{i=1}^{2j-1} U_{i,2k+2m+1}$. Define a character ψ_{2j-1} of R_{2j-1} by $\psi_{2j-1}(r) = \psi(r_{1,2} + r_{3,4} + \cdots + r_{2j-1,2j})$. Then a similar argument gives

Corollary 1 *The function*

$$f(z) = \int_{R_{2j-1}(F)\backslash R_{2j-1}(\mathbb{A})} \theta_{2k+2m+1}(rz)\, \psi_{2j-1}(r)\, dr$$

is left-invariant under the adelic points of $U_{2j,2k+2m+1}(\mathbb{A})$.

Next we prove

Proposition 5 *Suppose that* $\theta_{2k+2m+1}(\pi) = 0$. *Then* $\theta_{2k+2m-1}(\pi) = 0$.

Proof By assumption, the integral (2) is zero for all choices of data. Let V denote the unipotent radical of the maximal parabolic subgroup of SO_{2m} which preserves a line. Then the integral

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(v,g)}\, dv\, dg \qquad (6)$$

is zero for all choices of data. With the group V embedded inside $\mathrm{SO}_{2k+2m+1}$ via the embedding given in (1), we have $V \subset U_{1,2k+2m+1}$, and the quotient V $U_{1,2k+2m+1}$ may be identified with the subgroup of orthogonal matrices of the form

$$\begin{pmatrix} 1 & & {}^t u & & -\frac{1}{2}\langle u, u\rangle \\ & I_{m-1} & & & \\ & & I_{2k+1} & & -u \\ & & & I_{m-1} & \\ & & & & 1 \end{pmatrix} \cong F^{2k+1},$$

which is complementary to V in $U_{1,2k+2m+1}$. Let us expand the above integral along $(V\backslash U_{1,2k+2m+1})(\mathbb{A}/F)$. The group SO_{2k+1} acts on this quotient, and as in the proof of Lemma 1 we have three types of orbits. First we have the type which corresponds to vectors of nonzero length. Since these Fourier coefficients correspond to the unipotent class $(31^{2k+2m-2})$, one sees using Proposition 2 that they do not contribute to the integral. Next we consider the contribution to (6) from the terms which correspond to nonzero isotropic vectors. We get

$$\int_{Q^0(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(u,(1,g))} \times \psi_1(u)\, du\, dg .$$

Here Q is the parabolic subgroup of SO_{2k+1} which preserves a line and the upper zero indicates that we omit the GL_1, and ψ_1 is now the character $\psi_1(u) = \psi(u_{1,m+1})$. Let w_0 be the Weyl element

$$w_0 = \begin{pmatrix} \nu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu^* \end{pmatrix}, \qquad \nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & I_{k+m-2} & 0 \end{pmatrix}.$$

Conjugating by w_0 from left to right, the above integral equals

$$\int_{Q^0(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \varphi_\pi(g) \times \overline{\theta_{2k+2m+1}(uw_0(1,g))}\, \psi_2(u)\, du\, dg$$

where $\psi_2(u) = \psi(u_{1,2})$. Let L denote the unipotent radical of Q^0. Factoring the integration over this group and using Lemma 1 we obtain the integral of φ_π along the group $L(F)\backslash L(\mathbb{A})$ as inner integration. This integral is zero by the cuspidality of π. From this we deduce that the vanishing of $\theta_{2k+2m+1}(\pi)$ implies the vanishing of the integral

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(u,(1,g))}\, du\, dg.$$

for all choices of data. Using Proposition 1 with $r = 1$, this implies that $\theta_{2k+2m-1}(\pi) = 0$. $\square$

Proposition 6 *Let F be a nonarchimedean local field, and let π be a genuine irreducible admissible representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. If there exists no $\widetilde{\mathrm{SO}}_{2k+1}(F)$-invariant bilinear form on $\theta_{2k+2m+1} \otimes \pi$, then there exists no $\widetilde{\mathrm{SO}}_{2k+1}(F)$-invariant bilinear form on $\theta_{2k+2m-1} \otimes \pi$.*

Proof This is a local analog of Proposition 5, and the proof is parallel. Note that in the proof of Proposition 5 we make use of Proposition 2 which is Theorem 4.2 (i) of [3]. This result is stated globally, and indeed (ii) of Theorem 4.2 of [3] is essentially global. However (i) of Theorem 4.2, which is what is needed here, can be formulated and proved locally the same way as the global statement which is given in [3]. We omit further details. $\square$

Proof of Theorem 1: Let V_p denote the unipotent radical of the parabolic subgroup of SO_{2m} whose Levi part is $\mathrm{GL}_p \times \mathrm{SO}_{2m-2p}$. There are two associated parabolic subgroups of SO_{2m} whose Levi part is GL_m. With the embedding in (1) the unipotent radicals of these parabolic subgroups are conjugate. Hence we need only consider one of them. Let us write

$$V_p = \left\{ \begin{pmatrix} I_p & x & y \\ & I_{2m-2p} & x^* \\ & & I_p \end{pmatrix} \right\} \tag{7}$$

where $x \in \mathrm{Mat}_{p\times 2(m-p)}$ and $y \in \mathrm{Mat}^0_{p\times p} = \{A \in \mathrm{Mat}_{p\times p} : A^t J_p + J_p A = 0\}$.

We need to prove that if $\theta_{2k+2m-1}(\pi) = 0$ then the integral

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_p(F)\backslash V_p(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(v,g)}\, dv\, dg \tag{8}$$

is zero for all choices of data. We start by expanding (8) with respect to the characters of $U_{1,2k+2m+1}(\mathbb{A})$ which are trivial on $U_{1,2k+2m+1}(F)$. Once again the group $\mathrm{SO}_{2k+2m-1}(F)$ acts on the group of characters of $U_{1,2k+2m+1}(\mathbb{A})$

with three types of orbits. First are the orbits which correspond to vectors of non-zero length. The corresponding Fourier coefficients will correspond to the unipotent class $(31^{2k+2m-2})$. Hence by Proposition 2 the contribution of these orbits is zero. Next we consider the contribution of the constant term to the above expansion. As in the proof of Proposition 5 we see that this integral is $\theta_{2k+2m-1}(\pi)$. By our assumption this is zero. Thus (8) equals

$$\int \varphi_\pi(g) \sum_\gamma \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \overline{\theta_{2k+2m+1}(u\gamma(v,g))}\, \psi_1(u)\, du\, dv\, dg. \quad (9)$$

Here g and v are integrated as before, $\psi_1(u) = \psi(u_{1,2})$, and γ is summed over $Q^0_{2k+2m-1}(F)\backslash \mathrm{SO}_{2k+2m-1}(F)$, where $Q_{2k+2m-1}$ is the parabolic subgroup of $\mathrm{SO}_{2k+2m-1}$ which preserves a line and the upper zero indicates that we omit the GL_1.

To simplify notations we shall write θ for $\theta_{2k+2m+1}$ from now on. We shall also denote

$$\theta^{U_1,\psi_1}(z) = \int_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \theta(uz)\, \psi_1(u)\, du.$$

Write

$$\sum_{Q^0_{2k+2m-1}(F)\backslash \mathrm{SO}_{2k+2m-1}(F)} = \sum_{Q_{2k+2m-1}(F)\backslash \mathrm{SO}_{2k+2m-1}(F)} \sum_{\epsilon\in F^\times}$$

and denote

$$\vartheta^{U_1,\psi_1}(z) = \sum_{\epsilon\in F^\times} \theta^{U_1,\psi_1}(h_1(\epsilon)z) \quad (10)$$

where $h_1(\epsilon) = \mathrm{diag}(1, \epsilon, I_{2k+2m-3}, \epsilon^{-1}, 1)$. With these notations (9) equals

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_p(F)\backslash V_p(\mathbb{A})} \varphi_\pi(g) \sum_\gamma \overline{\vartheta^{U_1,\psi_1}(\gamma(v,g))}\, dv\, dg \quad (11)$$

where now γ is summed over $Q_{2k+2m-1}(F)\backslash \mathrm{SO}_{2k+2m-1}(F)$.

Consider the double cosets $Q_{2k+2m-1}\backslash \mathrm{SO}_{2k+2m-1} / Q_{2k+2m-1}$. This space has three representatives. They are e, w_2 and $\tilde{w}$ where

$$\tilde{w} = \begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & I & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix}.$$

We claim that the contributions to (11) from $\gamma = e$ and $\gamma = \tilde{w}$ are zero. Indeed, if the representative is e then we obtain

$$\int \varphi_\pi(g) \sum_{\epsilon} \int\limits_{U_{1,2k+2m+1}(F)\backslash U_{1,2k+2m+1}(\mathbb{A})} \overline{\theta_{2k+2m+1}(uh_1(\epsilon)(v,g))}\psi_1(u)\, du\, dv\, dg.$$

Using Lemma 1 the inner integration is left-invariant under the quotient $U_{2,2k+2m+1}(F)\backslash U_{2,2k+2m+1}(\mathbb{A})$. Notice that $U_{1,2k+2m+1}U_{2,2k+2m+1}$ contains the unipotent radical of the parabolic subgroup of $\mathrm{SO}_{2k+2m+1}$ whose Levi part is $\mathrm{GL}_2 \times \mathrm{SO}_{2k+2m-3}$. Denote this unipotent subgroup by L. Then $\psi_1(u)$ is trivial on L. Also $g \in \mathrm{SO}_{2k+1}$ commutes with $h_1(\epsilon)$. Thus, conjugating g to the left, after integrating over $L(F)\backslash L(\mathbb{A})$ we obtain zero, as can be seen by applying Proposition 1 and Proposition 6 at any nonarchimedean place. This shows that the contribution of $\gamma = e$ is zero.

Next we consider the contribution of $\tilde{w}$ to (11). Consider the root $\zeta = \beta_1 + \beta_2 + \cdots + \beta_m$ where the β_i are the simple roots of SO_{2m}. The one parameter subgroup $x_\zeta(r)$ is in V_p for all p. Using the embedding (1) we have $x_\zeta(r) = I_{2k+2m+1} + r(e_{1,2k+2m} - e_{2,2k+2m+1})$. We may write the integration

$$\int\limits_{V_p(F)x_\zeta(\mathbb{A})\backslash V_p(\mathbb{A})} \varphi_\pi(g) \left[\int_{A/F} \overline{\vartheta^{U_1,\psi_1}(\tilde{w}x_\zeta(r)(v,g))}\, dr\right] dv\, dg \;=$$

$$\int\limits_{V_p(F)x_\zeta(\mathbb{A})\backslash V_p(\mathbb{A})} \varphi_\pi(g) \left[\sum_{\epsilon\in F^\times} \int_{A/F} \psi(\epsilon r) dr\right] \overline{\theta^{U_1,\psi_1}(h_1(\epsilon)\tilde{w}(v,g))}\, dv\, dg \;=\; 0$$

by definition of ϑ^{U_1,ψ_1}, since $\psi_1(h_1(\epsilon)\tilde{w}x_\zeta(r)\tilde{w}h_1(\epsilon)^{-1}) = \psi(\epsilon r)$. Thus the contribution of $\tilde{w}$ is also zero.

Thus in (11) we are left with the contribution from w_2. This equals

$$\int\limits_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{V_p(F)\backslash V_p(\mathbb{A})} \varphi_\pi(g) \sum_{\gamma,\delta_2} \overline{\vartheta^{U_1,\psi_1}(w_2 x_{\alpha_2}(\delta_2)\gamma(v,g))}\, dv\, dg \qquad (12)$$

where the sum is over $\gamma \in Q_{2k+2m-3}(F)\backslash \mathrm{SO}_{2k+2m-3}(F)$ and $\delta_2 \in F$. Now we repeat this process. That is, we consider the space

$$Q_{2k+2m-3}\backslash \mathrm{SO}_{2k+2m-3} / Q_{2k+2m-3}.$$

As before there are three representatives. Using Proposition 5 one sees that the identity contributes zero to (12). As for the long Weyl element representative we use the one-parameter subgroup corresponding to the root $\beta_1 + \beta_2 + 2\beta_3 + \cdots + 2\beta_m$, which lies in any unipotent radical subgroup V_p of SO_{2m}, to show that this contributes zero. Continue inductively. At each stage we use Proposition 5 in order to show that the identity representative contributes zero and as for the long Weyl element, at the i-th step we use the one parameter subgroup which corresponds to the root $\beta_1 + \cdots + \beta_i + 2\beta_{i+1} \cdots + 2\beta_m$ which lies in any unipotent subgroup V_p of SO_{2m}. Doing so, we deduce that the integral (11) equals

$$\int_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_p(F)\backslash V_p(\mathbb{A})} \varphi_\pi(g) \\ \sum_{\delta_i} \overline{\vartheta^{U_1,\psi_1}(w_2 x_{\alpha_2}(\delta_2) \cdots w_m x_{\alpha_m}(\delta_m)(v,g))}\, dv\, dg \tag{13}$$

where the sum is over $\delta_i \in F$, $2 \le i \le m$, and where P_{2k+1} is the parabolic subgroup of SO_{2k+1} which preserves a line.

Let $e'_{i,j} = e_{i,j} - e_{2k+2m-j+2,2k+2m-i+2}$ and

$$z(\delta_2, \cdots, \delta_m) = I_{2k+2m+1} + \delta_2 e'_{2,m+1} + \cdots + \delta_m e'_{m,m+1}.$$

Also let $\tilde{w}_2 = w_2 \cdots w_m$. Then (13) equals

$$\int_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_p(F)\backslash V_p(\mathbb{A})} \varphi_\pi(g) \sum_{\delta_i} \overline{\vartheta^{U_1,\psi_1}(\tilde{w}_2 z(\delta_2, \cdots, \delta_m)(v,g))}\, dv\, dg. \tag{14}$$

In (7), if $x = (x_{i,j})$ let t be the first half of the first row of the matrix x, i.e. $t = (x_{1,1}, \cdots, x_{1,m-p})$. Embed t in V_p in the obvious way and view t as a subgroup of $\mathrm{SO}_{2k+2m+1}$ via (1). In (14) we may now conjugate t to the left, across $z(\delta_2, \cdots, \delta_m)$. When we do so, we obtain by the commutation relations the matrix

$$x_{\alpha_1+\cdots+\alpha_m}(x_{1,1}\delta_{p+1} + \cdots + x_{1,m-p}\delta_m).$$

Conjugating this matrix across $\tilde{w}_2$ and changing variables in $U_{1,2k+2m+1}$ we obtain the integral

$$\int \psi_1(x_{1,1}\delta_{p+1} + \cdots + x_{1,m-p}\delta_m)\, dx_{1,j}$$

as inner integration. This integral is zero unless $\delta_i = 0 \,\text{for}\, p+1 \leqslant i \leqslant m$. Thus the integral (14) equals

$$\int_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_{p,1}(F)\backslash V_{p,1}(\mathbb{A})} \varphi_\pi(g) \tag{15}$$
$$\sum_{\delta_i} \overline{\vartheta^{U_1,\psi_1}(\tilde{w}_2 z(\delta_2,\cdots,\delta_p,0,\cdots,0)(v,g))}\, dv\, dg.$$

Here $V_{p,1}$ is the subgroup of V_p where the first row of x and the first row of y are zero. If $p=1$, or if $p \geqslant 2$ and all the δ_i are zero, then this integral is zero. Indeed, let L_{2k+1} denote the unipotent radical of P_{2k+1}. We factor this group and we conjugate it to the right in $\tilde{\theta}^{U_1,\psi_1}$. Using Lemma 1 we obtain

$$\int_{L_{2k+1}(F)\backslash L_{2k+1}(\mathbb{A})} \varphi_\pi(lg)\, dl$$

as inner integration. By the cuspidality of π this is zero.

Henceforth we assume that $p \geq 2$ and that $z(\delta_2,\cdots,\delta_p,0,\cdots,0)$ is not zero. Embed the group GL_p in $\mathrm{SO}_{2k+2m+1}$ as $\hat{\zeta} = \mathrm{diag}(1,\zeta,I_{2k+2m-2p-1},\zeta^*,1)$. Let $z_1(1) = z(1,0,\cdots,0)$. The group $\mathrm{GL}_p(\mathrm{F})$ acts on the nonzero elements $z(\delta_2,\cdots,\delta_p,0,\cdots,0)$ with one orbit. We thus obtain

$$\int_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_{p,1}(F)\backslash V_{p,1}(\mathbb{A})} \varphi_\pi(g) \sum_{\zeta} \overline{\vartheta^{U_1,\psi_1}(\tilde{w}_2 z_1(1)(v,g)\hat{\zeta})}\, dv\, dg.$$

Here we have used the commutativity of $\hat{\zeta}$ with v and g, and also that if we conjugate $\hat{\zeta}^{-1}$ to the left by $\tilde{w}_2$ then $\tilde{\theta}^{U_1,\psi_1}$ is left-invariant under the matrix obtained after conjugation. Also ζ is summed over suitable matrices in $\mathrm{GL}_p(\mathrm{F})$. Thus to show that (16) is zero it is enough to prove that

$$\int_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_{p,1}(F)\backslash V_{p,1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\vartheta^{U_1,\psi_1}(\tilde{w}_2 z_1(1)(v,g))}\, dv\, dg \tag{16}$$

is zero. Recall that $\tilde{\vartheta}$ is a sum over $\epsilon \in F^\times$ (cf. (10)). We can collapse the summation over ϵ with the integration over the subgroup GL_1 contained in P_{2k+1}. Then (16) equals

$$\int_{P^0_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{V_{p,1}(F)\backslash V_{p,1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta^{U_1,\psi_1}(\tilde{w}_2 z_1(1)(v,g))}\, dv\, dg \tag{17}$$

where the superscript 0 in P^0_{2k+1} indicates that we omit the GL_1.

By Lemma 1 we can replace θ^{U_1,ψ_1} by $\theta^{U_1U_2,\psi_1}$, then repeat this process. In other words, we expand $\theta^{U_1U_2,\psi_1}$ along $U_{3,2k+2m+1}(F)\backslash U_{3,2k+2m+1}(\mathbb{A})$. The group $\mathrm{SO}_{2k+2m-5}(F)$ acts on the group characters of this quotient with three type of orbits. The ones which correspond to the vectors of nonzero length will contribute zero after applying Propositions 1 and 2. The constant term will also contribute zero. Indeed, if we factor the group L_{2k+1} as above, one can check that $\tilde{w}_2z_1(1)L_{2k+1}(\tilde{w}_2z_1(1))^{-1}\in U_{3,2k+2m+1}$. Thus we obtain zero by the cuspidality of π.

We are left with the orbit which corresponds to the nonzero isotropic vectors. This process is clearly inductive and depending on the relation between the numbers $2k+1$ and $2m$ we finally obtain the following integrals.

If $2m<2k+1$ then the integral (17) equals

$$\int_{P^0_{2k-2m+5}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\,\overline{\theta^{R_{2l+1},\psi_{2l+1}}(\tilde{w}z_0(1,g))}\,dg. \tag{18}$$

Here $\tilde{w}=\tilde{w}_{2m-2}\cdots\tilde{w}_2$ where $\tilde{w}_{2i}=w_{2i}\cdots w_{m+i}$, and

$$z_0=\mathrm{diag}(1,\tilde{z},I_{2k-2m+3},\tilde{z}^*,1),\qquad \tilde{z}=\begin{pmatrix} I & I\\ & I\end{pmatrix}$$

where I is the $(m-1)\times(m-1)$ identity matrix. We also have $l=(m-1)/2$ if m is odd and $l=m/2$ if m is even. The group $P^0_{2k-2m+5}$ is the subgroup of SO_{2k+1} given by $P^0_{2k-2m+5}=\mathrm{SO}_{2k-2m+5}\,L_{2k-2m+5}$ where $L_{2k-2m+5}$ is the unipotent radical of the standard nonmaximal parabolic subgroup whose Levi part is $\mathrm{GL}_1^{m-1}\times L_{2k-2m+5}$. We have the factorization $L_{2k-2m+5}=N_{m-1}L^0_{2k-2m+5}$ where N_{m-1} is the maximal unipotent subgroup of GL_{m-1} and $L^0_{2k-2m+5}$ is the unipotent radical of the maximal parabolic subgroup of SO_{2k+1} whose Levi part is $\mathrm{GL}_{m-1}\times\mathrm{SO}_{2k-2m+5}$. To show that this integral is zero we factor the measure and consider the inner integration over the group $L^0_{2k-2m+5}(F)\backslash L^0_{2k-2m+5}(\mathbb{A})$. Conjugating this matrix to the right, we see that the function $\theta^{R_{2l+1},\psi_{2l+1}}$ is left-invariant under this group. Thus we obtain zero by cuspidality.

A similar situation occurs if $2m>2k+1$. In this case we obtain

$$\int_{\tilde{L}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})}\int_{V_{p,j}(F)\backslash V_{p,j}(\mathbb{A})} \varphi_\pi(g)\,\overline{\theta^{R_{2l+1},\psi_{2l+1}}(\tilde{w}z_0(1,g))}\,dg. \tag{19}$$

Here $\tilde{w} = \tilde{w}_{2k} \cdots \tilde{w}_2$ where $\tilde{w}_{2i} = w_{2i} \cdots w_{k+i}$. Also $z_0 = \mathrm{diag}(1, \tilde{z}, 1, \tilde{z}^*, 1)$ where now

$$\tilde{z} = \begin{pmatrix} I_k & & I_k \\ & I_{m-k+1} & \\ & & I_k \end{pmatrix}.$$

The group $\tilde{L}$ is the maximal unipotent subgroup of SO_{2k+1} and $V_{p,j}$ is a certain subgroup of V_p. The number l equals $k/2$ if k is even and equals $(k-1)/2$ if k is odd.

To show that (19) is zero we factor the integration to obtain the integral over $\tilde{L}(F)\backslash\tilde{L}(\mathbb{A})$ as inner integration. As in the previous case observe that after conjugating this to the right, the function $\theta^{R_{2l+1},\psi_{2l+1}}$ is left-invariant by the matrix resulting from the conjugation. Thus we get zero by the cuspidality of π.

This completes the proof of Theorem 1.

4 The Nonvanishing of the Lift

We prove

Theorem 2 *Let π be a genuine cuspidal automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Then $\theta_{10k+1}(\pi)$ is nonzero. In other words, every π lifts nontrivially to an automorphic representation on $\widetilde{\mathrm{SO}}_{8k}(\mathbb{A})$.*

Remark. In fact we expect that the first occurrence will be before this. In the next section we will analyze the nonvanishing of the lift in the case when π is a generic automorphic cuspidal representation. We will find conditions for it to lift to $\widetilde{\mathrm{SO}}_{2k+2}$. We believe that every such π should lift nontrivially to $\widetilde{\mathrm{SO}}_{2k+4}$. At the end of Section 5 we give some computations which support this conjecture.

Proof Suppose that

$$\tilde{f}(h) = \int_{SO_{2k+1}(F)\backslash SO_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{10k+1}(h,g)}\, dg \tag{20}$$

is zero for all choices of data. We will derive a contradiction. Let V_{4k} denote the unipotent subgroup of $\widetilde{\mathrm{SO}}_{8k}$ as defined in (7) with $p = 4k$. From the assumption that the above integral vanishes for all choice of data, it follows that the integral

$$\int_{SO_{2k+1}(F)\backslash SO_{2k+1}(\mathbb{A})} \int_{V_{4k}(F)\backslash V_{4k}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{10k+1}(v,g)}\, \psi_V(v)\, dv\, dg \tag{21}$$

is zero for all choices of data. Here ψ_V is defined as follows. From (7) it follows that we can identify V_{4k} with all matrices

$$\mathrm{Mat}^0_{4k\times 4k} = \{y \in \mathrm{Mat}_{4k\times 4k} \mid yJ_{4k} + J_{4k}\,{}^t y = 0\}.$$

For $y = (y_{ij}) \in \mathrm{Mat}^0_{4k\times 4k}$ define $\psi_V(v) = \psi_V(y) = \psi(y_{1,1}+\ldots+y_{2k,2k})$. Notice that this character is uniquely defined in the following sense. Recall that V_{4k} is the unipotent radical of the maximal parabolic subgroup of SO_{8k} whose Levi part is GL_{4k}. The action of GL_{4k} on $\mathrm{Mat}^0_{4k\times 4k}$ is via the exterior square representation. This action has an open orbit, hence, up to conjugation by $\mathrm{GL}_{4k}(F)$, ψ_V is uniquely defined.

Before proceeding, let us explain the motivation for considering the integral (21). To derive a contradiction we need to show that as we vary the data in the space of the representation θ_{10k+1} we have enough information to deduce that (20) is nonzero. We will approach this in a way similar to [8] Section 4 using the structure of Fourier-Jacobi coefficients as described in [9].

More precisely, let R_{4k} denote the unipotent radical of the maximal parabolic subgroup of SO_{10k+1} whose Levi group is $\mathrm{GL}_{4k}\times \mathrm{SO}_{2k+1}$. Thus R_{4k} has the structure of a generalized Heisenberg group whose center is V_{4k}. Let l denote the homomorphism from the group R_{4k} onto the Heisenberg group with $4k(2k+1)+1$ variables.

Let $\tilde{\theta}^{\psi}_{\phi}$ denote the theta function on the double cover of $\mathrm{Sp}_{4k(2k+1)}$. Here ϕ is a Schwartz function. It follows from [9] that the space of functions

$$\tilde{\theta}^{\psi}_{\phi_1}((1,g)) \int_{R_{4k}(F)\backslash R_{4k}(\mathbb{A})} \theta_{10k+1}(r(1,g))\tilde{\theta}^{\psi}_{\phi_2}(l(r)(1,g))\, dr, \tag{22}$$

where ϕ_1 and ϕ_2 are Schwartz functions, is a dense subspace in the space of functions

$$\int_{V_{4k}(F)\backslash V_{4k}(\mathbb{A})} \theta_{10k+1}(v,g)\, \psi_V(v)\, dv. \tag{23}$$

From this we conclude that the vanishing of (21) for all choices of data is equivalent to the vanishing of

$$\int_{SO_{2k+1}(F)\backslash SO_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \tilde{\theta}^{\psi}_{\phi_1}((1,g)) \tag{24}$$
$$\times \int_{R_{4k}(F)\backslash R_{4k}(\mathbb{A})} \theta_{10k+1}(r(1,g))\, \tilde{\theta}^{\psi}_{\phi_2}(l(r)(1,g))\, dr\, dg$$

for all choices of data. Define

$$L(g) = \varphi_\pi(g) \int_{R_{4k}(F)\backslash R_{4k}(\mathbb{A})} \theta_{10k+1}(r(1,g))\, \tilde{\theta}^{\psi}_{\phi_2}(l(r)(1,g))\, dr.$$

Then it follows from (24) that

$$\int_{SO_{2k+1}(F)\backslash SO_{2k+1}(\mathbb{A})} L(g)\, \tilde{\theta}^{\psi}_{\phi_1}((1,g))\, dg$$

is zero for all choices of data. Arguing as in [12] Theorem I.2.1 we deduce that the vanishing of the above integral for all choices of data implies that the function $L(g)$ is zero for all choices of data. (We chose to consider the lift from $\widetilde{\mathrm{SO}}_{2k+1}$ to $\widetilde{\mathrm{SO}}_{8k}$ so that we would be able to use the result in [12]. Taking the lift to a smaller rank even orthogonal group would not guarantee the nonvanishing of the last integral.) However, if $L(g)$ is zero for all choices of data, this just means that (22) and hence (23) are zero for all choices of data.

In a way similar to that described in [3] formula (4.24), one can check that the Fourier coefficient written in (23) corresponds to the unipotent orbit $(2^{4k}1^{2k+1})$. We know from [3] Theorem 4.2 part 2, that θ_{10k+1} has a nonzero Fourier coefficient corresponding to the unipotent orbit $(2^{5k}1)$. From the description of these two Fourier coefficients, it follows that integral (23) is an inner integration to integral (4.24) in [3], which we know to be nonzero for some choice of data. Hence (23) is nonzero for some choice of data and we derived a contradiction. This completes the proof of Theorem 2. $\square$

5 The Whittaker Model of the Lift and the Nonvanishing of the Lift for Generic Representations

In this section we examine more carefully the question of the nonvanishing. We start by computing the Whittaker model of the lift and expressing it in terms of certain models of π. We first study the lift to $\widetilde{\mathrm{SO}}_{2(k+1)}$. In this case we show that the Whittaker model of the lift is nonzero if and only if π has a Bessel model. Then we consider the lift to $\widetilde{\mathrm{SO}}_{2(k+2)}$. In this case we show that if the Whittaker model of the lift is nonzero then π has a Whittaker model.

We start with the lift to $\widetilde{\mathrm{SO}}_{2(k+1)}$. Let U denote the maximal unipotent subgroup of $\widetilde{\mathrm{SO}}_{2(k+1)}$. We define the character $\psi_{U,a}$ of $U(F)\backslash U(\mathbb{A})$ as follows. If $u = (u_{i,j}) \in U$ define $\psi_{U,a}(u) = \psi(u_{1,2} + u_{2,3} + \ldots + u_{k,k+1} + au_{k,k+2})$ where $a \in F^{\times}$. It is easy to check that a may be multiplied by any square by conjugation. Via the embedding in (1) we consider the integral

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi_{\pi}(g)\, \theta_{4k+3}((u,g))\, \psi_{U,a}(u)\, du\, dg. \tag{25}$$

We shall now compute this integral and determine when it is nonzero.

The first part of the computation is similar to the computation done in the proof of Theorem 1, where we replace V_p by U. Indeed, following the same steps which led to the integral (14) we deduce that (25) equals

$$\int\limits_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi_{\pi}(g) \tag{26}$$
$$\sum_{\delta_i} \vartheta^{U_1,\psi_1}(\tilde{w}_2 z(\delta_2, \cdots, \delta_{k+1})(v,g))\psi_{U,a}(u) dv\, dg,$$

where $\tilde{w}_2 = w_2 \cdots w_m$. Notice that V_1 (introduced in the proof of Theorem 1) is a subgroup of U, hence if we carry out the same process which led from (14) to (16) we find that (26) equals

$$\int\limits_{P_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{U^1(F)\backslash U^1(\mathbb{A})} \varphi_{\pi}(g) \tag{27}$$
$$\textstyle\sum_{\epsilon} \theta^{U_1,\psi_1}(h_1(\epsilon)\tilde{w}_2 z(\epsilon^{-1}, 0, \cdots, 0)(u^1, g))\psi_{U,a}(u^1) du^1 dg.$$

Here $U^1 = U \bigcap \mathrm{SO}_{2k}$ where SO_{2k} is embedded in SO_{2k+2} in the middle block. The appearance of ϵ is due to (10) and to the fact that the character $\psi_{U,a}$ is not trivial on restriction to V_1 (whereas in the proof of Theorem 1 it was trivial). Collapsing summation with integration as in the proof of Theorem

1, we obtain that (27) equals

$$\int_{P^0_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U^1(F)\backslash U^1(\mathbb{A})} \varphi_\pi(g)\, \theta^{U_1,\psi_1}(\tilde{w}_2 z_1(1)(u^1,g))\, \psi_{U,a}(u^1)\, du^1\, dg. \tag{28}$$

Continuing in this way, as in the proof of Theorem 1 we deduce that (28) equals

$$\int_{P^0_3(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{U^k(F)\backslash U^k(\mathbb{A})} \sum_\gamma \varphi_\pi(g) \times \tag{29}$$
$$\theta^{R_{2k-1},\psi_{2k-1}}(\gamma\tilde{w}z_0(u^k,g))\, \psi_{U,a}(u^k)\, du^k\, dg$$

where we now explain the notations. Let P_3 denote the parabolic subgroup of SO_{2k+1} whose Levi part is $\mathrm{GL}_1^{k-1}\times \mathrm{SO}_3$. The group P^0_3 is the subgroup of P_3 where we omit the GL_1^{k-1} factor. Next, we define $U^k = U\bigcap \mathrm{SO}_4$ where SO_4 is embedded in SO_{2k+2} in the middle block. The group R_{2k-1}, which was also defined before Corollary 1, is the unipotent radical of the parabolic subgroup of SO_{4k+3} whose Levi part is $\mathrm{GL}_1^{2k-1}\times \mathrm{SO}_5$. The character ψ_{2k-1} was defined before Corollary 1. The function $\theta^{R_{2k-1},\psi_{2k-1}}$ is the Fourier coefficient along this unipotent subgroup with this character. The sum in (29) is over all γ in $Q^0_3(F)\backslash \mathrm{SO}_5(F)$ where Q^0_3 is the subgroup of the maximal parabolic which preserves a line, and the upper zero indicates that we omit the GL_1 factor. We also define $\tilde{w}=\tilde{w}_{2k}\cdots\tilde{w}_2$ where for all $1\leqslant i\leqslant k$ we have $\tilde{w}_{2i}=w_{2i}\cdots w_{k+i}$. Finally, we denote $z_0=\mathrm{diag}(1,\tilde{z},1,\tilde{z}^*,1)$ where

$$\tilde{z}=\begin{pmatrix} I_{k-1} & & I_{k-1} \\ & 1 & \\ & & I_{k-1}\end{pmatrix}.$$

The difference between this case and the cuspidality computation is that here we integrate also along the character $\psi_{U,a}$ which by definition is nontrivial on the entries $u_{k,k+1}$and $u_{k,k+2}$ of u^k. Hence at this point when we consider the space $Q_3\backslash \mathrm{SO}_5/Q_3$, we get a contribution of zero from the two small sets (in contrast to what happened in the cuspidality computation) and so we will only need to consider the contribution from the big cell. From this cell we obtain

$$\int \sum_{\gamma,\epsilon} \varphi_\pi(g)\theta^{R_{2k-1},\psi_{2k-1}}(h(\epsilon)w_0\gamma\tilde{w}z_0(u^k,g))\psi_{U,a}(u^k)du^k dg \tag{30}$$

where the sum is over $\epsilon \in F^\times$ and $\gamma \in U_{2k,4k+3}(F)$ where this last group was defined in (3). Also

$$h(\epsilon) = \begin{pmatrix} I_{2k-1} & & & & \\ & \epsilon & & & \\ & & I_3 & & \\ & & & \epsilon^{-1} & \\ & & & & I_{2k-1} \end{pmatrix}, \qquad w_0 = \begin{pmatrix} I_{2k-1} & & \\ & \nu & \\ & & I_{2k-1} \end{pmatrix},$$

where ν is a Weyl element in SO_5 which is a representative of the big cell as obtained from the above double coset factorization. All variables are integrated as in (29).

At this point we conjugate the matrix u^k to the left. Recall that the dimension of the group U^k is two. Via the embedding (1) this group consists of products of the matrices $I_{4k+3} + u_{k,k+1}e'_{k,k+1}$ and $I_{4k+3} + u_{k,k+2}e'_{k,3k+3}$ where the indices indicate the relation of these matrices to their embeddings in U^k. On this product we have $\psi_{U,a}(u^k) = \psi(u_{k,k+1} + au_{k,k+2})$. The above two matrices commute with z_0 and after conjugating them by $\tilde{w}$ we obtain the matrix $x(u_{k,k+1}, u_{k,k+2}) = I_{4k+3} + u_{k,k+1}e'_{2k-1,2k} + u_{k,k+2}e'_{2k-1,2k+4}$. Conjugating $x(0, u_{k,k+2})$ to the left and changing variables we obtain

$$\int \psi((\epsilon^{-1} - a)u_{k,k+2})\, du_{k,k+2}$$

as inner integration. Thus we obtain a nonzero contribution only if $\epsilon = a^{-1}$. Conjugating $x(u_{k,k+1}, 0)$ to the left and changing variables we obtain

$$\int \psi((1 - a^{-1}(\gamma, \gamma))u_{k,k+1}) du_{k,k+1}$$

as inner integration. Here (γ, γ) is the square of the length of the vector γ. Thus we get a nonzero contribution only if $(\gamma, \gamma) = a$.

The group $\mathrm{SO}_3(F)$ as embedded in $P_3^0(F)$ acts on the set of all $\gamma \in F^3$ which have fixed length, with one orbit. The stabilizer is a copy of $\mathrm{SO}_2(F)$ which is determined by the length of γ. Let γ_a be an element such that $(\gamma_a, \gamma_a) = a$. Then (30) equals

$$\int_{L_k(F)\,\mathrm{SO}_2^a(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0\gamma_a\tilde{w}z_0(1,g))\, dg. \tag{31}$$

Here L_k is the unipotent radical of P_3^0 and SO_2^a is the stabilizer of γ_a inside SO_3. We choose $\gamma_a = (1/2, 0, a)$ and if a is a square we may choose $\gamma_a = (0, 1, 0)$ (recall that a can be changed by any square by conjugation). Notice that in this last case $\mathrm{SO}_2^a \cong \mathrm{GL}_1$.

Conjugating this element to the right, (31) equals

$$\int_{L_k(F)\,\mathrm{SO}_2^a(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0 \tilde{w} z(1,g))\, dg \tag{32}$$

where

$$z = \begin{pmatrix} I_{k+1} & \alpha & * \\ & I_{2k+1} & \alpha^* \\ & & I_{k+1} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & a & 0 \end{pmatrix}.$$

Here α^* and $*$ indicate entries chosen so that the matrix is orthogonal. If a is a square then we can replace the last row of α by $(0, 0, 1, 0, 0)$.

We define the a-th Bessel model of π by

$$\mathcal{B}_a(\pi)(g) = \int_{\mathrm{SO}_2^a(F)\backslash \mathrm{SO}_2^a(\mathbb{A})} \int_{L_k(F)\backslash L_k(\mathbb{A})} \varphi_\pi(lhg)\, \psi_L(l)\, dl\, dh$$

where the definition of ψ_L is as follows. For $l = (l_{i,j}) \in L_k$ define $\psi_L(l) = \psi(l_{1,2} + \cdots + l_{k-2,k-1} + (\gamma_a, l'))$ where (γ_a, l') is the product of γ_a with $l' = (l_{k-1,k}, l_{k-1,k+1}, l_{k-1,k+2})$. This integral converges absolutely for all a. In the split case this is shown in [7].

Remark 1 In the case when π is a cuspidal automorphic representation of $\mathrm{SO}_{2k+1}(\mathbb{A})$and not the covering group, this Bessel model is related to the value of the standard L-Function at the center of symmetry. There is no reason to believe that a similar relation holds on the covering group.

With these notations (32) equals

$$\int_{L_k(\mathbb{A})\,\mathrm{SO}_2^a(\mathbb{A})\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0 \tilde{w} z(1,g))\, dg. \tag{33}$$

We record this as

Lemma 2 *Let π be a cuspidal automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Then the a-th Whittaker coefficient of $\theta_{4k+3}(\pi)$ can be expressed in terms of the Bessel model of the representation π. With the notations of (2), we have*

$$\begin{aligned}&\int_{U(F)\backslash U(\mathbb{A})} \tilde{f}(u)\,\psi_{U,a}(u)\,du \qquad (34)\\ = \quad &\int_{L_k(\mathbb{A})\,\mathrm{SO}_2^a(\mathbb{A})\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\,\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g))\,dg.\end{aligned}$$

Remark 2 It is interesting to note the similarity of (34) to the identity (9) in [6]. In that case the relation is between the ψ^a Whittaker coefficient of a representation on $\widetilde{Sp}_{2n}$ and a cuspidal representation on SO_{2n+1}. The comparison in [6] uses the theta representation on the double cover of the symplectic group.

Next we prove the following

Theorem 3 *The representation $\theta_{4k+3}(\pi)$ has a nonzero Whittaker model with respect to the character $\psi_{U,a}$ if and only if the representation π has a nonzero Bessel model $\mathcal{B}_a(\pi)$.*

Proof It follows from (34) that if $\theta_{4k+3}(\pi)$ has a nonzero $\psi_{U,a}$ Whittaker coefficient then $\mathcal{B}_a(\pi)$ is nonzero.

Conversely, assume that $\mathcal{B}_a(\pi)$ is nonzero and assume that the $\psi_{U,a}$ Whittaker coefficient of $\theta_{4k+3}(\pi)$ is zero for all choices of data. We will derive a contradiction. Indeed from (34) it follows that

$$\int_{L_k(\mathbb{A})\,\mathrm{SO}_2^a(\mathbb{A})\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\,\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g))\,dg \qquad (35)$$

is zero for all choices of data.

Let ϕ denote an arbitrary Schwartz function on $Y=\mathbb{A}^{2k+1}$. In the notation of Section 2 we identify Y with $U_{1,4k+3}/V_1$. In terms of coordinates we have the following embedding. If $y=(y_1,\cdots,y_{2k+1})\in Y$ then the embedding is given by $y\hookrightarrow I_{4k+3}+y_1e'_{1,k+2}+\cdots+y_{2k+1}e'_{1,3k+2}$.

From the vanishing of (35) we deduce that the integral

$$\int_{Y(\mathbb{A})}\int_{L_k(\mathbb{A})\,\mathrm{SO}_2^a(\mathbb{A})\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g)y)\,\phi(y)\,dg\,dy \qquad (36)$$

is zero for all choices of data. Conjugate the matrix y to the left and change variables in R_{2k-1}. Factoring the measure in the g variable we obtain that

$$\int_{P^0_{2k+1}(\mathbb{A})\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} J(h)\, \hat{\phi}(\xi_0 h)\, dh \tag{37}$$

is zero for all choices of data. Here $\xi_0 = (0, \cdots, 0, 1)$ and $\hat{\phi}$ denotes the Fourier transform of ϕ, and

$$J(h) = \int_{L_k(\mathbb{A})\, \mathrm{SO}_2^a(\mathbb{A})\backslash P^0_{2k+1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0 \tilde{w} z(1, gh))\, dg.$$

Since ϕ is arbitrary we deduce from the vanishing of (37) that $J(h)$ is zero for all choices of data. Substituting $h = 1$ and factoring the measure over g we deduce that

$$\int_{L_{k-1}(\mathbb{A})\, \mathrm{SO}_2^a(\mathbb{A})\backslash \mathrm{SO}_{2k-1}(\mathbb{A})} \mathcal{B}_a(\pi)(g)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0 \tilde{w} z(1, g))\, dg \tag{38}$$

is zero for all choices of data. Here $L_{k-1} = L_k \bigcap \mathrm{SO}_{2k-1}$.

Continue in this way, this time with $\mathcal{B}_a(\pi)(g)$. Let Y now denote the unipotent radical of the parabolic subgroup of SO_{2k+1} which preserves a line. Thus $Y \simeq U_{1,2k+1}$. Let ϕ be an arbitrary Schwartz function on $Y(\mathbb{A})$. From the vanishing of (38) we deduce that

$$\int_{Y(\mathbb{A})} \int_{L_{k-1}(\mathbb{A})\, \mathrm{SO}_2^a(\mathbb{A})\backslash \mathrm{SO}_{2k-1}(\mathbb{A})} \mathcal{B}_a(\pi)(gy)\phi(y)\, \theta^{R_{2k-1},\psi_{2k-1}}(w_0 \tilde{w} z(1, g))\, dg\, dy \tag{39}$$

is zero for all choices of data. Repeating the same argument as in (37) we can replace the domain of integration in (38) with $L_{k-1}(\mathbb{A})\, \mathrm{SO}_2^a(\mathbb{A})\backslash \mathrm{SO}_{2k-1}(\mathbb{A})$.

Repeating this process we finally obtain that $\mathcal{B}_a(\pi)(e)\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z)$ is zero for all choices of data. This is clearly a contradiction to our assumption. $\square$

When $a = 1$ we write ψ_U for $\psi_{U,1}$. From Theorem 3 we easily deduce

Corollary 2 *Suppose that the representation $\theta_{4k+3}(\pi)$ has a nonzero Whittaker model with respect to the character ψ_U. Then the representation π has a nonzero Whittaker model.*

Proof Let $W_{\varphi_\pi}(g)$ denote the Whittaker coefficient of the function $\varphi_\pi(g)$. It is easy to show that if $\mathcal{B}(\pi)(g)$ (the Bessel functional with $a = 1$) is nonzero then $W_{\varphi_\pi}(g)$ is nonzero (note that the converse need not be true). In fact this follows from [7]. $\square$

The Theorem is proved, but it is still of interest to express the Whittaker model of the lift in terms of the Whittaker model of π. To do so we go back to (32) with $a = 1$, and obtain that the ψ_U Whittaker coefficient of the lift equals

$$\int_{L_k(F)\,\mathrm{GL}_1(F)\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\,\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g))\,dg.$$

Factor the L_k integration to obtain

$$\int_{L_k(\mathbb{A})\,\mathrm{GL}_1(F)\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})}\int_{L_k(F)\backslash L_k(\mathbb{A})} \varphi_\pi(lg)\,\psi_L(l)\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g))dl\,dg. \tag{40}$$

Now we argue as in [7]. After conjugating by a suitable Weyl element ν of SO_{2k+1}, and after suitable Fourier expansions, (40) equals

$$\int_{L_k(\mathbb{A})\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})}\int_{Y(\mathbb{A})} W_{\varphi_\pi}(yg)\,\theta^{R_{2k-1},\psi_{2k-1}}(w_0\tilde{w}z(1,g))\,dy\,dg$$

where Y is a certain unipotent subgroup of SO_{2k+1}.

Next we consider the lift from $\widetilde{\mathrm{SO}}_{2k+1}$ to the group $\widetilde{\mathrm{SO}}_{2(k+2)}$. As in the previous case we shall compute the Whittaker coefficient of the lift and express it in terms of the representation π. To do this let U denote the maximal unipotent subgroup of $\widetilde{\mathrm{SO}}_{2(k+2)}$. For $u = (u_{i,j}) \in U(\mathbb{A})$ and $a \in F^\times$define the character $\psi_{U,a}$ of $U(F)\backslash U(\mathbb{A})$ by

$$\psi_{U,a}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{k,k+1} + u_{k+1,k+2}/2 + au_{k+1,k+3}).$$

Via the embedding (1) we consider the integral

$$\int_{\mathrm{SO}_{2k+1}(F)\backslash\,\mathrm{SO}_{2k+1}(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} \varphi_\pi(g)\theta_{4k+5}((u,g))\,\psi_{U,a}(u)\,du\,dg. \tag{41}$$

As before let us omit the subscript and write θ for θ_{4k+5}. The first steps of the computation are as in the case of the lifting to $\widetilde{\mathrm{SO}}_{2k+2}(\mathbb{A})$. Up to (29) there are no changes and then we continue to obtain

$$\int \sum_{\delta,\epsilon} \varphi_\pi(g)\,\theta^{R_{2k+1},\psi_{2k+1}}(h(\epsilon)w_{2k+2}x_{\alpha_{2k+2}}(\delta)\tilde{w}z_0(u^{k+1},g)) \\ \times\psi_{U,a}(u^{k+1})\,du^{k+1}\,dg. \tag{42}$$

Here g is integrated over $L_{k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})$, where L_{k+1} is the maximal unipotent of SO_{2k+1}, and u^{k+1} is integrated over $U^{k+1}(F)\backslash U^{k+1}(\mathbb{A})$, where this group is defined similarly to the definition of the group U^k immediately after (29). The sum is over $\delta \in F$ and $\epsilon \in F^\times$, and $h(\epsilon) = \mathrm{diag}(I_{2k}, \epsilon, I_3, \epsilon^{-1}, I_{2k})$. Also $\tilde{w} = \tilde{w}_{2k} \cdots \tilde{w}_2$ where for $1 \leqslant i \leqslant k$ we set $\tilde{w}_{2i} = w_{2i} \cdots w_{k+i+1}$. Finally,

$$z_0 = \begin{pmatrix} I_{k+2} & \alpha_1 & * \\ & I_{2k+1} & \alpha_1^* \\ & & I_{k+2} \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here α_1^* and $*$ indicate entries that are chosen so that the matrix is in SO_{4k+5}.

At this point let us conjugate the matrix u^{k+1} from right to left. After a change of variables we obtain as inner integrations the integrals $\int \psi(1/2r(1-\epsilon\delta^2))\,dr$ and $\int \psi(r(1-\epsilon a))\,dr$ with r integrated over $F\backslash\mathbb{A}$. From this we obtain that a must be a square, and since it was initially chosen modulo squares we may assume that $a=1$. Hence $\epsilon = 1$ and $\delta \in \{\pm 1\}$. Thus (42) equals

$$\int_{L_{k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \sum_{\delta} \varphi_\pi(g)\, \theta^{R_{2k+1},\psi_{2k+1}}(wz(\delta)(1,g))\,dg \tag{43}$$

with $w = w_{2k+2}\tilde{w}$, and

$$z(\delta) = \begin{pmatrix} I_{k+2} & \alpha & * \\ & I_{2k+1} & \alpha^* \\ & & I_{k+2} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} 0 & 0 & 0 \\ I_k & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix}.$$

Here α^* and $*$ again indicate entries chosen so that the matrix is orthogonal.

To continue the computation, pull out the adelic points of L_{k+1}. Doing that we obtain the Whittaker model of the representation π. If we assume that the lift is generic then it follows that π must also be generic.

6 The lift $\widetilde{\mathrm{SO}}_{2n} \longrightarrow \widetilde{\mathrm{SO}}_{2n+1}$

If G is a group and π is a representation of a subgroup P, we will denote by $\mathrm{Ind}_P^G(\pi)$ the unnormalizedly induced representation of G. If we intend normalized induction, we will explicitly write $\mathrm{Ind}_P^G(\delta_G^{-1/2}\delta_P^{1/2} \otimes \pi)$, of course omitting $\delta_G^{1/2}$ if G is unimodular. We denote compact induction by ind.

Take the embedding of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$ in SO_{4n+1} which puts the SO_{2n+1} in the odd numbered rows and columns, and the SO_{2n} in the even numbered rows and columns. For example, if $n = 2$, then $\mathrm{SO}_5 \times \mathrm{SO}_4$ is embedded in SO_9 as follows.

$$\begin{pmatrix} * & & * & & * & & * & & * \\ & \bullet & & \bullet & & \bullet & & \bullet & \\ * & & * & & * & & * & & * \\ & \bullet & & \bullet & & \bullet & & \bullet & \\ * & & * & & * & & * & & * \\ & \bullet & & \bullet & & \bullet & & \bullet & \\ * & & * & & * & & * & & * \\ & \bullet & & \bullet & & \bullet & & \bullet & \\ * & & * & & * & & * & & * \end{pmatrix} \qquad * = \mathrm{SO}_5, \bullet = \mathrm{SO}_4\,.$$

We consider the lifting $\widetilde{\mathrm{SO}}_{2n} \longrightarrow \widetilde{\mathrm{SO}}_{2n+1}$. Let F be a nonarchimedean local field. Let $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ be n-tuples of unramified characters of $F^\times$. These are to parametrize principal series representations of $\widetilde{\mathrm{SO}}_{2n+1}(F)$ and $\widetilde{\mathrm{SO}}_{2n}(F)$ to be denoted $\pi_{2n+1}(\nu)$ and $\pi_{2n}(\mu)$.

We next describe the parametrization of $\pi_{2n+1}(\nu)$ and $\pi_{2n}(\mu)$. For any k, let T_k denote the diagonal torus of SO_k, and let $\tilde{T}_k(F)$ denote the preimage in $\widetilde{\mathrm{SO}}_k(F)$ of $T_k(F)$. It is a 2-step nilpotent group, and its irreducible genuine representations are finite-dimensional. The first step is to parametrize an irreducible representation of $\tilde{T}_k(F)$ (where $k = 2n + 1$ or $2n$) by the data ν or μ. We will denote elements of $\widetilde{\mathrm{SO}}_k(F)$ by pairs $\langle g, \varepsilon \rangle$ with $g \in \mathrm{SO}_k(F)$ and $\varepsilon \in \mu_4$, with the multiplication $\langle g, \varepsilon \rangle \langle g', \varepsilon' \rangle = \langle gg', \varepsilon\varepsilon'\sigma(g, g') \rangle$ and the cocycle σ described in [3] Section 2. The center $Z(\tilde{T}_k(F))$ consists of elements of the form

$$\left\langle \begin{pmatrix} y_1 & & & & & & \\ & \ddots & & & & & \\ & & y_n & & & & \\ & & & 1 & & & \\ & & & & y_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & y_1^{-1} \end{pmatrix}, \varepsilon \right\rangle \qquad y_i \in (F^\times)^2 \quad \text{if } k = 2n+1,$$

$$\left\langle \begin{pmatrix} y_1 & & & & & \\ & \ddots & & & & \\ & & y_n & & & \\ & & & y_n^{-1} & & \\ & & & & \ddots & \\ & & & & & y_1^{-1} \end{pmatrix}, \varepsilon \right\rangle \qquad y_i \in (F^\times)^2 \qquad \text{if } k = 2n.$$

(In [3], page 1366 it is incorrectly stated that the y_i must be fourth powers to be in the center.) As in [3] Section 2 the quasicharacter

$$\left\langle \begin{pmatrix} y_1 & & & & & & \\ & \ddots & & & & & \\ & & y_n & & & & \\ & & & 1 & & & \\ & & & & y_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & y_1^{-1} \end{pmatrix}, \varepsilon \right\rangle \longmapsto \varepsilon \prod_{k=1}^{n} \nu_k(y_k), \qquad (y_k \in (F^\times)^2)$$

of $Z(\tilde{T}_{2n+1}(F))$ can be extended arbitrarily to a genuine character of an arbitrarily chosen maximal abelian subgroup of $\tilde{T}_{2n+1}(F)$, then normalizedly-induced to $\tilde{T}_{2n+1}(F)$ to obtain an irreducible representation of $\tilde{T}_{2n+1}(F)$. The resulting representation does not depend on the choice of maximal abelian subgroup, nor the extension to it. See Bump and Ginzburg [4] Proposition 1.1 and Kazhdan and Patterson [10] Section 0.3 for further discussion of the point (essentially Clifford's Theorem) that the parametrized representation is independent of the choice of maximal abelian subgroup or extension to it.

Having parametrized a genuine irreducible representation of $\tilde{T}_{2n+1}(F)$, it may be extended to the inverse image $\tilde{B}_{\mathrm{SO}_{2n+1}}(F)$ of the standard Borel subgroup, so that the unipotent elements act trivially, then induced normalizedly to $\widetilde{\mathrm{SO}}_{2n+1}(F)$. This representation is $\pi_{2n+1}(\nu)$.

Similarly $\pi_{2k}(\mu)$ is induced from the character

$$\begin{pmatrix} y_1 & & & & & \\ & \ddots & & & & \\ & & y_n & & & \\ & & & y_n^{-1} & & \\ & & & & \ddots & \\ & & & & & y_1^{-1} \end{pmatrix} \longmapsto \prod_{k=1}^{n} \mu_k(y_k). \tag{44}$$

We will say that μ and ν are in *general position* if they are in the complement of an effectively computable subset of measure zero in the unitary dual of $(F^\times)^n$. We will not describe this subset explicitly since conditions on μ and ν can appear in different places of the argument. If μ and ν are in general position, then $\pi_{2n}(\mu)$ and $\pi_{2n+1}(\nu)$ are irreducible.

Let W be the SO_{2n+1} Weyl group, a group of order $2^n \cdot n!$ generated by permutations of the ν_i and 2^n transformations which map each $\nu_k \longrightarrow \nu_k^{\pm 1}$. Applying an element of W does not affect the isomorphism class of $\pi_{2n+1}(\nu)$ if $\pi_{2n+1}(\nu)$ is irreducible, which is true when ν is in general position.

Theorem 4 *Assume that μ and ν are in general position and that there exists a nonzero $\widetilde{\mathrm{SO}}_{2n}(F) \times \widetilde{\mathrm{SO}}_{2n+1}(F)$-equivariant map $\theta_{4n+1} \otimes \pi_{2n}(\mu) \longrightarrow \pi_{2n+1}(\nu)$. Then after applying an element of W to ν, we may arrange that each $\nu_k = \mu_k$.*

This means that if we associate to ν and μ the conjugacy classes A_ν and A_μ in the "L-groups" $\mathrm{SO}_{2n+1}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{C})$ of $\widetilde{\mathrm{SO}}_{2n+1}$ and $\widetilde{\mathrm{SO}}_{2n}$ having eigenvalues $\nu_k^{\pm 1}, 1$ and $\mu_k^{\pm 1}$, then A_ν is the image of A_μ under the obvious inclusion. As we have explained in the introduction, this means that the formalism of Langlands functoriality applies in this metaplectic setting, and the lift is functorial.

The proof will occupy the rest of the section. We claim that it is sufficient to show that

$$\nu_1 \in \{\mu_1, \cdots, \mu_n, \mu_1^{-1}, \cdots, \mu_n^{-1}\}. \tag{45}$$

Indeed, we are assuming that μ and ν are in general position, so we may assume $\mu_1, \cdots, \mu_n, \mu_1^{-1}, \cdots, \mu_n^{-1}$ are distinct, as are $\nu_1, \cdots, \nu_n, \nu_1^{-1}, \cdots, \nu_n^{-1}$. If we prove (45), then without loss of generality we may assume $\nu_1 =$

μ_1. Since $\pi_{2n+1}(\nu) = \pi_{2n+1}(\nu')$ where ν' is the image of ν under any element of the Weyl group, the same argument then shows that ν_2 is one of $\mu_1, \cdots, \mu_n, \mu_1^{-1}, \cdots, \mu_n^{-1}$ but it cannot equal μ_1 or μ_1^{-1}, since the characters $\mu_1, \cdots, \mu_n, \mu_1^{-1}, \cdots, \mu_n^{-1}$ are distinct, as are $\nu_1, \cdots, \nu_n, \nu_1^{-1}, \cdots, \nu_n^{-1}$. Applying another Weyl group element, we may thus assume that $\nu_2 = \mu_2$. Continuing in this fashion, the theorem is proved.

Let V (previously denoted U_{2n+1}) be the unipotent radical of the parabolic subgroup $P_{1,2n-1}$ of SO_{2n+1} with Levi $\mathrm{GL}(1) \times \mathrm{SO}(2n-1)$. Then $\pi_{2n+1}(\nu)$ is parabolically induced from the representation $\nu_1 \otimes \pi_{2n-1}(\nu')$ of $\widetilde{\mathrm{GL}}(1) \times \widetilde{\mathrm{SO}}(2n-1)$, where now $\nu' = (\nu_2, \cdots, \nu_n)$. Let R and Q be the groups of matrices of the form

$$R = \begin{pmatrix} a & * & * & * & * \\ & a & * & * & * \\ & & \boxed{\mathrm{SO}_{4n-3}} & * & * \\ & & & a^{-1} & * \\ & & & & a^{-1} \end{pmatrix}, \qquad Q = \begin{pmatrix} a & * & * \\ & \boxed{\mathrm{SO}_{4n-1}} & * \\ & & a^{-1} \end{pmatrix},$$

respectively. In particular $Q = (\mathrm{GL}_1 \times \mathrm{SO}_{4n-1})U$, where U is the unipotent radical consisting of upper triangular unipotent elements of SO_{4n+1} with nonzero off-diagonal entries in the first row and last columns only. Let θ_U denote the Jacquet module of $\theta = \theta_{4n+1}$ with respect to U. Also let $\psi_U : U \longrightarrow \mathbb{C}$ be $\psi_U(u) = \psi(u_{12})$, so that R is the stabilizer of ψ_U in Q. Let θ_{U,ψ_U} denote the twisted Jacquet functor with respect to this character.

We note that any character of U is of the form $\psi(\langle r, u\rangle)$ where $r \in F^{4n-1}$. By Proposition 3 the Jacquet module of θ with respect to such a character vanishes if r has nonzero length. The kernel of the natural map $\theta \longrightarrow \theta_U$ is glued from the Jacquet modules of nonzero characters of U, and by Proposition 3, only those corresponding to r of length zero are nonvanishing. The group Q acts transitively on these, and a typical one of these is ψ_U, with stabilizer R in Q. It follows as in Proposition 5.12 (d) of Bernstein and Zelevinsky [1] that there is an exact sequence

$$0 \longrightarrow \mathrm{ind}_R^Q(\theta_{U,\psi_U}) \longrightarrow \theta \longrightarrow \theta_U \longrightarrow 0.$$

Note that $\mathrm{ind}_R^Q(\theta_{U,\psi_U})$ is compactly induced. Regarding these as modules for $\tilde{P}_{1,2n-1} \times \widetilde{\mathrm{SO}}_{2n}$ we may then apply the ordinary Jacquet functor with respect to V and obtain an exact sequence

$$0 \longrightarrow \mathrm{ind}_R^Q(\theta_{U,\psi_U})_V \longrightarrow \theta_V \longrightarrow \theta_U \longrightarrow 0. \tag{46}$$

By Frobenius reciprocity, there exists a nonzero $\widetilde{\mathrm{GL}}_1 \times \widetilde{\mathrm{SO}}_{2n-1} \times \widetilde{\mathrm{SO}}_{2n}$-equivariant bilinear map

$$\theta_V \otimes \pi_{2n}(\mu) \longrightarrow \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}.$$

We note that

$$\mathrm{Hom}_{\mathrm{GL}_1 \times \mathrm{SO}_{2n-1} \times \mathrm{SO}_{2n}}(\theta_U \otimes \pi_{2n}(\mu), \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}) = 0. \tag{47}$$

Indeed the $\widetilde{\mathrm{GL}}_1$ acts by a (computable) fixed character on θ_U, and by ν_1 on the right; since we are assuming ν is in general position, we obtain the vanishing statement (47). Hence by (46) there is a nonzero $\widetilde{\mathrm{GL}}_1 \times \widetilde{\mathrm{SO}}_{2n-1} \times \widetilde{\mathrm{SO}}_{2n}$-equivariant map

$$\mathrm{ind}_R^Q(\theta_{U,\psi_U})_V \otimes \pi_{2n}(\mu) \longrightarrow \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}.$$

Let $H = P_{1,2n-1} \times \mathrm{SO}_{2n} = (\mathrm{GL}_1 \times \mathrm{SO}_{2n-1} \times \mathrm{SO}_{2n})V$. By Mackey theory in the form of Bernstein and Zelevinsky [2] Theorem 5.2, if τ and σ are representations of R and H respectively, the space $\mathrm{Hom}_H(\mathrm{ind}_R^Q(\tau), \sigma)$ is glued from the spaces $\mathrm{Hom}_{S_\gamma}({}^\gamma\tau, \delta_H^{-1}\delta_{S_\gamma} \otimes \sigma)$ where γ runs through a set of representatives of the double cosets $R\backslash Q/H$, and $S_\gamma = H \cap \gamma^{-1}R\gamma$. In the case at hand, it may be checked that there is only one double coset $R\gamma H$ such that

$$\mathrm{Hom}_{S_\gamma}({}^\gamma(\theta_{U,\psi_U}) \otimes \pi_{2n}(\mu), \delta_H^{-1}\delta_{S_\gamma} \otimes \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}) \neq 0. \tag{48}$$

We may take $\gamma = 1$ as a representative of this double coset. Then $S_\gamma = H \cap R$, when $\gamma = 1$, and we denote this group by S. It is the image of $P_{1,2n-1} \times P_{1,2n-2}$ in $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$. For example if $n = 2$, S consists of matrices of the form

$$\begin{pmatrix} a & & * & & * & & * & & \circledast \\ & a & & * & & * & & \circledast & \\ & & * & & * & & \circledast & & * \\ & & & b & & & & * & \\ & & * & & \circledast & & * & & * \\ & & & & & b^{-1} & & * & \\ & & \circledast & & * & & * & & * \\ & & & & & & & a^{-1} & \\ & & & & & & & & a^{-1} \end{pmatrix}.$$

(The locations marked $\circledast$ are zero in the Lie algebra of SO_9.) Thus we have a nonzero element of

$$\mathrm{Hom}_S(\theta_{U,\psi_U} \otimes \pi_{2n}(\mu), \delta_H^{-1}\delta_S \otimes \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}).$$

Now the Proposition 4 means that θ_{U,ψ_U} factors factors through the Jacquet module with respect to the parabolic subgroup $P_{2,4n-3}$ of SO_{4n+1} whose Levi factor is $\mathrm{GL}_2 \times \mathrm{SO}_{4n-3}$, because the unipotent radical of this parabolic is generated by its first row, which is contained in the kernel of ψ_U, and the second row, which is dealt with by the Lemma. If $\Theta_2 \otimes \theta_{4n-3}$ is this theta representation of $\widetilde{\mathrm{GL}}_2 \times \widetilde{\mathrm{SO}}_{4n-3}$ then we can identify θ_{U,ψ_U} with $\omega \otimes \theta_{4n-3}$ where ω is the twisted Jacquet module with respect to the standard maximal unipotent of GL_2 of Θ_2. All we care about is the value of ω on the center of GL_2, which is a subgroup we will denote by GL_1^Δ. It corresponds to the locations marked a in the definition of R. This can be read off from (2.21) of [BFG] by taking $m = 2$ and n replaced by our $2n$. We have

$$\omega \begin{pmatrix} a & \\ & a \end{pmatrix} = |a|^{2n-3/2}.$$

Thus we have a nonzero element in

$$\mathrm{Hom}_{\mathrm{SO}_{2n-1} \times (\mathrm{GL}_1^\Delta \times \mathrm{SO}_{2n-2}) N_{1,2n-2}}(\omega \otimes \pi_{2n}(\mu), \delta_H^{-1}\delta_S \otimes \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}),$$

where $N_{1,2n-2}$ is the unipotent radical of the parabolic subgroup $P_{1,2n-2}$ of SO_{2n}. Since $N_{1,2n-2}$ acts nontrivially only on $\pi_{2n}(\mu)$, we may replace $\pi_{2n}(\mu)$ by its ordinary Jacquet module with respect to this parabolic, which, since μ is in general position, is a direct sum of irreducible representations of $\widetilde{\mathrm{GL}}_1 \times \widetilde{\mathrm{SO}}_{2n-2}$, or which a typical one is $\mu_1 \otimes \pi_{2n-2}(\mu') \otimes \delta_{P_{1,2n-2}}^{1/2}$, where $\mu' = (\mu_2, \cdots, \mu_n)$. At least one of these has a nonzero contribution. To prove (45), because we are only asserting that ν_1 is one of the $\mu_k^{\pm 1}$, we may assume that this nonzero contribution is $\mu_1 \otimes \pi_{2n-2}(\mu') \otimes \delta_{P_{1,2n-2}}^{1/2}$. In this case, we will prove $\nu_1 = \mu_1$; if the nonzero contribution is one of the other constituents of this Jacquet module, we would obtain some other $\mu_k^{\pm 1}$.

We obtain a nonzero $\mathrm{SO}_{2n-1} \times (\mathrm{GL}_1^\Delta \times \mathrm{SO}_{2n-2})$ equivariant map

$$\omega \otimes \mu_1 \otimes \pi_{2n-2}(\mu') \otimes \delta_{P_{1,2n-2}}^{1/2} \longrightarrow \delta_H^{-1}\delta_S \otimes \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2}.$$

We note that $\delta_H = \delta_{P_{1,2n-1}}$ and $\delta_S = \delta_{P_{1,2n-1}}\delta_{P_{1,2n-2}}$, so this gives us an equivariant map

$$\omega \otimes \mu_1 \otimes \pi_{2n-2}(\mu') \longrightarrow \nu_1 \otimes \pi_{2n-1}(\nu') \otimes \delta_{P_{1,2n-1}}^{1/2} \delta_{P_{1,2n-2}}^{1/2}$$

On $a \in \mathrm{GL}_1^\Delta$, we have

$$\omega = |a|^{2n-3/2}, \qquad \delta_{1,2n-2}^{1/2} = |a|^{n-1}, \qquad \delta_{1,2n-1}^{1/2} = |a|^{n-1/2}.$$

These precisely cancel, so $\mu_1(a) = \nu_1(a)$. This completes the proof of the Theorem.

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