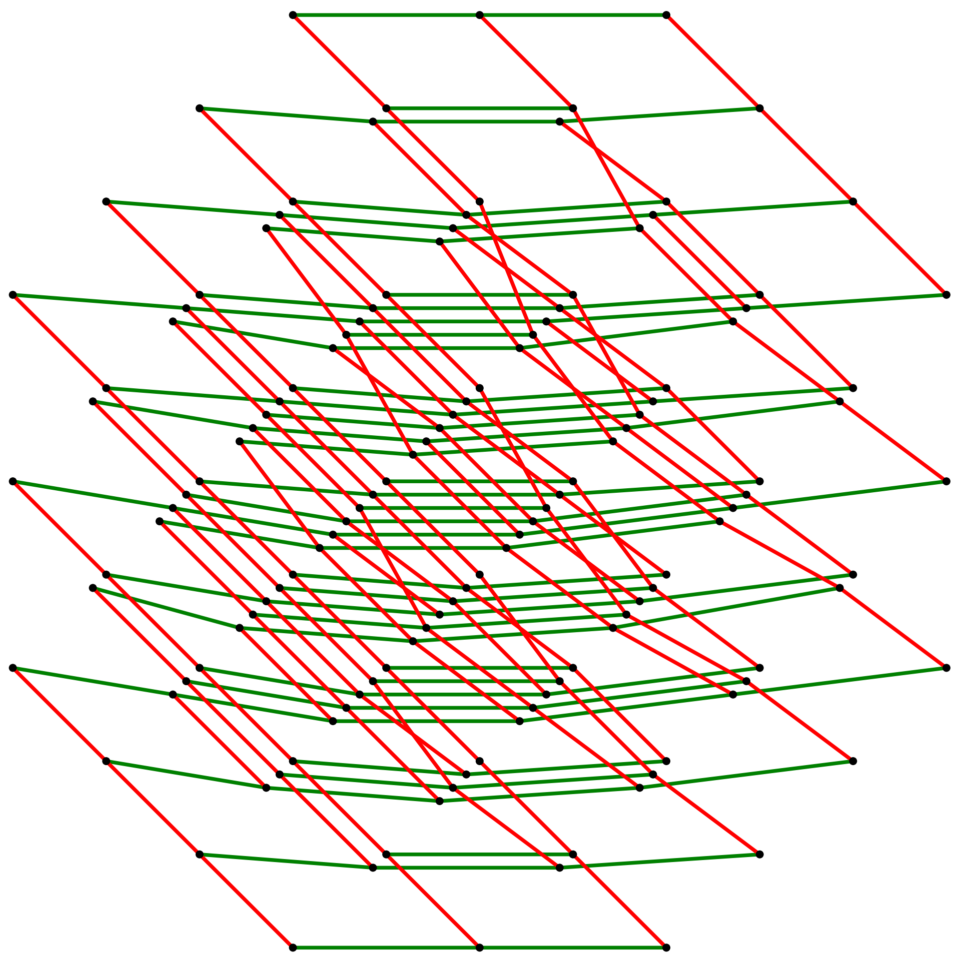
Crystals and Multiple Dirichlet Series



Brubaker, Bump, Chinta, Friedberg, Gunnells

Work In Progress

Quantum groups and Crystal Bases

- **Quantum groups** (Drinfeld, Jimbo) are deformations (in a suitable category) of Lie groups.
- **Crystal Bases** were introduced by Kashiwara in connection with quantum groups.

From Groups to Quantum Groups

- Lie groups do not have deformations in their own category, so one works in the category of Hopf algebras.

If G is a Lie group, we have maps:

$$\begin{array}{ll} \text{multiplication:} & m: G \times G \longrightarrow G \\ \text{diagonal:} & \delta: G \longrightarrow G \times G \end{array}$$

satisfying the associative laws and a compatibility property

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ \Big\downarrow 1 \times m & & \Big\downarrow m \\ G \times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\delta} & G \times G \\ \Big\downarrow \delta & & \Big\downarrow 1 \times \delta \\ G \times G & \xrightarrow{\delta \times 1} & G \times G \times G \end{array}$$

$$\begin{array}{ccc} G \times G & \xrightarrow{\delta \times \delta} & G \times G \times G \times G \\ \Big\downarrow m & & \Big\downarrow \\ G & \xrightarrow{\delta} & G \times G \end{array}$$

rightmost vertical map:
$(x, y, z, w) \mapsto (m(x, z), m(y, w))$

The **universal enveloping algebra** $U(\mathfrak{g})$ of $\mathfrak{g} = \mathrm{Lie}(G)$ is a functor from groups to $\mathbb{C}$-algebras. So $U = U(\mathfrak{g})$ becomes an algebra and a co-algebra, with associative maps

$$\begin{array}{ll} \text{multiplication:} & m: U \otimes U \longrightarrow U \\ \text{diagonal:} & \delta: U \longrightarrow U \otimes U \end{array}$$

and a compatibility that makes it a **Hopf algebra.**

Deformation

- The deformation (Drinfeld, Jimbo) $U_q(\mathfrak{g})$ is a Hopf algebra depending on a parameter $q \in \mathbb{C}$.
- If $q = 1$ it is U.
- $q \to 1$ is like classical limit $\hbar \longrightarrow 0$ of quantum mechanics.

Representations

- The irreducible modules of G or U are the **same** as the irreducible modules of $U_q(\mathfrak{g})$. If $\pi\colon G \longrightarrow \mathrm{GL}(V)$ is a representation, V is a module for U and also for $U_q(\mathfrak{g})$ for all q.
- However $U_q(\mathfrak{g})$-module structure on $V_1 \otimes V_2$ (two modules) varies with q.

The significance of the comultiplication is that it determines a multiplicative structure on the tensor category of representations.

- If V is a G-module it is a U-module.
- If V_1, V_2 are modules then $V_1 \otimes V_2$ is too. Comultiplication $\delta\colon U \longrightarrow U \otimes U$ implements this for the Hopf algebra U.
- For $U_q(\mathfrak{g})$ comult δ is **required** for this. (Group is gone.)
- The Hopf algebra U is **cocommutative** — a property it inherits from the diagonal map:

<table>
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<td>

$$\begin{tikzcd} G \arrow[rr, "\delta"] \arrow[dr, "\delta"'] & & G \times G \arrow[dl, "\tau"] \\ & G \times G & \end{tikzcd}$$

</td>
<td>

$$\begin{tikzcd} U \arrow[rr, "\delta"] \arrow[dr, "\delta"'] & & U \otimes U \arrow[dl, "\tau"] \\ & U \otimes U & \end{tikzcd}$$

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<tr>
<td>$\tau\colon (x, y) \longmapsto (y, x)$</td>
<td>$\tau\colon x \otimes y \longmapsto y \otimes x$</td>
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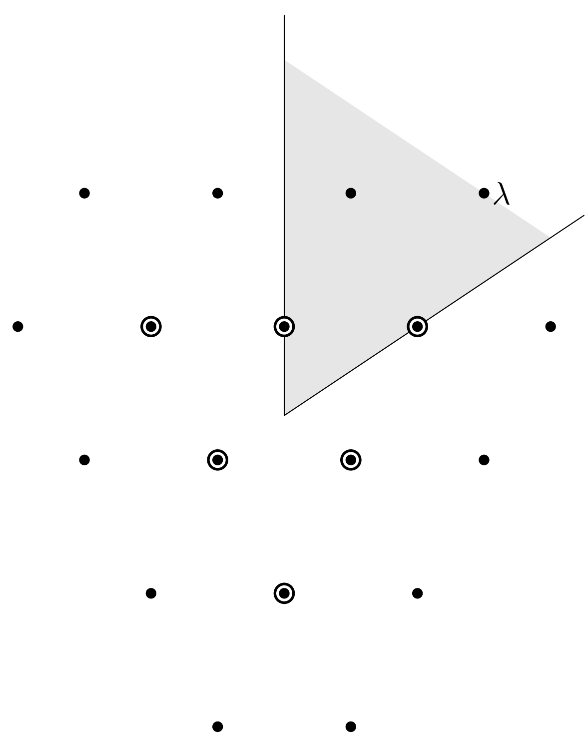
- The deformation $U_q(\mathfrak{g})$ is no longer cocommutative.

Highest Weight Modules (Weyl)

Let G be a semisimple complex Lie group of rank r. Let T be a maximal torus. Then $X^*(T) \cong \mathbb{Z}^r$.

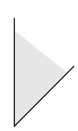
- Elements of $X^*(T) \cong \mathbb{Z}^r$ are called **weights**
- $\mathbb{R} \otimes X^*(T) \cong \mathbb{R}^r$ has a fundamental domain $\mathcal{C}^+$ for the Weyl group W called the **positive Weyl chamber**.
- If (π, V) is a representation then restricting to T, the module V decomposes into a direct sum of weight eigenspaces $V(\mu)$ with multiplicity $m(\mu)$ for weight μ.
- There is a unique highest weight λ wrt partial order. We have $\lambda \in \mathcal{C}^+$ and $m(\lambda) = 1$.
- $V \longleftrightarrow \lambda$ gives a **bijection** between irreducible representations and weights λ in $\mathcal{C}^+$.



Here are the weights of an irreducible V for $G = \mathrm{SL}_3$, showing multiplicities and the pos. Weyl chamber.

• $m(\mu) = 1$

⊙ $m(\mu) = 2$

 $\mathcal{C}^+$

Legend

Root Operators

Let $\alpha_1, \cdots, \alpha_r$ be the simple positive roots. For each there is a pair of **root operators** in the Lie algebra $\mathfrak{g} \subset U$:

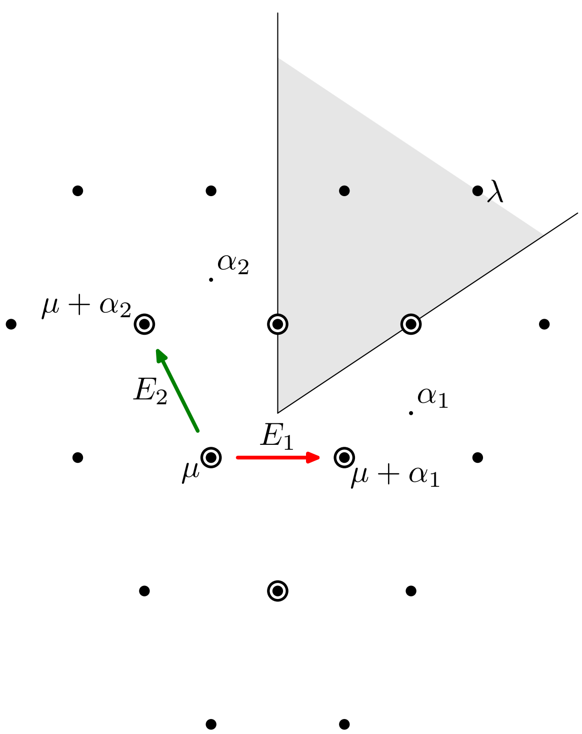
$$\begin{array}{ll} E_i & \text{corresponding to } \ \alpha_i, \\ F_i & \text{corresponding to } \ -\alpha_i. \end{array}$$

For example if $G = \mathrm{SL}(3)$:

$$E_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ F_i = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- The root operators shift the weight. Thus

$$E_i: V(\mu) \longrightarrow V(\mu + \alpha_i)$$
$$F_i: V(\mu) \longrightarrow V(\mu - \alpha_i)$$



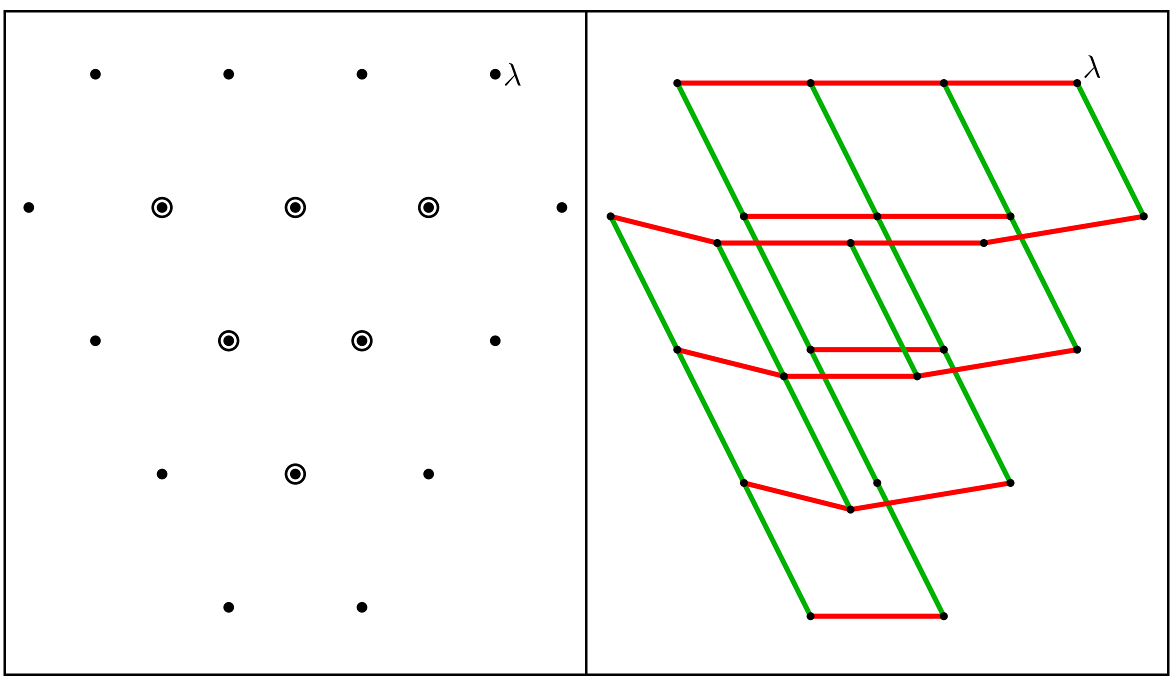
The root operator E_1 maps the 2-dimensional vector space $V(\mu)$ into $V(\mu + \alpha_1)$, and E_2 maps $V(\mu) \longrightarrow V(\mu + \alpha_2)$.

For pictorial purposes associate a color with each root operator.

Crystalization (Kashiwara)

It is not actually possible to take $q=0$ in the Hopf algebra $U_q(\mathfrak{g})$.

- In the limit $q \longrightarrow 0$ the semisimple part of $\mathfrak{g}$ breaks down.
- Still E_i and F_i can be continued to $q=0$.
- Their effect beomes very simple.
- It is possible to choose a **crystal basis** v_i such that In the limit $q = 0$: **each E_i or F_i takes a basis element to another basis element or 0**.
- The action of the E_i and F_i on the crystal basis can be illustrated very simply by giving the **crystal graph** $\mathfrak{C}_\lambda$.
- Existence proofs were given by Kashiwara, Kashiwara and Nakashima, Lusztig and Littelmann (path model).



Weyl Group Multiple Dirichlet Series

Given a root system Φ and a field F containing the n-th roots of unity, we may (always or often) construct a family of multiple Dirichlet series

$$\sum_{c_i} H(c_1, \cdots, c_r; m_1, \cdots, m_r) \prod_{i=1}^{r} \mathbb{N}c_i^{-2s_i}$$

having analytic continuation and functional equations in the s_i.

- Here c_i are ideals of S-integers $\mathfrak{o}_S$ (a PID for suitable S).
- The coefficients are **twisted multiplicative**. Let $\left(\frac{*}{*}\right)$ be the n-th power residue symbol. If $\gcd(\prod C_i, \prod C_i') = 1$:

$$\frac{H(C_1C_1', \cdots, C_rC_r', m_1, \cdots, m_r)}{H(C_1, \cdots, C_r, m_1, \cdots, m_r)\; H(C_1', \cdots, C_r', m_1, \cdots, m_r)} =$$
$$\prod_{i=1}^{r} \left(\frac{C_i}{C_i'}\right)^{\|\alpha_i\|^2} \left(\frac{C_i'}{C_i}\right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{C_i}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{C_i'}{C_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}$$

- When $\gcd(C_1 \cdots C_r, m_1' \cdots m_r') = 1$:

$$H(C_1, \cdots, C_r; m_1m_1', \cdots, m_rm_r') =$$
$$\left(\frac{m_1'}{C_1}\right)^{-1} \cdots \left(\frac{m_r'}{C_r}\right)^{-1} H(C_1, \cdots C_r; m_1, \cdots, m_r).$$

- These rules reduce the specification if H's to the case C_i and m_i are all powers of the same prime.
- If n is large a satisfactory theory is in place (Brubaker, Bump and Friedberg)
- If n is small the story is complex and interesting and under development (Brubaker, Bump, Chinta, Friedberg and Gunnells). This is the **unstable case**.

Three Approaches

There are about 3 approaches to defining the functions H. As noted above, we only need to define the ***p*-part**

$$H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}).$$

- The **Crystal Description** generalizes the **Gelfand-Tsetlin description** found for type A_r by Brubaker, Bump, Friedberg and Hoffstein. This definition may be translated into sums over crystals. This works well sometimes, and in other cases will require refinement.
- The **Chinta-Gunnells Description** involves alternating sums over Weyl groups analogous to the Weyl character formula.
- The **Eisenstein-Whittaker Description** describes the p-part in terms of metaplectic Whittaker coefficients. In approach, the global MDS is regarded as a Whittaker coefficient of an Eisenstein series.

It is not clear at the outset that these three approaches are equivalent. However progress is being made towards unifying them.

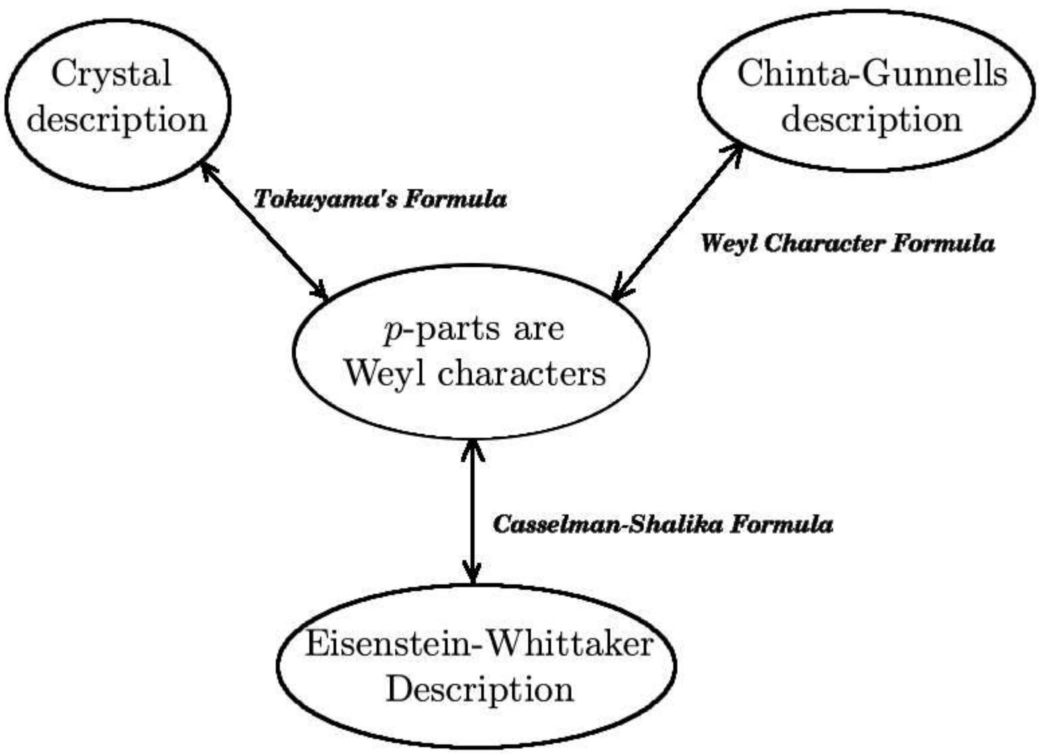
- For the Gelfand-Tsetlin (Crystal) description Brubaker, Bump, Friedberg proved equivalence of two versions of the definition, related by the Schützenberger involution. This implies analytic continuation of the Weyl Group MDS.
- The method of Chinta and Gunnells now works in the most general case but only in certain cases are the coefficients very explicitly known. They can be computed by computer and checked to agree with other definitions.
- The third method awoke in 2008 after a long nap.
- In at least one case all three methods are known to agree.

The Nonmetaplectic Case

When $n = 1$ all three methods agree in at least two cases:

- Type A: Tokuyama + Casselman-Shalika
- Type C: Hamel and King and Beineke, Brubaker, Frechette

Cartan type A or C when $n = 1$ (nonmetaplectic)



- For other Cartan types, the Crystal description needs more investigation but the picture should remain valid.
- The other links are valid for all Cartan types when $n = 1$.
- The underlying Lie group is the Langlands L-group.
- These Weyl Group MDS are Euler products.

Tokuyama's formula (Type A)

- Tokuyama's formula (1988) predates crystals. It was stated in the language of Gelfand-Tsetlin patterns.

The following data are equivalent (for type A):

Tableaux	Gelfand-Tsetlin Patterns	Vertices in Crystal Graphs

We can translate Tokuyama's formula into the crystal language.

- It is clear that a Weyl character χ_λ with highest weight λ can be expressed as a sum over the crystal graph $\mathfrak{C}_\lambda$. **This is not Tokuyama's formula**.

- Tokuyama's formula gives χ_λ as a ratio with **numerator** a sum over $\mathfrak{C}_{\lambda+\rho}$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. $\quad (\Phi_+ = \text{pos. roots.})$

Weyl Character Formula (WCF)

We return to a Lie group G with maximal torus T. The Weyl character formula takes place in the character ring $\mathbb{C}[\Lambda]$ of T. If $\mu \in \Lambda = X^*(T)$ is a weight, we will denote its image in this character ring as e^μ. Thus $e^\mu e^\nu = e^{\mu+\nu}$ since we are writing the group law in $\Lambda \cong \mathbb{Z}^r$ additively.

Let $\lambda \in \mathcal{C}_+$ be weight in the positive Weyl chamber, and let χ_λ be the character of its highest weight module. WCF asserts:

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{l(w)} w(e^{\lambda+\rho})}{\sum_{w \in W} (-1)^{l(w)} w(e^\rho)} \qquad (W = \text{Weyl group}).$$

- Tokuyama's formula gives a deformation of the numerator, which may be expressed as a sum over the crystal $\mathfrak{C}_{\lambda+\rho}$.

Berenstein-Zelevinsky-Littelmann strings

Fix a reduced decomposition of the long Weyl group element into simple reflections.

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}, \qquad s_i = s_{\alpha_i}.$$

Now place a turtle at a vertex v of the crystal. The sequence

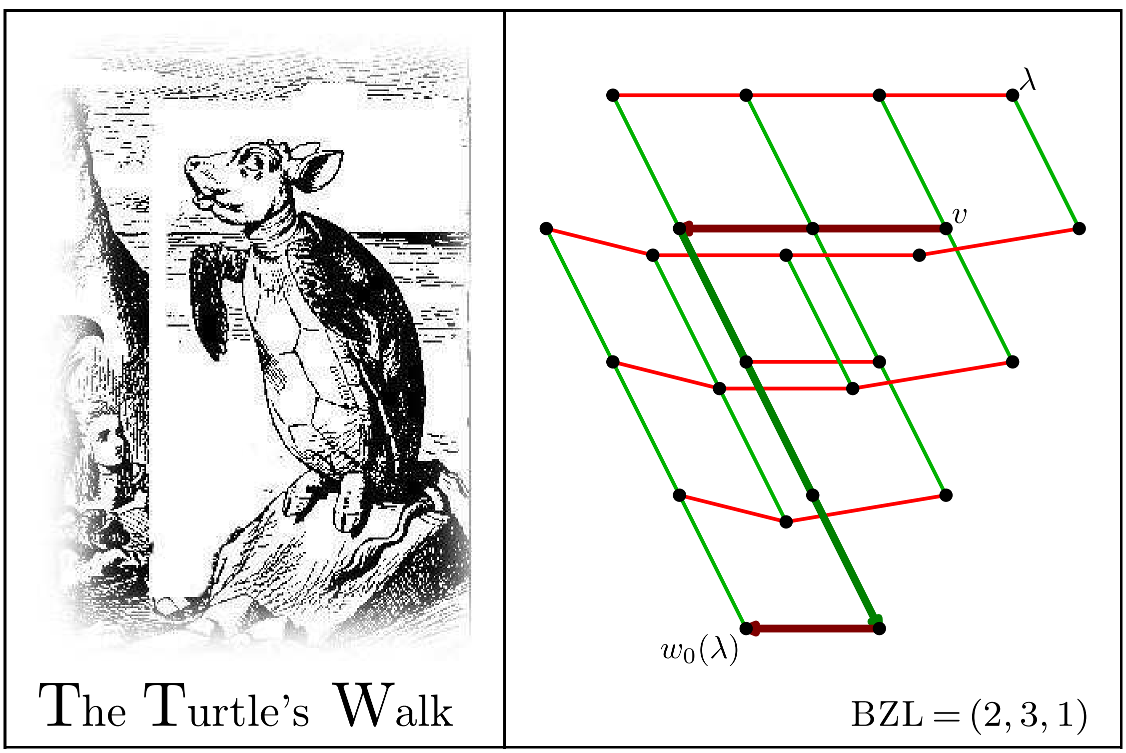
$$i_1, i_2, \cdots, i_N$$

will be a code telling the turtle the order in which to follow the root operators.

- Let k_1 be the largest integer such that $F_{i_1}^{k_1}(v) \neq 0$.
- In other words, the turtle can move along the α_{i_1}-colored edge a distance of k_1 then no further.

Then the process is repeated.

- k_2 is the largest integer such that $F_{i_2}^{k_2} F_{i_1}^{k_1} v \neq 0$.
- The turtle winds up at the vertex with weight $w_0(\lambda)$.



- The **BZL string** is $(k_1, k_2, \cdots)$. It uniquely determines v.

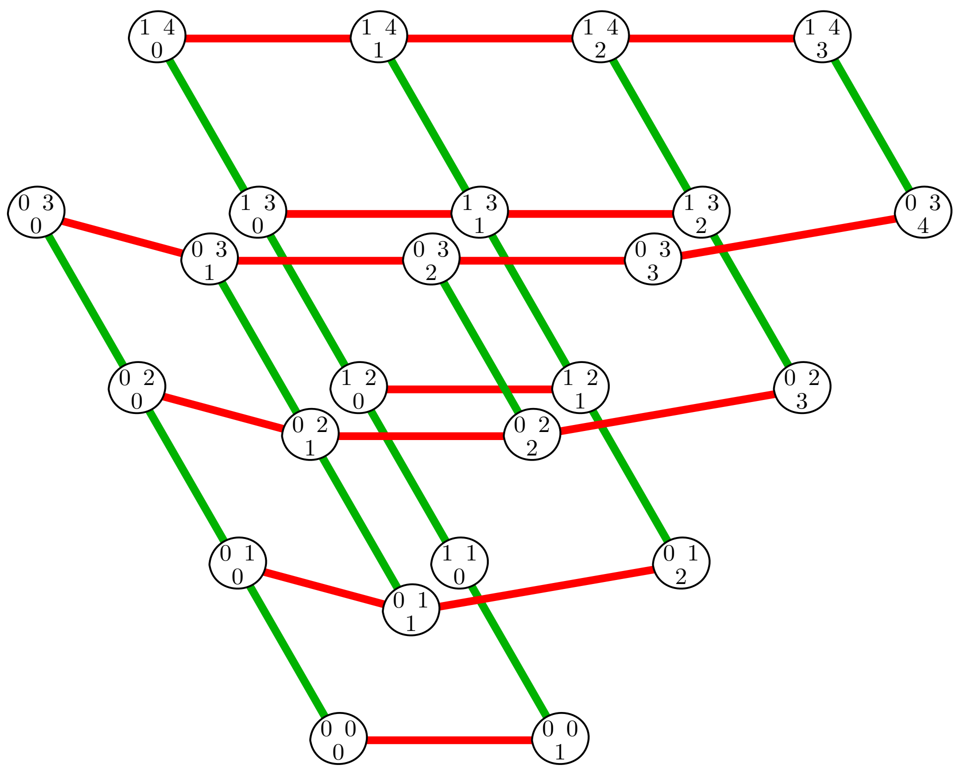
Decorating the BZL string

Before we can state Tokuyama's theorem we must **decorate** the BZL string. We are in type A_r and use the decomposition

$$w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_rs_{r-1}\cdots s_2s_1).$$

We follow Littelmann, arranging the BZL by inserting entries into a triangular array from top to bottom and right to left.

- Thus $(2,3,1)$ becomes $\left\{\begin{array}{ccc} 1 & & 2 \\ & 3 & \end{array}\right\}$.



- We will decorate certain entries by boxing or circling them:

$$\left\{\begin{array}{ccc} 0 & & 2 \\ & \boxed{3} & \end{array}\right\}$$

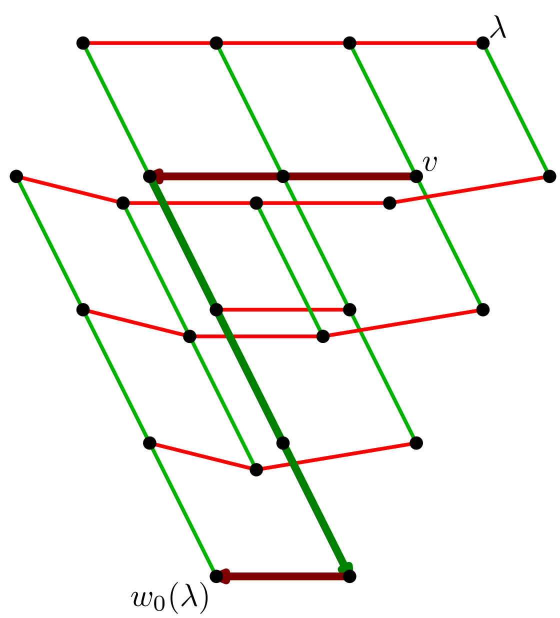
Boxing

The boxing rule is quite simple to understand.

- If $(k_1,\ k_2,\ k_3,\ \cdots)$ is the BZL, so that $F_{i_m}^{k_m} \cdots F_{i_1}^{k_1}(v)$ is the location of the turtle after m moves, then we box k_m if

$$e_{i_m} f_{i_{m-1}}^{k_{m-1}} \cdots f_{i_1}^{k_1}(v) = 0.$$

- This means the turtle travels the entire length of the root string corresponding to the m-th move.



- In this example, the turtle travels the entire length of the first segement (of length 2) and the last (of length 1) but not the second. So we box 2 and 1 but not 3:

$$\left\{ \begin{array}{ccc} \boxed{1} & & 3 \\ & \boxed{2} & \end{array} \right\}$$

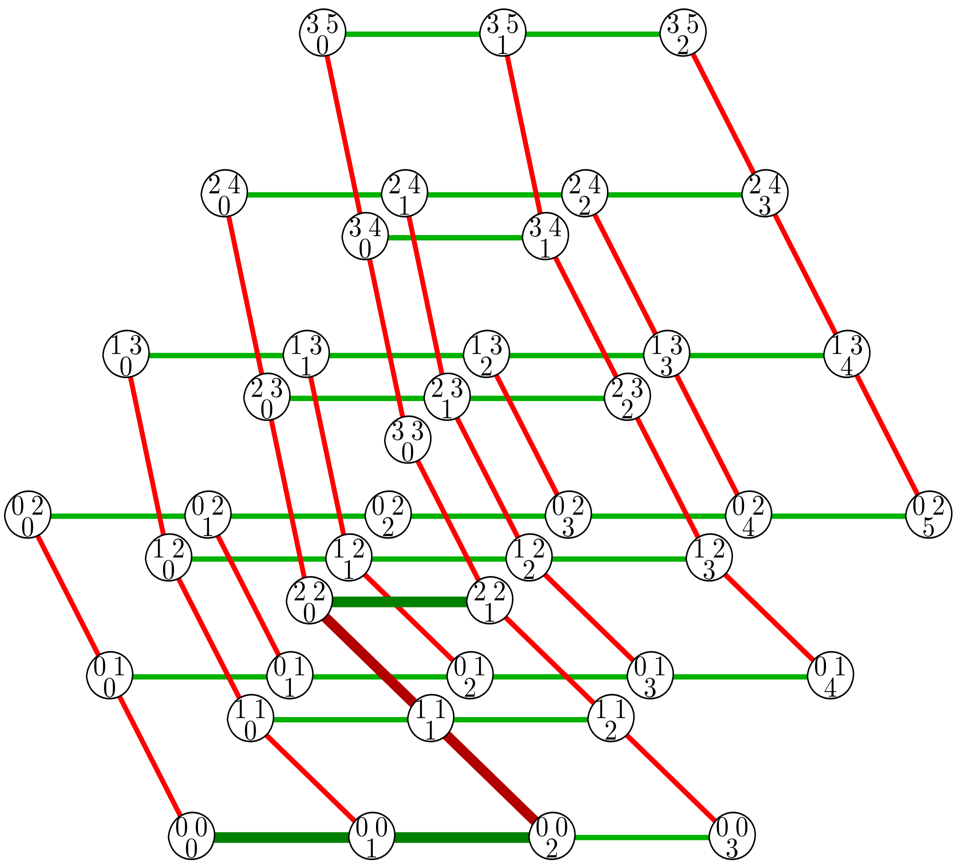
Circling

- Littelmann proved that if the the BZL pattern is arranged in a triangle as above, the **rows are nonincreasing.** So

$$a_{i\,1} \geqslant a_{i\,2} \geqslant \cdots \geqslant a_{i,r+1-i} \geqslant 0$$

  for the i-th row $(a_{i\,1}, a_{i\,2}, \cdots, a_{i,r+1-i})$.

- If any of these inequalities is an equality, we circle the corresponding entry. Thus if $a_{i,j} = a_{i,j+1}$ we circle $a_{i,j}$ or if the last entry vanishes we circle it.



The second row is $(2, 2)$ with equality $2 = 2$ so we circle the first 2:

$$\left\{\begin{array}{ccc} 2 & & 2 \\ & \boxed{1} & \end{array}\right\}$$

Tokuyama's Theorem

We may now formulate Tokuyama's theorem in crystal terms. If $a > 0$ let $g(a) = -q^{a-1}$, $h(a) = (q-1)q^{a-1}$. If $v \in \mathfrak{C}_{\lambda+\rho}$ define

$$G(v) = \prod_{a \in \mathrm{BZL}(v)} \begin{cases} q^a & \text{if } a \text{ is circled,} \\ g(a) & \text{if } z \text{ is boxed,} \\ h(a) & \text{if neither,} \\ 0 & \text{if both.} \end{cases}$$

Let s_λ be the Schur polynomial so that if $g \in \mathrm{GL}_{r+1}(\mathbb{C})$ has eigenvalues $\alpha_1, \cdots, \alpha_{r+1}$ then

$$\chi_\lambda(g) = s_\lambda(\alpha_1, \cdots, \alpha_{r+1}).$$

Tokuyama's theorem may be stated as follows:

$$q^{-\lambda_r - 2\lambda_{r-1} - \ldots - r\lambda_1} s_\lambda(\alpha_1, q\alpha_2, q^2\alpha_3, \cdots, q^r\alpha_{r+1}) = \frac{\displaystyle\sum_{v \in \mathfrak{C}_{\lambda+\rho}} G(v) e^{\mathrm{wt}(v)}}{\displaystyle\prod_{i<j} (1 - q^{i-j-1}\alpha_i\alpha_j^{-1})},$$

where $\mathrm{wt}\colon \mathfrak{C}_{\lambda+\rho} \longrightarrow \Lambda$ is the weight map to the weight lattice.

- If $q \longrightarrow 1$ all but $(r+1)!$ terms in the numerator become zero.
- Those that survive have are those with weight $w(\lambda+\rho)$ for some $w \in W$.
- In this specialization, Tokuyama's formula becomes the Weyl character formula.

The Gelfand-Tsetlin Description

- Brubaker, Bump, Friedberg and Hoffstein gave a description of Weyl group MDS for the n-fold cover of Type A_r in terms of Gelfand-Tsetlin patterns.
- If $n=1$ then this boils down to Tokuyama's formula.
- Brubaker, Bump and Friedberg gave a proof of the functional equations of the MDS by proving a difficult combinatorial statement.
- The Chinta-Gunnells description also produces a Weyl group MDS with functional equations. Conjecturally they are equivalent but this has only been proved in certain cases.
- Eisenstein series on the n-fold metaplectic cover of GL_{r+1} give yet another Weyl group MDS. **Very recently it was related to Gelfand-Tsetlin description by pushing through a formerly intractable computation (BBF).**
- These are cases where Whittaker models are not unique.
- The reformulation of the Gelfand-Tsetlin description for type A is just a paraphrase but it is a very suggestive one that points the way to generalizations.
- **Crystals are well adapted to Kac-Moody framework and there is good hope that Weyl group multiple Dirichlet series can be extended to this setting.**
- **A recent preprint of Bucur and Diaconu** in function field case shows that a Weyl group multiple Dirichlet series for the affine Weyl group $D_4^{(1)}$ produces an object resembling the infinite Weyl-denominator, supporting this hope.

The p-part in Type A

Let the ground field contain the n-th roots of unity, and define:

$$g(m,c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} \left(\frac{a}{c}\right)\psi\left(\frac{ma}{c}\right), \qquad \psi = \text{additive char.} \qquad \left(\frac{a}{c}\right) = \begin{array}{c}\text{power}\\ \text{residue}\\ \text{symbol.}\end{array}$$

Fix p and for $a>0$ let $g(a) = g(p^{a-1}, p^a)$, $h(a) = g(p^a, p^a)$.

- If $n=1$ these reduce to $g(a) = -\,q^{a-1}$, $h(a) = (q-1)q^{a-1}$ as in Tokuyama's formula. **Now $q = \mathbb{N}p$**.

The p-part of the Weyl group MDS is given by exactly the same formula as in Tokuyama's recipe, multiplied by $e^{-w_0(\lambda)}$. It is

$$\sum_{v \in \mathfrak{C}_{\lambda+\rho}} G(v) e^{\mathrm{wt}(v+w_0(\lambda))}, \qquad G(v) = \prod_{a \in \mathrm{BZL}(v)} \left\{\begin{array}{ll} q^a & \text{if } a \text{ is circled,} \\ g(a) & \text{if } z \text{ is boxed,} \\ h(a) & \text{if neither,} \\ 0 & \text{if both.} \end{array}\right\}$$

The weight $e^{\mathrm{wt}(v+w_0(\lambda))}$ may be interpreted as $\prod_{i=1}^{r} \mathbb{N}p^{-2\mu_i s_i}$ where μ_i is the number of steps of "color" i in the turtle's walk.

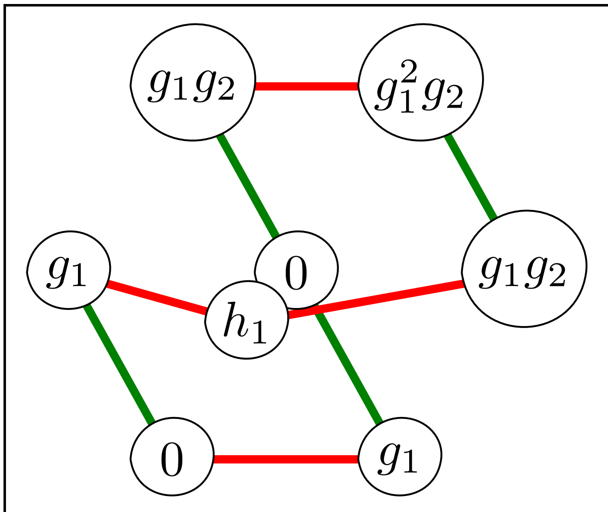
- The "Weyl denominator" as in Tokuyama's formula becomes the normalizing factor.

$$\prod_{\alpha \in \Phi^+} (1 - q^{-n k_i(\alpha)(2s_i - 1) - 1})^{-1},$$

where $\alpha = \sum k_i(\alpha) \cdot \alpha_i$ ($\alpha_i =$ simple roots).

Example

Let $\Phi = A_2$ and $\lambda = 1$. Here is the $\mathfrak{C}_\rho$ crystal labeled with the contributions to the p-part.



$$g_1 = g(1) = g(1, p)$$
$$g_2 = g(2) = g(p, p^2)$$
$$h_1 = h(1) = h(p, p)$$

p-part:

$$\begin{gathered} 1 + g_1 q^{-2s_1} + g_1 q^{-2s_2} \\ + h_1 q^{-2s_1 - 2s_1} + g_1 g_2 q^{-2s_1 - 4s_2} \\ + g_1 g_2 q^{-4s_1 - 2s_2} + g_1^2 g_2 q^{-4s_1 - 4s_2} \end{gathered}$$

The General Case

- The construction depended on a decomposition of the long Weyl group element into simple reflections.
- There are several such decompositions.
- Some are better than others. Littelmann investigated the different choices and exhibited good choices for the classical root system types.
- For types B_r and n even, or types C_r and n odd, the above scheme works with only **minor modifications**.
- For type B_2 and n odd, we have a scheme that works, but there a strange detail involving moving boxes and circles.

A Riddle

The B_r theory when $n = 2$ is particularly striking so we will describe it. This is related to Whittaker models on the metaplectic double cover $\widetilde{\mathrm{Sp}}(2r, \mathbb{Q}_p)$ of $\mathrm{Sp}(2r)$. But first, a riddle.

What is the L-group of $\widetilde{\mathrm{Sp}}(2r, \mathbb{Q}_p)$?

- The L-group of $\mathrm{Sp}(2r)$ is $\mathrm{SO}(2r+1)$.
- Langlands did not define an L-group for metaplectic groups, but there are reasons to say that if $G = \widetilde{\mathrm{Sp}}^{(n)}(2r)$ is the n-th metaplectic cover, then

$$^L G = \begin{cases} \mathrm{SO}(2r+1) & \text{if } n \text{ is odd,} \\ \mathrm{Sp}(2r) & \text{if } n \text{ is even.} \end{cases}$$

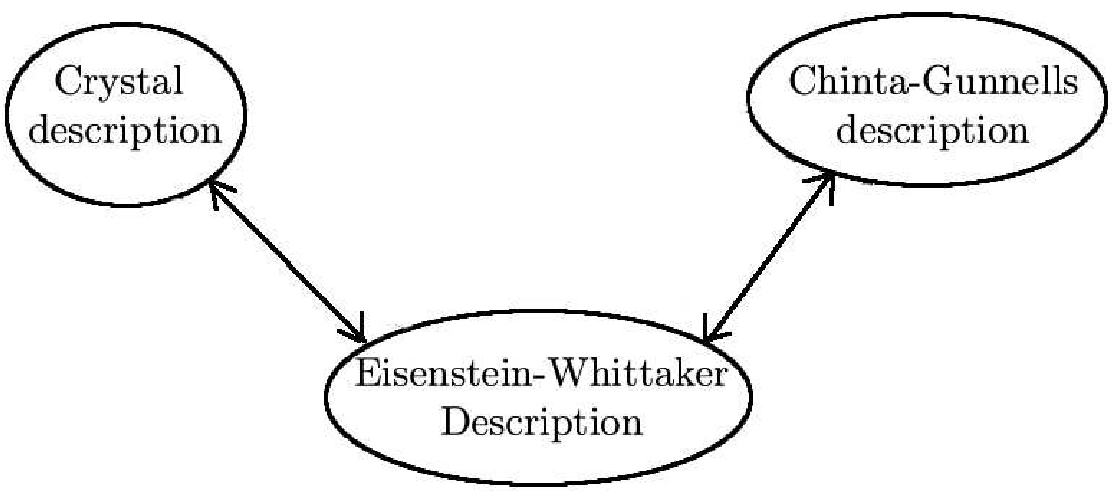
- **Evidence:** Savin computed (affine) Iwahori Hecke algebras (IHA) of the n-fold covers of semisimple split groups, and the IHA of the n cover of $\mathrm{Sp}(2r)$ is isomorphic to the IHA of $\mathrm{Sp}(2r)$ if n is odd or $\mathrm{spin}(2r+1)$ if if n is even.
- **Evidence:** Andrianov and Zhuravlev gave Rankin-Selberg constructions that on $\widetilde{\mathrm{Sp}}^{(n)}(2r)$ produce a degree $2r+1$ L-function when $n=1$ and a degree $2r$ L-function when $n=2$.

A Paradox

- Yet for $\widetilde{\mathrm{Sp}}^{(2)}(2r)$ it is “orthogonal” B_r crystals we employ and we will encounter representations of $\mathrm{SO}(2r+1, \mathbb{C})$.
- This is most unexpected for the above reasons.
- Both representations of $\mathrm{SO}(2r+1, \mathbb{C})$ and $\mathrm{Sp}(2r, \mathbb{C})$ will appear.

Three Descriptions

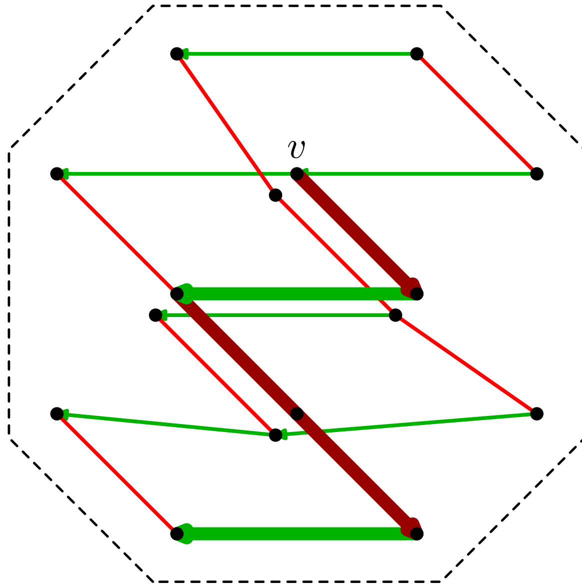
- Equivalence of the three descriptions is (nearly) proved. In preparation: Brubaker, Bump, Chinta and Gunnells. Prior work: Bump, Friedberg, Hoffstein (Duke 1991).
- In this case, Whittaker models are unique.
- Eisenstein-Whittaker description (BFH 1991) expresses the spherical Whittaker function as sum over the Weyl group of either $B_r = \mathrm{SO}(2r+1)$ or $C_r = \mathrm{Sp}(2r)$.
- The Weyl groups are the same so take your pick.
- But the ρ in the Weyl character formula tells us the computation is related to the representation theory of $\mathrm{Sp}(2r)$, not $\mathrm{SO}(2r+1)$.
- **Or does it?** We will come back to this point.



- The Chinta-Gunnells description is also a sum over the same Weyl group, and is equivalent to BFH 1991 formula.
- The equivalence of the Crystal description to Eisenstein-Whittaker description amenable to proof by induction.

The B_r Crystal

Here is a B_2 crystal.
(B_r works the same way.)



$w_0 = s_1 s_2 s_1 s_2$
(red, green, red, green)

$$\mathrm{BZL}(v) = \left\{ \begin{array}{ccc} \boxed{1} & 2 & \boxed{1} \\ & \boxed{1} & \end{array} \right\}$$

Apply root lowering operators in the order red, green, red, green, returning to the lowest weight vector in a_1, a_2, a_3, a_4 steps.

$$\mathrm{BZL} = \left\{ \begin{array}{ccccc} \ddots & & \vdots & & \cdots \\ & a_2 & a_3 & a_4 & \\ & & a_1 & & \end{array} \right\}.$$

Littelmann:

$$\begin{array}{c} \vdots \\ 2a_2 \geqslant a_3 \geqslant 2a_4 \\ a_1 \geqslant 0. \end{array}$$

The boxing rule is as before. The circling rule:

$$\begin{array}{ccccc} \ddots & & \vdots & & \cdots \\ & \boxed{\begin{array}{l} 2a_2 = a_3 \\ \text{circle } a_2 \end{array}} & \boxed{\begin{array}{l} a_3 = 2a_4 \\ \text{circle } a_3 \end{array}} & \boxed{\begin{array}{l} a_4 = 0 \\ \text{circle } a_4 \end{array}} & \\ & & \boxed{\begin{array}{l} a_1 = 0 \\ \text{circle } a_1 \end{array}} & & \end{array}$$

The p-Part for Type B_r

- The r axial entries (here a_1 and a_3) correspond to the short roots.

$$
\begin{array}{ccccccc}
\ddots & & & \vdots & & & \iddots \\
 & \text{long} & \text{long} & \text{short} & \text{long} & \text{long} & \\
 & & \text{long} & \text{short} & \text{long} & & \\
 & & & \text{short} & & &
\end{array}
$$

If $\alpha = \alpha(x)$ is the root correspond to the turtle walk of length x:

$$
\mathcal{G}(v) = \prod_{x \in L} \begin{cases} g_\alpha(x) & \text{if } x \text{ is boxed but not circled,} \\ h_\alpha(x) & \text{if } x \text{ neither circled nor boxed,} \\ q & \text{if } x \text{ is circled but not boxed,} \\ 0 & \text{if } x \text{ is both circled and boxed.} \end{cases}
$$

- Here g_α is a quadratic Gauss sum for the short roots but a Ramanujan sum for the long roots.
- Both can be evaluated explicitly. Assuming the ground field contains the 4-th roots of unity,

$$
g_\alpha(m) = \begin{cases} -\,q^{m-1} & \text{for long roots } \alpha, \\ -\,q^{m-1} & \text{for short roots } \alpha,\ m \text{ even,} \\ q^{(m-\frac{1}{2})} & \text{for short roots } \alpha,\ m \text{ odd,} \end{cases}
$$

$$
h_\alpha(m) = \begin{cases} (q-1)q^{m-1} & \text{for long roots } \alpha, \\ (q-1)q^{m-1} & \text{for short roots } \alpha,\ m \text{ even,} \\ 0 & \text{for short roots } \alpha,\ m \text{ odd,} \end{cases}
$$

- Note that square roots of q appear: $g_\alpha(1) = \sqrt{q}$ if α short.

Type B_2

Let λ be a dominant weight, and take the B_2 crystal with highest weight vector $\lambda+\rho$. Take the sum of

$$\mathcal{G}(\Delta)x^{a_2+a_4}y^{a_1+a_3}.$$

Then we obtain

$$(1-x)(1+q^{1/2}y)(1+q^{3/2}xy)(1-q^2xy^2)$$

times a polynomial $P(x,\ y;\ q^{1/2})$ that is given by the following table (ε_i are the fundamental weights).

λ	$P_\lambda(x,y;\sqrt{q})$
0	1
ε_1	$1+qx-q^{3/2}xy+q^3xy^2+q^4x^2y^2$
ε_2	$(1-q^{1/2}y)(1-q^{3/2}xy)$
$\varepsilon_1+\varepsilon_2$	$(1+qx)(1-q^{1/2}y)(1-q^{3/2}xy)(1+q^3xy^2)$
$2\varepsilon_1$	$x^4y^4q^8+x^3y^4q^7+x^2y^4q^6-x^3y^3q^{11/2}$ $+x^3y^2q^5-x^2y^3q^{9/2}+2x^2y^2q^4$ $+xy^2q^3-x^2yq^{5/2}+x^2q^2$ $-xyq^{3/2}+xq+1$
$2\varepsilon_2$	$x^2y^4q^6-x^2y^3q^{9/2}+x^2y^2q^4$ $-xy^3q^{7/2}+xy^2q^3+xy^2q^2$ $+y^2q^2-xyq^{3/2}-yq^{1/2}+1$

- When we specialize $\sqrt{q}\longrightarrow -1$, these polynomials become characters of spin(5).
- **Paradoxical:** This is in contrast with the popular belief that the L-group of $\widetilde{\mathrm{Sp}}(2r)$ is $\mathrm{Sp}(2r)$.
- **Paradoxical:** Also in contrast with this popular belief is the fact that we used a type B (orthogonal) crystal.

Type B_3

Here are the corresponding polynomials for B_3:

λ	$P_\lambda(x,y,z,\sqrt{q})$
0	1
$\varepsilon_1 = (1,0,0)$	$1 + qz + q^2yz - q^{5/2}xyz + q^4x^2yz + q^5x^2y^2z + q^6x^2y^2z^2$
$\varepsilon_2 = (1,1,0)$	$1 + qy - q^{3/2}xy + q^3x^2y + q^4x^2y^2 + q^2yz - q^{5/2}xyz + q^4x^2yz - q^{7/2}xy^2z + q^4x^2y^2z + 2q^5x^2y^2z - q^{11/2}x^3y^2z + q^6x^2y^3z - q^{13/2}x^3y^3z + q^8x^4y^3z + q^6x^2y^2z^2 + q^7x^2y^3z^2 - q^{15/2}x^3y^3z^2 + q^9x^4y^3z^2 + q^{10}x^4y^4z^2$
$\varepsilon_3 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$(1-\sqrt{q}x)(1-q^{3/2}xy)(1-q^{5/2}xyz)$

Again, when $\sqrt{q} \longrightarrow -1$, these become the characters of irreducible representations of spin(7).

- The paradox is that these polynomials are defined in terms of the representation theory of Sp$(2r)$ yet their crystal interpretation involves type B_r (orthogonal) crystals and the expressions specialize to characters of Spin$(2r+1)$ when $\sqrt{q} \longrightarrow -1$.
- We "explain" the paradox by proving that this always happens for all r.
- The explanation does not dispel the mystery.

Alternator formulation (BFH 1991)

Change notation: $x_i = \alpha_i / \alpha_{i+1}$. B_r / C_r Weyl group W acts on $\mathbb{C}[\alpha_1^{\pm 1}, \cdots, \alpha_r^{\pm 1}]$ and $\mathbb{C}[\alpha_1^{\pm 1/2}, \cdots, \alpha_r^{\pm 1/2}]$.

$$\text{The alternator is} \qquad \mathcal{A} = \sum_{w \in W} (-1)^{l(w)} \cdot w.$$

$$\text{Let} \qquad \Delta_{\mathrm{Sp}(2r)} = \mathcal{A}(\alpha_1^r \alpha_2^{r-1} \cdots \alpha_r),$$

$$\Delta_{\mathrm{SO}(2r+1)} = \mathcal{A}(\alpha_1^{r-\frac{1}{2}} \alpha_2^{r-\frac{3}{2}} \cdots \alpha_r^{\frac{1}{2}}).$$

By WCF, if $k_1 \geqslant \cdots \geqslant k_r \geqslant 0$, $k_i \in \mathbb{Z}$, then

$$\chi_{\mathrm{Sp}}(k_1, \cdots, k_r) = \Delta_{\mathrm{Sp}(2r)}^{-1} \mathcal{A}(\alpha_1^{k_1+r} \alpha_2^{k_2+r-1} \cdots \alpha_r^{k_r+1})$$

is a character of $\mathrm{Sp}(2r, \mathbb{C})$ on the conjugacy class with eigenvalues $\alpha_1^{\pm 1}, \cdots, \alpha_r^{\pm 1}$. Similarly

$$\chi_{\mathrm{SO}(2r+1)}(k_1, \cdots, k_r) = \frac{\mathcal{A}(\alpha_1^{k_1+r-\frac{1}{2}} \alpha_2^{k_2+r-\frac{3}{2}} \cdots \alpha_r^{k_r+\frac{1}{2}})}{\Delta_{\mathrm{SO}(2r+1)}}.$$

is a character of $\mathrm{SO}(2r+1)$ on conjugacy class with eigenvalues $\alpha_1^{\pm 1}, \cdots, \alpha_r^{\pm 1}, 1$. If k_i are half-integral, this same formula would give characters of $\mathrm{spin}(2r+1)$, but we specializing to the integral case, $P_\lambda(x; \lambda)$ is

$$\Delta_{\mathrm{Sp}(2r)}^{-1} \mathcal{A}\left(\alpha_1^{k_1+r} \alpha_2^{k_2+r-1} \cdots \alpha_r^{k_r+1} \prod_{i=1}^{r} (1 - q^{-1/2} \alpha_i^{-1}) \right),$$

Expanding and using WCF it is a sum of up 2^r terms, each an irreducible character of $\mathrm{Sp}_{2r}(\mathbb{C})$.

Paradox Explained, Mystery not Dispelled

After $q^{1/2} \to -1$ specialization P_λ becomes

$$\Delta^{-1}_{\mathrm{Sp}(2r)} \mathcal{A}\left(\alpha_1^{k_1+r} \alpha_2^{k_2+r-1} \cdots \alpha_r^{k_r+r} \prod_{i=1}^{r} (1+\alpha_i^{-1}) \right) =$$

$$\frac{\mathcal{A}\left(\alpha_1^{k_1+r-\frac{1}{2}} \alpha_2^{k_2+r-\frac{3}{2}} \cdots \alpha_r^{k_r+\frac{1}{2}} \prod_{i=1}^{r} (\alpha^{1/2}+\alpha_i^{-1/2}) \right)}{\Delta_{\mathrm{Sp}(2r)}}.$$

The product inside is invariant under the Weyl group action and so we can pull it out, and using

$$\prod_{i=1}^{r} (\alpha^{1/2}+\alpha_i^{-1/2}) \frac{\Delta_{\mathrm{SO}(2r+1)}}{\Delta_{\mathrm{Sp}(2r)}} = 1$$

we get

$$\boxed{\chi_{\mathrm{SO}(2r+1)}(k_1, \cdots, k_r).}$$

- The BFH formula will need to be extended to GSp before we see the spin characters.
- This shows that when $q^{1/2} \longrightarrow -1$ this “symplectic” alternating sum becomes orthogonal.
- This “explanation” of the paradox raises more questions than it answers.
- **Wanted:** Quantum interpretation.
- **Wanted:** Connection with symmetric function theory (Hall-Littlewood polynomials, etc.)