ON THE AVERAGES OF CHARACTERISTIC POLYNOMIALS FROM CLASSICAL GROUPS

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Abstract. We provide an elementary and self-contained derivation of formulae for averages of products and ratios of characteristic polynomials of random matrices from classical groups using classical results due to Weyl and Littlewood.

1. Introduction

The study of averages of characteristic polynomials of random matrices has attracted considerable attention in recent years. The interest has been motivated, in part, by connections with number theory, following the pioneering work of Keating and Snaith [39]; and, in part, by importance of these averages in quantum chaos, first discussed by Andreev and Simons [1]. Over the ensuing years it has become increasingly clear that averages of characteristic polynomials are a fundamental characteristic of random matrix models (see, for example, discussion in [8, 1.6] were the authors argue that they might be more fundamental than correlation functions). Results in the case of Hermitian ensembles were obtained in Baik, Deift and Strahov [2], Borodin and Strahov [8], Brézin and Hikami [9], [10], [11], [12], Forrester and Keating [27], Fyodorov [28], Fyodorov and Keating [29], Fyodorov and Strahov [31], [32], [30], Mehta and Normand [47] and Strahov and Fyodorov [52]. Averages of products in the case of compact classical groups were considered by Conrey, Farmer, Keating, Rubinstein, and Snaith in [17] in connection with conjectures for integral moments of zeta and L-functions [16].

Recently the averages of ratios in the case of compact classical groups were considered by Conrey, Farmer, and Zirnbauer [15] and by Conrey, Forrester and Snaith [18]. The approach in [15] is based on using supersymmetry and the theory of dual reductive pairs. The approach in [18] is based on reducing the orthogonal and symplectic case to the case of unitary invariant Hermitian matrices and then invoking the results obtained by Fyodorov and Strahov [32] and by Baik, Deift and Strahov [2]; the case of unitary group is treated by appealing to the formula of Day [21] for Toeplitz determinants,

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for which the authors give a self-contained derivation using the method of Basor and Forrester [5].

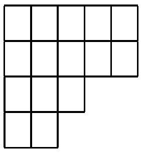
The goal of this paper is to provide an elementary and self-contained derivation of formulas for averages of products of characteristic polynomials from classical groups, obtained in [17], and for averages of their ratios, obtained in [15] and [18] (we note that in [15] no restriction on the dimension of the group is imposed; see the remark at the beginning of section 3.3). We also obtain an elementary derivation of the formulas for integral moments of characteristic polynomials derived by Keating and Snaith [39, 40] using Selberg's integral. Our proofs use classical results due to Weyl [54] and Littlewood [44] and can be viewed as application of symmetric function theory in random matrix theory along the lines pioneered by Diaconis and Shahshahani [24] and applied in Rains [49], Bump and Diaconis [14], Baik and Rains [3], Diaconis [22] and Diaconis and Gamburd [23]. We begin with review of symmetric function theory in Section 2 and consider the unitary group in Section 3.

In a nutshell, our method consists of expressing the mean value of a product or ratio of characteristic polynomials on a group G in terms of a symmetric function (such as a Schur polynomial) related to a character of an irreducible representation whose highest weight vector is a partition of "rectangular shape," then reducing that value as a sum over elements of W/W_M, where W is the Weyl group of G, M is a subgroup of G and W_M is the Weyl group of M. This point is explained in Section 4. The symplectic group is considered in Section 5 and the orthogonal group is considered in section 6. Our results imply simple derivations of formulas for classical group characters of rectangular shape due to Okada [48] and Krattenthaler [42] and yield several generalizations; this is considered in Section 7.

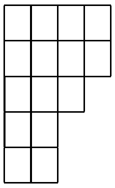
2. Review of symmetric functions theory

2.1. **Schur functions.** A *partition* λ is a sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ of non-negative integers, eventually zero. By abuse of notation, we write $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ for any n such that $\lambda_{n+1} = 0$. There is a unique n such that $\lambda_n > 0$ but $\lambda_{n+1} = 0$ and this $n = l(\lambda)$ is the *length* of λ. We call $|\lambda| = \sum \lambda_i$ the *size* of λ. If $i > 0$ let $m_i = m_i(\lambda)$ be the number of parts λ_j of λ equal to i; in this case, we write $\lambda = \langle 1^{m_1} 2^{m_2} 3^{m_3} \cdots \rangle$.

The Young diagram of a partition λ is defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leqslant i \leqslant \lambda_j$; it is often convenient to replace the set of points above by squares. The conjugate partition λ' of λ is defined by the condition that the Young diagram of λ' is the transpose of the Young diagram of λ; equivalently $m_i(\lambda') = \lambda_i - \lambda_{i+1}$.



Young diagram of λ



Young diagram of λ'

In the figure we exhibited a partition $\lambda = (5,5,3,2) = \langle 1^0 2^1 3^1 5^2 \rangle$; $|\lambda| = 15$ and $l(\lambda) = 4$.

Let λ and μ be partitions. We define $\lambda + \mu$ to be the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \cdots)$. On the other hand we define $\lambda \cup \mu$ to be the partition whose parts are the union of the parts of λ and μ, arranged in descending order. For example if $\lambda = (321)$ and $\mu = (22)$ then $\lambda + \mu = (541)$ and $\lambda \cup \mu = (32221)$. The operations $+$ and $\cup$ are dual to each other:

$$(\lambda + \mu)' = \lambda' \cup \mu'.$$

We write $\lambda \supset \mu$ if the diagram of λ contains the diagram of μ, or equivalently, if $\lambda_i \geqslant \mu_i$ for all i.

The elementary symmetric functions $e_r(x_1, \dots, x_n)$ are defined by

$$e_r(x_1, \dots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \dots x_{i_r}; \tag{1}$$

complete symmetric functions $h_r(x_1, \dots, x_n)$ are defined by

$$h_r(x_1, \dots, x_n) = \sum_{i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \dots x_{i_r}. \tag{2}$$

Now given a partition λ, we define

$$e_\lambda(x_1, \dots, x_n) = \prod_{j=1}^{n} e_{\lambda_j}(x_1, \dots, x_n) \tag{3}$$

and similarly

$$h_\lambda(x_1, \dots, x_n) = \prod_{j=1}^{n} h_{\lambda_j}(x_1, \dots, x_n). \tag{4}$$

Schur functions are symmetric polynomials indexed by partitions. If λ is any partition of length $\leqslant n$ we define

$$s_\lambda(x_1,\cdots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n} =$$

$$\frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix}} \tag{5}$$

If $l(\lambda) > n$ we define $s_\lambda(x_1,\cdots,x_n) = 0$.

The denominator in (5) is a Vandermonde determinant admitting the following simple evaluation:

$$\det(x_i^{n-j}) = \prod_{1\leqslant j<k\leqslant n}(x_j - x_k). \tag{6}$$

Let $\Lambda^{(n)} = \Lambda^{(n)}(x)$ denote the ring of symmetric polynomials with integer coefficients in $x_1,\cdots,x_n$. Then $s_\lambda(x_1,\cdots,x_n) \in \Lambda^{(n)}$. We have a homomorphism $\Lambda^{(n+1)} \longrightarrow \Lambda^{(n)}$ in which the last variable x_{n+1} is specialized to 0, and under this homomorphism it is easy to see from the definition that this specialization takes s_λ to s_λ; that is,

$$s_\lambda(x_1,\cdots,x_n,0) = s_\lambda(x_1,\cdots,x_n). \tag{7}$$

This means that s_λ may be regarded as an element of the ring Λ

$$\Lambda = \Lambda(x) = \lim_{\longleftarrow} \Lambda^{(n)}(x).$$

The ring Λ is a polynomial ring in either $h_1,h_2,\cdots$ or $e_1,e_2,\cdots$ where $h_n = s_{\langle n\rangle}$ and $e_n = s_{\langle 1^n\rangle}$ (Bump [13], Proposition 36.5 or Macdonald [46], p. 22). It is a free abelian group with basis s_λ, as λ runs through all partitions. The ring $\Lambda^{(n)}$ is a free abelian group with basis s_λ, as λ runs through the partitions of length $\leqslant n$.

The ring Λ has an involution ι which interchanges $h_n \longleftrightarrow e_n$ (Bump [13], Theorem 36.3). This involution takes s_λ to $s_{\lambda'}$ (Bump [13], Theorem 37.2 or Macdonald [46], (3.8) on p. 42).

We will make crucial use of the following classical result (see Weyl [54], Lemma 7.6.A; or, for modern presentations, Bump [13], Chapter 43 or Macdonald [46], (4.3) on p. 63.)

Proposition 1. (Cauchy identity) *Let $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ be two sets of variables. Then*

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_p) s_\lambda(y_1, \ldots, y_q), \tag{8}$$

where the sum is over all partitions λ of length $\leqslant \min(p, q)$.

Both sides are convergent provided the $|x_i|$ and $|y_j|$ are all less than 1; otherwise, this may be regarded as a formal identity.

We have also the *dual Cauchy identity*

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(1 + x_i y_j) = \sum_{\lambda} s_\lambda(x_1, \ldots, x_p) s_{\lambda'}(y_1, \ldots, y_q). \tag{9}$$

The right hand side is a finite sum, since unless $l(\lambda) \leqslant p$ and $l(\lambda') \leqslant q$ we have either $s_\lambda = 0$ or $s_{\lambda'} = 0$, so the diagram of λ must fit inside a $p \times q$ rectangle.

The dual Cauchy identity can be deduced from the standard Cauchy identity by making use of the involution on Λ. See Bump [13], Theorem 43.5 or Macdonald [46], (4.3′) on p. 65.

If μ and ν are partitions, then $s_\mu s_\nu$ can be decomposed into Schur polynomials; we write

$$s_\mu s_\nu = \sum_{\lambda} c^\lambda_{\mu\nu} s_\lambda. \tag{10}$$

where the sum is over partitions λ of size $|\mu| + |\nu|$. It follows from (7) that the *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$ are independent of n, and they are determined by this equation provided $n \geqslant |\lambda| + |\mu|$ (so that the s_ν are linearly independent).

Let $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ be two sets of variables. The Littlewood-Richardson coefficients also appear in

$$s_\lambda(x_1, \cdots, x_p, y_1, \cdots, y_q) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_1, \cdots, x_p) s_\nu(y_1, \cdots, y_q). \tag{11}$$

See Macdonald [46], (5.9) on p. 72, who proves (11) from (10) using the Cauchy identity. We will also give a proof (which is closely related to Macdonald's) in Section 2.4.

Particular cases of (10) are furnished by Pieri's formula (see Bump [13], Theorem 42.4 or Macdonald [46], (5.16) on p. 73.) If $\lambda \supseteq \mu$ are partitions, we say that $\lambda - \mu$ is a *horizontal strip* if no two elements of the set theoretic difference of their diagrams lie in the same column; and we say that $\lambda - \mu$ is a *vertical strip* no two elements of the set theoretic difference of their diagrams lie in the same row.

The content of Pieri's formula is that

$$c^\lambda_{\mu\langle r\rangle} = \begin{cases} 1 & \text{if } \lambda - \mu \text{ is a horizontal strip and } |\lambda| - |\mu| = r; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Equivalently, we can write

$$s_\mu h_r = \sum_\lambda s_\lambda, \tag{13}$$

where the summation is over all partitions λ such that $\lambda - \mu$ is a horizontal r-strip.

Similarly we have

$$s_\mu e_r = \sum_\lambda s_\lambda, \tag{14}$$

where the summation is over all partitions λ such that $\lambda - \mu$ is a vertical r-strip.

From (11) and (13) it follows that

$$s_\lambda(x_1, \ldots, x_k, y) = \sum_\mu s_\mu(x_1, \ldots, x_k) y^{|\lambda-\mu|}, \tag{15}$$

where the sum is over partitions μ such that $\lambda - \mu$ is a horizontal strip.

We denote by $k^n = \langle k^n \rangle$ the partition $(k, \ldots, k)$ of length n; assuming that λ is a partition of length less than or equal to n we denote

$$\lambda + k^n = (\lambda_1 + k, \ldots, \lambda_n + k).$$

We have

$$s_{\lambda+k^n}(x_1, \cdots, x_n) = e_n^k(x_1, \cdots, x_n)\, s_\lambda(x_1, \cdots, x_n). \tag{16}$$

This is immediate from the definition of the Schur polynomial together with the fact that $e_n(x_1, \cdots, x_n) = x_1 \cdots x_n$.

2.2. **Connection with unitary group.** Let $g \in \mathrm{GL}(n, \mathbb{C})$ have eigenvalues $t_1, \cdots, t_n$. Assuming that the length $l(\lambda) \leqslant n$, we define $\chi_\lambda(g) = \chi_\lambda^{(n)}(g) = s_\lambda(t_1, \cdots, t_n)$. (The assumption on $l(\lambda)$ is necessary since otherwise $s_\lambda(t_1, \cdots, t_n) = 0$.) The following result is a special case of the Weyl character formula [54], which expresses the characters of compact Lie groups as ratios of alternating functions; see, for example, Bump [13], Theorem 38.2. for a modern exposition.

Proposition 2. *If λ is a partition of length $\leqslant n$, the function χ_λ is the character of an irreducible analytic representation $\pi_\lambda = \pi_\lambda^{(n)}$ of $\mathrm{GL}(n, \mathbb{C})$. It is irreducible; in fact, its restriction to $U(n)$ is irreducible.*

As a consequence, we have Schur orthogonality for the functions χ_λ. Let $\langle\ ,\ \rangle$ denote the inner product with respect to the Haar measure on $G = U(n)$, normalized so that the volume of G is 1. If $\mu = (\mu_1, \cdots, \mu_n)$ where μ_i are integers (possibly negative) and $\mu_1 \geqslant \cdots \geqslant \mu_n$, then we call μ a *dominant weight*. If λ and μ are dominant weights, we have

$$\langle \chi_\lambda, \chi_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Now (16) implies that if $\mu = \lambda + k^n$, then $\chi_\mu = \det^k \otimes \chi_\lambda$. From this observation we may extend the definition of χ_λ as follows. The dominant weight μ is a partition if and only if $\mu_n \geqslant 0$. We can always write $\mu = \lambda + \langle k^n \rangle$ where λ is a partition for some k; for example, we can take $k = \mu_n$. Then we define $\chi_\mu = \det^k \otimes \chi_\lambda$. It is of course the character of an irreducible analytic representation of $\mathrm{GL}(n, \mathbb{C})$, or its compact subgroup $U(n)$. The following result is due to Weyl [54]; see Bump [13], Theorem 38.3. for a modern presentation.

Theorem 1. *The characters of the irreducible analytic representations of $\mathrm{GL}(n, \mathbb{C})$ are precisely the χ_λ as λ runs through the dominant weights. Their restrictions to $U(n)$ are precisely the irreducible characters of $U(n)$.*

If λ is a partition, we call the representation π_λ of $\mathrm{GL}(n, \mathbb{C})$ with character χ_λ a *polynomial representation* since its matrix coefficients are polynomials of the coordinate functions g_{ij} of $g \in \mathrm{GL}(n, \mathbb{C})$. If λ is a dominant weight that is not necessarily a partition, we call π_λ a *rational representation.* Its matrix coefficients may have denominators that are powers of the determinant.

The value of $s_\lambda(1^n)$ gives the dimension of the irreducible representation of $U(n)$ with highest weight λ. From (5) using l'Hopital rule one easily derives for $\lambda = (\lambda_1, \ldots, \lambda_n)$ the following formula, known as the Weyl dimension formula [54]:

$$s_\lambda(1, \ldots, 1) = \frac{\prod_{i<j}(\mu_i - \mu_j)}{\prod_{i<j}(i - j)}, \tag{18}$$

where $\mu_i = \lambda_i + n - i$.

8	7	5	3	2
7	6	4	2	1
4	3	1		
2	1			

FIGURE 1

It is easy to deduce that the right-hand side of (18) can be expressed in the following equivalent form (see [46] or [51]):

$$s_\lambda(1^n) = \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}, \tag{19}$$

where for a box u in a diagram λ, $h(u)$ is a hook number of u and $c(u)$ is a content number of u, which we now define. Given a diagram λ and a square $u = (i, j) \in \lambda$, the content of λ at u is defined by $c(u) = j - i$. A hook with a vertex u is a set of squares in λ directly to the right or directly below u. We define hook-length (also referred to as hook number) $h(u)$ of λ at u by

$$h(u) = \lambda_i + \lambda'_j - i - j + 1.$$

Equivalently, $h(u)$ is the number of squares directly to the right or directly below u, counting u itself once. For instance, in figure 1 we display hook lengths for partition $\lambda = (5,5,3,2)$.

2.3. **Laplace expansion.** The following classical result from linear algebra will be used repeatedly in what follows; its deeper meaning and significance will be discussed in Section 4.

Proposition 3. (Laplace expansion) *Fix L rows of the matrix A. Then the sum of products of the minors of order L that belong to these rows by their cofactors is equal to the determinant of A.*

Let $\Xi_{L,K}$ consist of all permutations $\sigma \in S_{K+L}$ such that

$$\sigma(1) < \cdots < \sigma(L), \qquad \sigma(L+1) < \cdots < \sigma(L+K). \tag{20}$$

Given $(K+L) \times (K+L)$ matrix $A = (a_{ij})$ Laplace expansion in the first L rows can be written as follows:

$$\det(a_{ij}) = \sum_{\sigma \in \Xi_{L,K}} \operatorname{sgn}(\sigma) \begin{vmatrix} a_{1,\sigma(1)} & \cdots & a_{1,\sigma(L)} \\ \vdots & & \vdots \\ a_{L,\sigma(1)} & \cdots & a_{L,\sigma(L)} \end{vmatrix} \cdot \begin{vmatrix} a_{L+1,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \\ \vdots & & \vdots \\ a_{L+K,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \end{vmatrix}.$$

We now record two simple applications of the Laplace expansion.

Lemma 1. *Suppose $\lambda = \tau \cup \rho$ with $\lambda = (\lambda_1, \ldots, \lambda_{L+K})$, $\tau = (\lambda_1, \ldots, \lambda_L)$ and $\rho = (\lambda_{L+1}, \ldots, \lambda_{L+K})$. Then*

$$s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) = \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} \left(\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)}\right)^{-1}$$

$$\times s_{\tau + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)})\, s_\rho(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}) \quad . \tag{21}$$

Proof We apply the Laplace expansion in the first L rows to the determinant in

$$s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) = \Delta^{-1} \begin{vmatrix} \alpha_1^{K+\lambda_1+L-1} & \cdots & \alpha_{K+L}^{K+\lambda_1+L-1} \\ \vdots & & \vdots \\ \alpha_1^{K+\lambda_L} & \cdots & \alpha_{K+L}^{K+\lambda_L} \\ \alpha_1^{K+\lambda_{L+1}-1} & \cdots & \alpha_{K+L}^{K+\lambda_{L+1}-1} \\ \vdots & & \vdots \\ \alpha_1^{\lambda_{K+L}} & \cdots & \alpha_{K+L}^{\lambda_{K+L}} \end{vmatrix}$$

where

$$\Delta = \begin{vmatrix} \alpha_1^{K+L-1} & \cdots & \alpha_{K+L}^{K+L-1} \\ \alpha_1^{K+L-2} & \cdots & \alpha_{K+L}^{K+L-2} \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = \prod_{i<j} (\alpha_i - \alpha_j) = \operatorname{sgn}(\sigma) \prod_{i<j} (\alpha_{\sigma(i)} - \alpha_{\sigma(j)})$$

and simplify. $\square$

Lemma 2. *For $\lambda \subseteq \langle N^k \rangle$ let $\tilde\lambda = (k-\lambda'_N, \cdots, k-\lambda'_1)$. Then we have*

$$\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i - t_n) = \sum_{\lambda\subseteq N^k} (-1)^{|\tilde\lambda|} s_\lambda(x_1,\cdots,x_k) s_{\tilde\lambda}(t_1,\cdots,t_N). \tag{22}$$

Using the fact that

$$s_\mu(-t_1,\cdots,-t_N) = (-1)^{|\mu|} s_\mu(t_1,\cdots,t_N)$$

the formula may be written

$$\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i + t_n) = \sum_{\lambda\subseteq N^k} s_\lambda(x_1,\cdots,x_k) s_{\tilde\lambda}(t_1,\cdots,t_N). \tag{23}$$

Generalizations of this Lemma will be given below in Proposition 9 and also in Lemma 4, Lemma 6, and Lemma 5.

Proof Using the definition of Schur functions (5) and the Laplace expansion we can rewrite the right-hand side of (22) as follows:

$$\det\begin{vmatrix} x_1^{N+k-1} & x_1^{N+k-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k-1} & x_k^{N+k-2} & \dots & 1 \\ t_1^{N+k-1} & t_1^{N+k-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ t_N^{N+k-1} & t_N^{N+k-2} & \dots & 1 \end{vmatrix} \times \frac{1}{\prod_{1\leqslant i<j\leqslant k}(x_i - x_j)} \frac{1}{\prod_{1\leqslant i<j\leqslant N}(t_i - t_j)}. \tag{24}$$

Now the determinant in the equation (24) can be evaluated using the formula for Vandermonde determinant formula (6) to be equal to

$$\prod_{1\leqslant i<j\leqslant k}(x_i - x_j) \prod_{1\leqslant i<j\leq N}(t_i - t_j) \prod_{i=1}^{k}\prod_{n=1}^{N}(x_i - t_n). \tag{25}$$

Combining (24) and (25) completes the proof. See Remark 1 below. □

Remark 1. The following combinatorial fact is used implicitly in the last proof in determining which pair of Schur functions appears: if λ is a partition such that $\lambda_1 \leqslant N$ and $\lambda'_1 \leqslant k$, then the $N+k$ numbers $\lambda_i + k - i$ (for $1 \leqslant i \leqslant k$) and $k - 1 + j - \lambda'_j$ (for $1 \leqslant j \leqslant N$) are a permutation of $\{0, 1, 2, \ldots, N+k-1\}$. See Macdonald [46], (1.7) on p.3., or Bump [13], Proposition 37.2.

Remark 2. Lemma 2 can by proved by a simple application of dual Cauchy identity; conversely dual Cauchy is an immediate consequence of this Lemma.

2.4. **On the role of Howe duality in symmetric function theory.** The Cauchy identity (8) and dual Cauchy identity (9) play a crucial role in our derivations. While (as indicated in Remark 2) they admit simple formal or combinatorial proofs (see for example Macdonald [46] or Stanley [51]), their deeper meaning is revealed by Howe's theory of dual pairs which we briefly review in this section.

Let G and H be groups, and let ω be a representation of $G\times H$. Following Howe [35], [34], [37], we say that ω is a *correspondence* if $\omega = \bigoplus_{i\in I} \pi_i \otimes \sigma_i$ where the π_i are irreducible representations of G and the σ_i are irreducible representations of H, and if there are no repetitions among the (isomorphism classes of the) π_i, nor among the σ_i. In this case $\pi_i \longleftrightarrow \sigma_i$ is the graph of a bijection between a set of irreducible representations of G and a set of irreducible representations of H. We call this bijection the *graph of the correspondence*. We refer to $G \times H$ as the *dual pair* of the correspondence.

A classical example is furnished by *Frobenius-Schur duality*, going back to Schur's dissertation and emphasized by Weyl [54]. This is the fact that if $V = \mathbb{C}^n$, then $U(n)$ and the symmetric group S_k act on $\bigotimes^k V$; the group $U(n)$ acts by $g : v_1 \otimes \cdots \otimes v_k \longrightarrow (gv_1) \otimes \cdots \otimes (gv_k)$, while S_k acts by permuting the components. These actions commute with each other, and they therefore give rise to a representation of $U(n) \times S_k$. Frobenius-Schur duality is the fact that this representation is a correspondence. Frobenius-Schur duality allows computations on the unitary group sometimes to be transferred to the symmetric group, a principle that has applications to random matrix theory. See, for example Diaconis and Shahshahani [24] and Bump and Diaconis [14].

As as second example, the representation of $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ on the symmetric algebra $\bigvee \mathrm{Mat}_{p\times q}(\mathbb{C})$, or equivalently of the subgroups $U(p) \times U(q)$ is a correspondence. We now have the following interpretation of Cauchy identity. Let the group $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ act on $\mathrm{Mat}_{p\times q}(\mathbb{C})$ by left and right translation; that is,

$$(g_1, g_2) : X \longrightarrow g_1\, X\, {}^t g_2.$$

Thus $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ acts on the symmetric algebra $\bigvee \mathrm{Mat}_{p\times q}(\mathbb{C})$, and we consider the trace of

$$(g_1, g_2) = \left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_p \end{pmatrix}, \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_q \end{pmatrix} \right), \qquad 0 < |x_i|,\, |y_j| < 1.$$

Since the eigenvalues of (g_1, g_2) on $\mathrm{Mat}_{p\times q}(\mathbb{C})$ are the pq quantities $x_i y_j$,

$$\begin{aligned} \prod_{i=1}^{p}\prod_{j=1}^{q}(1 - x_i y_j)^{-1} &= \sum_{k=0}^{\infty} h_k(x_1 y_1, \cdots, x_p y_q) \\ &= \mathrm{tr}\left((g_1, g_2) \mid \bigvee \mathrm{Mat}_{p\times q}(\mathbb{C}) \right). \end{aligned}$$

Hence the Cauchy identity amounts to the statement that as $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ modules,

$$\bigvee \mathrm{Mat}_{p\times q}(\mathbb{C}) \cong \bigoplus_{\lambda} \chi_\lambda^{(p)}(g_1) \otimes \chi_\lambda^{(q)}(g_2), \tag{26}$$

where the sum is over all partitions λ of length $\leqslant \min(p,q)$. This concrete interpretation is the basis of the proofs in Bump [13], Chapter 43 and Howe [37].

The dual Cauchy identity describes the decomposition of the exterior algebra over $\mathrm{Mat}_{p\times q}(\mathbb{C})$, which (unlike the symmetric algebra) is finite-dimensional; it is discussed from the point of view of dual pairs in Howe [37].

We now turn to the discussion of (10) and (11) from the point of view of the theory of dual pairs. Equation (10) describes the decomposition rule for tensor products of representations of $\mathrm{GL}(n,\mathbb{C})$, or equivalently, the compact subgroup $U(n)$ of $\mathrm{GL}(n,\mathbb{C})$:

$$\chi_\lambda^{(n)}\chi_\mu^{(n)} = \sum_\nu c_{\lambda\mu}^\nu \chi_\nu^{(n)}, \qquad \text{or} \qquad \pi_\lambda \otimes \pi_\mu = \bigoplus_\nu c_{\lambda\mu}^\nu \pi_\nu.$$

Formula (11) is a reflection of the fact that Littlewood-Richardson coefficients appear in a different context, namely in the branching rule from $U(p+q,\mathbb{C})$ to the subgroup $U(p)\times U(q)$.

Given completely reducible representations, V and W of groups G and H respectively, together with an embedding $H \hookrightarrow G$, we let $[V,W] = \dim \mathrm{Hom}_H(W,V)$, where V is regarded as a representation of H by restriction. If W is irreducible, then $[V,W]$ is the multiplicity of W in V. A description of the numbers $[V,W]$ is referred in the mathematics and physics literature as a *branching rule*. We refer to Howe, Tan and Willenbring [36] for discussion of the problem of obtaining branching rules from the point of view of dual reductive pairs, and to King [41] for an extremely useful survey of known branching rules.

Now (11) follows from the following result by evaluating χ_λ on a matrix of $U(p+q)$ with eigenvalues $x_1,\cdots,x_p$ in $U(p)$ and $y_1,\cdots,y_q$ in $U(q)$.

Theorem 2. *We have*

$$\begin{aligned} \chi_\nu^{(p+q)}|_{U(p)\times U(q)}(g_1,g_2) &= \sum_{\lambda,\mu} c_{\lambda\mu}^\nu \chi_\lambda(g_1)\chi_\mu(g_2), \\ \pi_\nu^{(p+q)}(g_1,g_2) &= \bigoplus_{\lambda,\mu} c_{\lambda\mu}^\nu \pi_\lambda(g_1)\otimes\pi_\mu(g_2). \end{aligned} \tag{27}$$

Whippman [55] attributes this fact to Coleman and (independently) Robinson. See also King [41]. The following proof is essentially the same as the one in Howe, Tan and Willenbring [36].

Proof Let Ω be a group and let ω be a representation of Ω. Let G_1 be a subgroup of Ω, and let H_2 be its centralizer. We assume that G_1 is the centralizer of H_2. We allow the possibility that G_1 and H_2 have nontrivial intersection. Even so, $(g,h) \longrightarrow gh$ is a homomorphism $G_1 \times H_2 \longrightarrow \Omega$, and so ω gives rise to a representation of $G \times H$. We are interested in the case where this restriction is a correspondence, so we may write

$$\omega|_{G_1 \times H_2} = \bigoplus_{i \in I} \pi_i^{(1)} \otimes \sigma_i^{(2)}$$

where $\pi_i^{(1)}$ and $\sigma_i^{(2)}$ are irreducible representations of G_1 and H_2, respectively, and $\pi_i^{(1)} \longleftrightarrow \sigma_j^{(2)}$ is the graph of the correspondence.

Now suppose that H_1 is a subgroup of G_1. Thus the centralizer G_2 of H_1 contains H_2, and we assume that H_1 is the centralizer of G_2. We assume that $\omega|_{H_1 \times G_2}$ is also a correspondence, so we may write

$$\omega|_{H_1 \times G_2} = \bigoplus_{j \in J} \sigma_j^{(1)} \otimes \pi_j^{(2)},$$

where $\sigma_j^{(1)}$ and $\pi_j^{(2)}$ are irreducible representations of H_1 and G_2, respectively. We thus have a *seesaw*:

$$\begin{tikzcd} G_1 \arrow[d, dash] \arrow[dr, dash] & G_2 \arrow[d, dash] \arrow[dl, dash] \\ H_1 & H_2 \end{tikzcd}$$

In this diagram, the vertical lines are inclusions, and the diagonal are the dual pairs of the two correspondences.

Lemma 3. *Let $G_1 \supset H_1$ and $G_2 \supset H_2$ be as above. If $i \in I$ then only representations from the set $\{\sigma_j^{(1)} \,|\, j \in J\}$ occur in the restriction of $\pi_i^{(1)}$ to H_1, so we have the "branching rule"*

$$\pi_i^{(1)} = \sum_{j \in J} c_{ij} \sigma_j^{(1)} \tag{28}$$

for some multiplicities c_{ij}. We have also for $j \in J$ the branching rule

$$\pi_j^{(2)} = \sum_{i \in I} c_{ji} \sigma_i^{(2)}.$$

Proof Any representation in the restriction of $\pi_i^{(1)}$ to H_1 occurs in the restriction of ω to H_1, so it is among the $\sigma_j^{(1)}$. Thus we may write (28). Similarly we may write

$$\pi_j^{(2)} = \sum_{i \in I} c'_{ji} \sigma_i^{(2)}$$

for some multiplicities c'_{ji}, and is just a matter of showing that the $c_{ij} = c'_{ij}$. We restrict ω to $H_1 \times H_2$ and get isomorphisms of $H_1 \times H_2$-modules:

$$\bigoplus_{i,j} c_{ij} \sigma_j^{(1)} \otimes \sigma_i^{(2)} \cong \bigoplus_i \pi_i^{(1)} \otimes \sigma_i^{(1)} \cong \omega \cong \bigoplus_j \sigma_j^{(1)} \otimes \pi_j^{(2)} \cong \bigoplus_{i,j} c'_{ij} \sigma_j^{(1)} \otimes \sigma_i^{(2)}$$

and the statement follows. □

We may now complete the proof of Theorem 2. We will exhibit a see-saw:

$$\begin{tikzcd} U(p+q) \arrow[d, no head] \arrow[drr, no head] & & U(n)\times U(n) \arrow[d, no head] \arrow[dll, no head] \\ U(p)\times U(q) & & U(n) \end{tikzcd}$$

On the right side, $U(n)$ is embedded diagonally in $U(n)\times U(n)$. The representation ω is the action of $U((p+q)n)$ on the symmetric algebra on $\mathrm{Mat}_{(p+q)\times n}(\mathbb{C})$. The actions are as follows. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathrm{Mat}_{(p+q)\times n}(\mathbb{C}), \qquad X_1 \in \mathrm{Mat}_{p\times n}(\mathbb{C}),\ X_2 \in \mathrm{Mat}_{q\times n}(\mathbb{C}).$$

The action of $U(p+q)$ is by left multiplication, and the action of $U(n)\times U(n)$ is by right multiplication on X_1 and X_2 individually. The centralizer of $U(p+q)$ is the diagonal subgroup $U(n)$ of $U(n)\times U(n)$ and the centralizer of $U(n)\times U(n)$ is the subgroup $U(p)\times U(q)$ of $U(p+q)$. As we have already explained (26) shows that the restriction of ω to either of these dual pairs is a correspondence, and the graphs of the correspondences are $\pi_\nu^{(p+q)} \longleftrightarrow \pi_\nu^{(n)}$ for the dual pair $(U(p+q), U(n))$, and $\pi_\lambda^{(p)} \otimes \pi_\mu^{(q)} \longleftrightarrow \pi_\lambda^{(n)} \otimes \pi_\mu^{(n)}$ for the dual pair $(U(p)\times U(q), U(n)\times U(n))$. The statement now follows from Lemma 3. □

3. Unitary group

3.1. **Products.** The goal of this section is to give a simple proof of Proposition 4, first derived by Conrey, Farmer, Keating, Rubinstein, and Snaith in [17], and of Corollary 1, first derived by Keating and Snaith [40].

We always normalize the Haar measure on a compact group so the total volume is 1. All integrations will be with respect to Haar measure. We will also sometimes denote by $\mathbb{E}_G f$ the integral of a function f over the group, its expected value with respect to the Haar probability measure.

Proposition 4. *Let $\alpha_1, \cdots, \alpha_{K+L}$ be complex numbers and let $\Xi_{L,K}$ consist of all permutations $\sigma \in S_{K+L}$ described in* (20).
Then

$$\begin{aligned} &\int_{U(N)} \left\{ \prod_{l=1}^{L} \det(I + \alpha_l^{-1}\cdot g^{-1}) \cdot \prod_{k=1}^{K} \det(I + \alpha_{L+k} g) \right\} dg = \\ &\frac{s_{\langle N^L \rangle}(\alpha_1, \cdots, \alpha_{K+L})}{\prod_{l=1}^{L}(\alpha_l)^N} = \\ &\sum_{\sigma\in\Xi_{L,K}} \frac{\prod_{k=1}^{K}(\alpha_{\sigma(L+k)}^{-1}\alpha_{L+k})^N}{\prod_{k=1}^{K}\prod_{l=1}^{L}(1-\alpha_{\sigma(l)}^{-1}\alpha_{\sigma(L+k)})}. \end{aligned} \tag{29}$$

Proof We have

$$\int_{U(N)} \left\{ \prod_{l=1}^{L} \det(I + \alpha_l^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det(I + \alpha_{L+k} g) \right\} dg =$$

$$\prod_{l=1}^{L} \alpha_l^{-N} \int_{U(N)} \prod_{k=1}^{K+L} \det(I + \alpha_k g) \overline{\det(g)^L} \, dg.$$

By the dual Cauchy identity, if $t_1, \cdots, t_N$ are the eigenvalues of g,

$$\prod_{k=1}^{K+L} \det(I + \alpha_k g) = \sum_{\lambda} s_\lambda(\alpha_1, \cdots, \alpha_{K+L}) \, s_{\lambda'}(t_1, \cdots, t_N),$$

where λ runs through all partitions.

Note that

$$\det(g)^L = s_{\lambda'}(t_1, \cdots, t_N)$$

where $\lambda = \langle N^L \rangle$ and $\lambda' = \langle L^N \rangle$. Thus integrating over g gives just this term. This proves the first line of (29). Now the second line follows by an application of Lemma 1. □

Remark 3. In view of discussion in Section 2.4 , we may characterize the preceding proof as using the dual pair $U(N), U(k)$ to transfer the computation from $U(N)$ to $U(k)$. This is fully analogous to the method used (for example) in Diaconis and Shahshahani [24] of using Frobenius-Schur duality to transfer computations from the unitary group to the symmetric group.

Keating and Snaith [40] (see also Baker and Forrester [4]) proved Corollary 1 below (without restriction that k be an integer) using the Selberg integral (see Forrester [26] for a comprehensive and insightful discussion of the Selberg integral and its generalizations.) This result was important because of its relationship to conjectures of Conrey and Ghosh [19], Conrey and Gonek [20], and Keating and Snaith [40] for the moments of the Riemann zeta function, due to which a dramatic new aspect of predictive power of random matrix theory for the zeta function was established .

Corollary 1. *We have*

$$\int_{U(N)} |\det(g - I)|^{2k} \, dg = \prod_{j=0}^{N-1} \frac{j!(j+2k)!}{(j+k)!^2}. \tag{30}$$

Proof Proposition 4 applied with $\alpha_i = 1$ and $L = K = k$ implies

$$\int_{U(N)} |\det(g - I)|^{2k} \, dg = s_{\langle N^k \rangle}(1^{2k}).$$

We now apply (19). It is easy to see, that for a partition $\lambda = N^k$ the product of hook numbers is given by

$$\prod_{j=0}^{N-1} \frac{(j+k)!}{j!}, \tag{31}$$

whereas the product $\prod_{u\in\lambda}(2k+c(u))$ is given by

$$\prod_{i=1}^{k}\prod_{j=1}^{N}(2k-i+j)=\prod_{j=1}^{N-1}\frac{(j+2k)!}{(j+k)!};$$

the result follows. □

3.2. **Littlewood-Schur symmetric functions.** While the proof of the result for ratios in the unitary case can be given using Schur functions only, it becomes more neat and transparent if we use the following generalization of Schur functions. These are called *hook Schur functions* by Berele and Regev [6] who denoted them by $\mathrm{HS}_\lambda(x;y)$. Macdonald [46, p.27, ex. 5; p.45, ex. 3]) denotes them $s_\lambda(x/y)$. These functions were considered earlier by Littlewood [43], pp. 66-70 in 1936 (see also Littlewood [44] p. 114–118, the section entitled "Extension to rational fractions of the formula for the S-function as a quotient of determinants"). Littlewood gave a formula for the LS_λ when $\lambda=\langle p^q\rangle$, which is important for us (a particular case of his formula is given in Proposition 9 below). We have not seen reference to this work of Littlewood in any of the post 1980-s papers. Perhaps an appropriate name for these functions is *Littlewood-Schur symmetric functions*, so we will denote them LS_λ.

Let $x_1,\cdots,x_k$ and $y_1,\cdots,y_l$ be two sets of variables. Define

$$\mathrm{LS}_\lambda(x_1,\cdots,x_k;\,y_1,\cdots,y_l)=\sum_{\mu,\nu}c^\lambda_{\mu\nu}s_\mu(x_1,\cdots,x_k)s_{\nu'}(\,y_1,\cdots,y_l).$$

Since ι is an automorphism of Λ we have $c^\lambda_{\mu\nu}=c^{\lambda'}_{\mu'\nu'}$ and so

$$\mathrm{LS}_\lambda(x_1,\cdots,x_k;\,y_1,\cdots,y_l)=\mathrm{LS}_{\lambda'}(y_1,\cdots,y_l;x_1,\cdots,x_k). \tag{32}$$

With possible exception of Proposition 8 the results in this section are not new, though the proofs may be. We start with the following generalization of Cauchy identity, due to Berele and Remmel [7], who also gave a bijective proof.

Proposition 5. *Let $\alpha_1,\cdots,\alpha_m$, $\beta_1,\cdots,\beta_n$, $\gamma_1,\cdots,\gamma_s$ and $\delta_1,\cdots,\delta_t$ be four sets of variables. We have*

$$\begin{aligned}&\sum\mathrm{LS}_\lambda(\alpha_1,\cdots,\alpha_m;\beta_1,\cdots,\beta_n)\ \mathrm{LS}_\lambda(\gamma_1,\cdots,\gamma_s;\delta_1,\cdots,\delta_t)\quad=\\ &\prod_{i,k}(1-\alpha_i\gamma_k)^{-1}\prod_{i,l}(1+\alpha_i\delta_l)\prod_{j,k}(1+\beta_j\gamma_k)\prod_{j,l}(1-\beta_j\delta_l)^{-1}.\end{aligned} \tag{33}$$

Proof By (11) and the Cauchy identity we have

$$\sum_\lambda \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1,\cdots,\alpha_m) s_\nu(\beta_1,\cdots,\beta_n) \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\gamma_1,\cdots,\gamma_s) s_\tau(\delta_1,\cdots,\delta_t) =$$

$$\sum_\lambda s_\lambda(\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n)\, s_\lambda(\gamma_1,\cdots,\gamma_s,\delta_1,\cdots,\delta_t) =$$

$$\prod_{i,k}(1-\alpha_i\gamma_k)^{-1} \prod_{i,l}(1-\alpha_i\delta_l)^{-1} \prod_{j,k}(1-\beta_j\gamma_k)^{-1} \prod_{j,l}(1-\beta_j\delta_l)^{-1}.$$

Formally, the identity follows on applying the involution ι in the variables β and δ. To make a rigorous proof from this idea, we may proceed as follows. Rewrite the last identity

$$\sum_\lambda \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1,\cdots,\alpha_m) s_\nu(\beta_1,\cdots,\beta_n) \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\gamma_1,\cdots,\gamma_s) s_\tau(\delta_1,\cdots,\delta_t) =$$

$$\prod_{i,k}(1-\alpha_i\gamma_k)^{-1} \prod_{i=1}^m \left(\sum_{N=0}^\infty \alpha_i^N h_N(\delta_1,\cdots,\delta_t)\right)$$

$$\times \prod_{j,k}(1-\beta_i\gamma_k)^{-1} \prod_{j=1}^n \left(\sum_{N=0}^\infty \beta_j^N h_N(\delta_1,\cdots,\delta_t)\right).$$

Since this is true for all t we may write

$$\sum_\lambda \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1,\cdots,\alpha_m) s_\nu(\beta_1,\cdots,\beta_n) \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\gamma_1,\cdots,\gamma_s) s_\tau =$$

$$\prod_{i,k}(1-\alpha_i\gamma_k)^{-1} \prod_{i=1}^m \left(\sum_{N=0}^\infty \alpha_i^N h_N\right)$$

$$\times \prod_{j,k}(1-\beta_i\gamma_k)^{-1} \prod_{j=1}^n \left(\sum_{N=0}^\infty \beta_j^N h_N\right),$$

where now s_τ and h_N are regarded as elements of the ring Λ which, we recall, is the inverse limit of the $\Lambda^{(N)}$. Applying the involution, which replaces s_τ by $s_{\tau'}$ and h_N by e_N, then specializing $s_\tau \longrightarrow s_\tau(\delta_1,\cdots,\delta_t)$ gives

$$\sum_\lambda \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1,\cdots,\alpha_m) s_\nu(\beta_1,\cdots,\beta_n)\, \mathrm{LS}_\lambda(\gamma_1,\cdots,\gamma_s;\delta_1,\cdots,\delta_t) =$$

$$\prod_{i,k}(1-\alpha_i\gamma_k)^{-1} \prod_{i,l}(1+\alpha_i\delta_l) \prod_{j,k}(1-\beta_i\gamma_k)^{-1} \prod_{j,l}(1+\beta_j\delta_l).$$

Now applying the same process again in the β_i gives the result. $\square$

Proposition 6. *We have*

$$\mathrm{LS}_\lambda(x_1,\cdots,x_{p+q};y_1,\cdots,y_l) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, \mathrm{LS}_\mu(x_1,\cdots,x_p;y_1,\cdots,y_l) s_\nu(x_{p+1},\cdots,x_{p+q}).$$

Proof By the definition of LS_λ and (11), we have

$$\begin{aligned}
& \mathrm{LS}_\lambda(x_1,\cdots,x_{p+q};y_1,\cdots,y_l) \\
= & \sum_{\theta,\phi} c^\lambda_{\theta\phi} s_\theta(x_1,\cdots,x_{p+q}) s_{\phi'}(y_1,\cdots,y_l) \\
= & \sum_{\theta,\phi,\psi,\nu} c^\lambda_{\theta\phi} c^\theta_{\psi\nu} s_\psi(x_1,\cdots,x_p) s_{\phi'}(y_1,\cdots,y_l) s_\nu(x_{p+1},\cdots,x_{p+q}) \\
= & \sum_{\mu,\phi,\psi,\nu} c^\lambda_{\mu\nu} c^\mu_{\psi\phi} s_\psi(x_1,\cdots,x_p) s_{\phi'}(y_1,\cdots,y_l) s_\nu(x_{p+1},\cdots,x_{p+q}),
\end{aligned}$$

where we have used the identity

$$\sum_\theta c^\lambda_{\theta\phi} c^\theta_{\psi\nu} = \sum_\mu c^\lambda_{\mu\nu} c^\mu_{\psi\phi},$$

which is a reflection of the fact that Λ is a commutative ring. (Both sides equal $\langle \chi_\lambda, \chi_\phi\chi_\psi\chi_\nu\rangle$ where the product is taken over $U(n)$ for any sufficiently large n.) The statement follows. ☐

A special case of this is a generalization of Pieri's formula.

Proposition 7. *We have*

$$\begin{aligned}
& \mathrm{LS}_\lambda(x_1,\cdots,x_k;y_1,\cdots,y_l) \\
= & \sum_{\substack{\mu\subseteq\lambda \\ \lambda-\mu \text{ a horizontal strip}}} \mathrm{LS}_\lambda(x_1,\cdots,x_{k-1};y_1,\cdots,y_l)\, x_k^{|\lambda|-|\mu|} \\
= & \sum_{\substack{\mu\subseteq\lambda \\ \lambda-\mu \text{ a vertical strip}}} \mathrm{LS}_\lambda(x_1,\cdots,x_k;y_1,\cdots,y_{l-1})\, y_k^{|\lambda|-|\mu|}.
\end{aligned}$$

Proof The second formula follows from the first on applying (32). We prove the first. If $l=0$, this is Pieri's formula (12). Applying this fact in Proposition 6 (with $p=k-1$ and $q=1$) gives the result. ☐

We also have the following generalization of Lemma 1:

Proposition 8. *Suppose λ of length $\leqslant K$ such that $\lambda_L \geqslant \lambda_{L+1}+Q$, let $\lambda=\tau\cup\rho$ with*

$$\tau=(\lambda_1,\ldots,\lambda_L), \qquad \rho=(\lambda_{L+1},\cdots,\lambda_{L+K}).$$

Then

$$\begin{aligned}
& \mathrm{LS}_\lambda(\alpha_1,\cdots,\alpha_{L+K};\gamma_1,\ldots,\gamma_Q) \\
= & \sum_{\sigma\in\Xi_{L,K}} \prod_{\substack{1\leqslant l\leqslant L \\ 1\leqslant k\leqslant K}} \left(\alpha_{\sigma(l)}-\alpha_{\sigma(L+k)}\right)^{-1} \\
\times & \ \mathrm{LS}_{\tau+\langle K^L\rangle}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(L)};\gamma_1,\ldots,\gamma_Q) \\
\times & \ \mathrm{LS}_\rho(\alpha_{\sigma(L+1)},\cdots,\alpha_{\sigma(L+K)};\gamma_1,\ldots,\gamma_Q) \qquad (34)
\end{aligned}$$

Proof We prove this by induction on Q. If $Q = 0$ this is Lemma 1. We assume that $Q > 0$ and that the statement is true for $Q - 1$. We enumerate the partitions $\mu \subseteq \lambda$ such that $\lambda - \mu$ is a vertical strip as follows. Let $\tau_1 \subseteq \tau$ be a partition such that $\tau - \tau_1$ is a vertical strip, and let $\rho_1 \subseteq \rho$ be a partition such that $\rho - \rho_1$ is a vertical strip. Then we let $\mu = \tau_1 \cup \rho_1$ where

$$\tau_1 = (\mu_1, \cdots, \mu_L) \qquad \rho_1 = (\mu_{L+1}, \cdots \mu_{L+K}).$$

Since $\lambda_L - \lambda_{L+1} \geqslant Q$ we have $\mu_L - \mu_{L+1} \geqslant Q - 1$, so μ is a partition, and the induction hypothesis is also satisfied. By Proposition 7 we have

$$\begin{aligned}
&\mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) \\
= &\sum_{\substack{\tau_1 \subseteq \tau \\ \rho_1 \subseteq \rho \\ \tau - \tau_1, \rho - \rho_1 \\ \text{vertical strips}}} \mathrm{LS}_\mu(\alpha_1, \cdots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_{Q-1})\, \gamma_Q^{|\lambda| - |\mu|} \\
= &\sum_{\substack{\tau_1 \subseteq \tau \\ \rho_1 \subseteq \rho \\ \tau - \tau_1, \rho - \rho_1 \\ \text{vertical strips}}} \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1} \\
\times\ &\mathrm{LS}_{\tau_1 + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_{Q-1}) \gamma_Q^{|\tau| - |\tau_1|} \\
&\mathrm{LS}_{\rho_1}(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_{Q-1}) \gamma_Q^{|\rho| - |\rho_1|}.
\end{aligned}$$

Applying Proposition 7 again, the statement follows. $\square$

Proposition 9. (Littlewood) *We have*

$$\mathrm{LS}_{\langle (l+m)^k \rangle}(x_1, \cdots, x_k; y_1, \cdots, y_l) = \left(\prod_{i=1}^k x_i \right)^m \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant l}} (x_i + y_j).$$

Proof We claim that if $N > k$ and $\mu \subseteq N^k$ then

$$c_{\mu\nu}^{\langle N^k \rangle} = \begin{cases} 1 & \text{if } \nu = (N - \mu_k, N - \mu_{k-1}, \cdots, N - \mu_1); \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let χ_μ, χ_ν and $\chi_{\langle N^k \rangle}$ be the corresponding associated characters of $U(k)$. We have $\chi_{\langle N^k \rangle} = \det^N$. The Littlewood-Richardson coefficient

$$c_{\mu\nu}^{\langle N^k \rangle} = \left\langle \chi_\mu \chi_\nu, \det{}^N \right\rangle = \left\langle \chi_\nu, \det{}^N \otimes \overline{\chi_\mu} \right\rangle.$$

Now $\det^N \otimes \overline{\chi_\mu}$ is the character of an irreducible representation; if $x_1, \cdots, x_k$ are the eigenvalues of g, its value at g is $s_{\nu_0}(x_1, \cdots, x_k)$, where

$$\nu_0 = (N - \mu_k, N - \mu_{k-1}, \cdots, N - \mu_1).$$

This may be seen directly from the definition of the Schur polynomial s_μ. The statement follows.

Now taking $N = l + m$,

$$\mathrm{LS}_{\langle (l+m)^k \rangle}(x_1, \cdots, x_k; y_1, \cdots, y_l) = \sum_\mu s_\mu(x_1, \cdots, x_k) s_{\nu'}(y_1, \cdots, y_l),$$

where we sum over all $\mu \in N^k$ and $\nu = (N - \mu_k, N - \mu_{k-1}, \cdots, N - \mu_1)$. We may restrict ourselves to μ for which $s_{\nu'} \neq 0$. This means that $\nu_1 = l(\nu') \leqslant l$, which implies that $\mu_k \geqslant m$. Thus $\langle m^k \rangle \subseteq \mu \subseteq \langle N^k \rangle$, and we may write $\mu = \langle m^k \rangle + \tilde{\mu}$ where now $\tilde{\mu} \subseteq \langle l^k \rangle$, $\tilde{\mu}' \subseteq \langle k^l \rangle$ and

$$\nu = (l - \tilde{\mu}_k, \cdots, l - \tilde{\mu}_1), \qquad \nu' = (k - \tilde{\mu}'_l, \cdots, k - \tilde{\mu}'_1).$$

We have

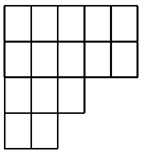
$$s_{\nu'}(y_1, \cdots, y_l) = (y_1 \cdots y_l)^k s_{\tilde{\mu}'}(y_1^{-1}, \cdots, y_l^{-1}),$$

so by the dual Cauchy identity

$$\begin{aligned} &\mathrm{LS}_{\langle (l+m)^k \rangle}(x_1, \cdots, x_k; y_1, \cdots, y_l) \\ = &\left(\prod_{i=1}^k x_i\right)^m (y_1 \cdots y_l)^k \sum s_\mu(x_1, \cdots, x_k) s_{\tilde{\mu}'}(y_1^{-1}, \cdots, y_l^{-1}) \\ = &\left(\prod_{i=1}^k x_i\right)^m (y_1 \cdots y_l)^k \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant l}} (1 + x_i y_j^{-1}), \end{aligned}$$

and the statement follows. ☐

Remark 4. A substantial generalization of Proposition 9 is furnished by the following result, known as Sergeev-Pragacz Formula (see Macdonald [46, p. 60 ex. 24]). To describe it, let λ be a partition with $\lambda_{k+1} \leqslant l$. Let μ be the part of λ that falls within the $k \times l$ rectangle, let ν and η be the remaining parts to the right and underneath this rectangle, that is $\lambda = (\mu + \nu) \cup \eta$. For example, if $\lambda = (5, 5, 3, 2)$, $k = 2$, $l = 3$ we have $\mu = (3, 3)$, $\nu = (2, 2)$ and $\eta = (2, 2, 1)$.



Then

$$\mathrm{LS}_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = \frac{1}{\prod_{i<j}(x_i - x_j) \prod_{i<j}(y_i - y_j)} \times \sum_{w \in S_k \times S_l} \varepsilon(w) w \left(x^{\nu + \delta_k} y^{\eta' + \delta_l} \prod_{(i,j) \in \mu} (x_i + y_j) \right), \tag{35}$$

where

$$\delta_k = (k-1, k-2, \ldots, 1, 0).$$

In the special case when λ contains l^k (35) factorizes as follows:
(36)

$$\mathrm{LS}_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = s_\nu(x_1, \ldots, x_k) s_{\eta'}(y_1, \ldots, y_l) \prod_{i=1}^{k} \prod_{j=1}^{l} (x_i + y_j).$$

Formula (36) is due to Berele and Regev [6, (6.20)]; it will be used in Section 7.

Remark 5. We conclude this section by remarking that Proposition 5 encodes an important property of the ring Λ, namely, the fact that it is a *Hopf algebra*. See Geissinger [33] and Zelevinsky [56]. We will not require this fact, and the reader may skip it. It seems important enough to include. The multiplication in Λ induces a map $m : \Lambda \otimes \Lambda \longrightarrow \Lambda$, whose adjoint with respect to the basis for which the s_λ are orthonormal is a map $m^* : \Lambda \longrightarrow \Lambda \otimes \Lambda$. Specifically we have

$$m(s_\mu \otimes s_\nu) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda, \qquad m^*(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu \otimes s_\nu.$$

The map m^* is a comultiplication making Λ a coalgebra. The *Hopf axiom* is the commutativity of the following diagram:

$$\begin{array}{ccccc} \Lambda \otimes \Lambda & \xrightarrow{m^* \otimes m^*} & \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda & \xrightarrow{1 \otimes \tau \otimes 1} & \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \\ \downarrow m & & & & \downarrow m \otimes m \\ \Lambda & & \xrightarrow{m^*} & & \Lambda \otimes \Lambda \end{array},$$

where $\tau : R \otimes R \longrightarrow R \otimes R$ is the map $\tau(u \otimes v) = v \otimes u$.

Proposition 10. (Geissinger [33]**)** *The Hopf axiom is satisfied.*

Proof The Hopf axiom reduces to the formula

$$\sum_\lambda c^\lambda_{\mu\nu} c^\lambda_{\sigma\tau} = \sum_{\varphi,\eta} c^\sigma_{\varphi\eta} c^\tau_{\psi\xi} c^\mu_{\varphi\xi} c^\nu_{\psi\eta}, \tag{37}$$

since if we apply $m^* \circ m$ to $s_\mu \otimes s_\nu$, then extract the coefficient of $s_\sigma \otimes s_\tau$ we obtain the left-hand side in (37), while if we perform the same computation with the map $(m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (m^* \otimes m^*)$ we obtain the right-hand side.

To deduce (37) from the generalized Cauchy identity we note that (in an obvious notation) the left-hand side of (33) equals

$$\sum c^\lambda_{\mu\nu} s_\mu(\alpha) s_{\nu'}(\beta) c^\lambda_{\sigma\tau} s_\sigma(\gamma) s_{\tau'}(\delta)$$

while the right-hand side equals

$$\begin{aligned}
&\sum s_\varphi(\alpha)s_\varphi(\gamma)s_{\psi'}(\beta)s_{\psi'}(\delta)s_\xi(\alpha)s_{\xi'}(\delta)s_{\eta'}(b)s_\eta(\gamma)\\
=\ &\sum c^\sigma_{\varphi\eta}c^\tau_{\psi\xi}s_\varphi(\alpha)s_\xi(\alpha)s_{\psi'}(\beta)s_{\eta'}(\beta)s_\sigma(\gamma)s_{\tau'}(\delta)\\
=\ &\sum c^\sigma_{\varphi\eta}c^\tau_{\psi\xi}c^\mu_{\varphi\xi}c^\nu_{\psi\eta}s_\mu(\alpha)s_{\nu'}(\beta)c^\lambda_{\sigma\tau}s_\sigma(\gamma)s_{\tau'}(\delta).
\end{aligned}$$

Comparing, we obtain the result. □

3.3. **Ratios.** The goal of this section is to give a simple proof of Theorem 3, which was established by Conrey, Farmer and Zirnbauer [15] and by Conrey, Forrester and Snaith [18]. We remark that in [15] the validity of Theorem 3 is established for all N, whereas in [18] it is established under the assumption $N \geq K \geq Q$.

Theorem 3. *Let $\alpha_1, \cdots, \alpha_{K+L}$ be complex numbers; let $\gamma_1, \ldots, \gamma_Q$ and $\delta_1, \ldots \delta_R$ be complex numbers satisfying $|\gamma_q| < 1$ and $|\delta_r| < 1$. Let $\Xi_{L,K}$ consist of all permutations $\sigma \in S_{K+L}$ described in* (20)*. If $N \geqslant Q, R$ we have*

$$\int_{U(N)} \frac{\prod_{l=1}^L \det(I+\alpha_l^{-1}\cdot g^{-1})\cdot\prod_{k=1}^K \det(I+\alpha_{L+k}\cdot g)}{\prod_{q=1}^Q \det(I-\gamma_q\cdot g)\prod_{r=1}^R\det(I-\delta_r\cdot g^{-1})}\,dg =$$

$$\sum_{\sigma\in\Xi_{L,K}}\prod_{k=1}^K(\alpha^{-1}_{\sigma(L+k)}\alpha_{L+k})^N \times$$

$$\frac{\prod_{q=1}^Q\prod_{l=1}^L(1+\gamma_q\alpha^{-1}_{\sigma(l)})\prod_{r=1}^R\prod_{k=1}^K(1+\delta_r\alpha_{\sigma(L+k)})}{\prod_{k=1}^K\prod_{l=1}^L(1-\alpha^{-1}_{\sigma(l)}\alpha_{\sigma(L+k)})\prod_{r=1}^R\prod_{q=1}^Q(1-\gamma_q\delta_r)}. \tag{38}$$

Proof By the dual Cauchy identity,

$$\begin{aligned}
&\prod_{l=1}^L \det(I+\alpha_l^{-1}\cdot g^{-1})\cdot\prod_{k=1}^K\det(I+\alpha_{L+k}\cdot g)\\
=\ &\overline{\det(g)^L}\ \prod_{l=1}^L\alpha_l^{-N}\prod_{k=1}^{K+L}\det(I+\alpha_k g)\\
=\ &\overline{\det(g)^L}\ \prod_{l=1}^L\alpha_l^{-N}\sum_\lambda s_\lambda(\alpha_1,\cdots,\alpha_{K+L})\chi_{\lambda'}(g)
\end{aligned}\tag{39}$$

On the other hand by the Cauchy identity

$$\prod_{q=1}^Q\det(I-\gamma_q g)^{-1}=\sum_\mu s_\mu(\gamma_1,\cdots,\gamma_Q)\ \chi_\mu(g)$$

and

$$\prod_{r=1}^R\det(I-\delta_r\cdot g^{-1})^{-1}=\sum_\nu s_\nu(\delta_1,\cdots,\delta_R)\ \overline{\chi_\nu(g)}.$$

Since we are assuming that $N \geqslant Q, R$, the sums in these identities is over all partitions μ of length $\leqslant Q$ and ν of length $\leqslant R$.

By Schur orthogonality the left hand side in (38) equals

$$\sum_{\lambda,\mu,\nu} \left\langle \chi_{\lambda'}\chi_\mu, \det^L \otimes \chi_\nu \right\rangle \prod_{l=1}^{L} \alpha_l^{-N} s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) s_\mu(\gamma_1, \cdots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R).$$

We rewrite this as

$$\begin{aligned} \prod_{l=1}^{L} \alpha_l^{-N} \sum_{\lambda,\mu,\nu} c_{\lambda'\mu}^{\tilde{\nu}} s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) s_\mu(\gamma_1, \cdots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R) &= \\ \prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} \mathrm{LS}_{\tilde{\nu}}(\gamma_1, \ldots, \gamma_Q; \alpha_1, \ldots, \alpha_{L+K}) s_\nu(\delta_1, \cdots, \delta_R) &= \\ \prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} \mathrm{LS}_{\hat{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R), & \end{aligned}$$

where $\tilde{\nu} = \nu + \left\langle L^N \right\rangle$ and $\hat{\nu} = \tilde{\nu}' = N^L \cup \nu'$. Now Proposition 8 is applicable since $N \geqslant Q$. It gives

$$\mathrm{LS}_{\hat{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) = \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} \left(\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)}\right)^{-1} \times$$

$$\mathrm{LS}_{\langle (N+K)^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q)\, \mathrm{LS}_{\nu'}(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q).$$

Substituting this expression, then using Propositions 5 and 9, and the obvious fact that

$$\prod_{l=1}^{L} \alpha_l^{-N} \alpha_{\sigma(l)}^{N} = \prod_{k=1}^{K} (\alpha_{\sigma(L+k)}^{-1} \alpha_{L+k})^N, \tag{40}$$

completes the proof. □

Remark 6. We conclude this section by remarking that Theorem 3 in combination with Heine identity (see [53] or [14]) easily implies a formula of Day [21]. The derivation is parallel to that in [23, Section 3] where the formula due to Schmidt and Spitzer [50], of which Day's formula is a generalization, is deduced from Proposition 4. A simple derivation of Day's formula was given by Conrey, Forrester and Snaith [18] using the method of Basor and Forrester [5].

4. Common Features of the formulae

This section contains a remark about the nature of the proofs of the formulae for mean values of products and ratios of characteristic polynomials. In each of the formulas for mean values of products or ratios of characteristic polynomials, a sum appears over certain Weyl group elements. For example, in Theorem 3, this is the sum over $\Xi_{L,K}$, while in Theorem 4 below, it is the sum over the ε_i. In each case, the summation appears from the Laplace expansion of a determinant, as in Lemma 1.

Our method may be summarized in the following terms:

- The quantity in question (a mean value of a product or ratio of characteristic polynomials) is expressed in terms of the character χ_λ of an irreducible representation of the group G, typically one whose highest weight vector is a partition of rectangular type;
- The shape of the highest weight vector suggests reduction with respect to a particular parabolic subgroup $P = MU$. If W and W_M denote the Weyl groups of G and the Levi factor M, the irreducible character χ is a sum over $W_M \backslash M$.

To make the paper as elementary as possible we always express these reductions by an application of the Laplace expansion of a determinant, but it seems worthwhile to note a more general reason such reductions are possible.

Let G be a reductive complex analytic Lie group, and let P be a parabolic subgroup with Levi decomposition $P = MU$, where M is the Levi factor and U the unipotent radical of P. Let T be a maximal torus of M, which is therefore also a maximal torus of G. We will denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T.

Let $\Phi \subset \mathfrak{t}^*$ be the root system of G, and let $\Phi_M \subset \Phi$ be the root system of M. We choose an ordering of the roots so that the roots in U are positive. Let ρ, ρ_M be half the sum of the positive root in Φ and Φ_M, respectively, and let ρ_U be half the sum of the positive roots in U, so that $\rho = \rho_M + \rho_U$.

Let W and W_M be the Weyl groups of G and M with respect to $\mathfrak{t}$. Let $\mathcal{C}$ and $\mathcal{C}_M$ be the positive Weyl chambers, so

$$\begin{aligned} \mathcal{C} &= \{x \in \mathfrak{t}^* \mid \langle \alpha, x\rangle \geqslant 0 \text{ for all } \alpha \in \Phi^+ \}, \\ \mathcal{C}_M &= \{x \in \mathfrak{t}^* \mid \langle \alpha, x\rangle \geqslant 0 \text{ for all } \alpha \in \Phi^+_M \} . \end{aligned}$$

If $\Lambda \subset \mathfrak{t}^*$ is the lattice of weights, that is, differentials of rational characters of T, then it is known that

$$\begin{aligned} \Lambda \cap \mathcal{C}^\circ &= \rho + (\Lambda \cap \mathcal{C}), \\ \Lambda \cap \mathcal{C}^\circ_M &= \rho_M + (\Lambda \cap \mathcal{C}_M). \end{aligned} \tag{41}$$

Each coset in $W_M \backslash W$ has a unique representative w such that $w\mathcal{C} \subset \mathcal{C}_M$. Let Ξ be this set of coset representatives.

Now let $\lambda \subset \Lambda \cap \mathcal{C}$ be a dominant weight. If $w \in \Xi$ then $w(\lambda+\rho) = \lambda_w + \rho_M$ where by (41) we have $\lambda_w \in \Lambda \cap \mathcal{C}_M$.

The Weyl character formula expresses the character χ^G_λ as

$$\frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)}}{e^{-\rho}\prod_{\alpha\in\Phi^+}(1-e^\alpha)} = \frac{\sum_{w\in\Xi}\sum_{\sigma\in W_M}(-1)^{l(\sigma w)}e^{\sigma w(\lambda+\rho)}}{e^{-\rho}\prod_{\alpha\in\Phi^+}(1-e^\alpha)}.$$

Here l is the length function on the Weyl group.

We note that if $w \in W_M$ then $w(\rho_U) = \rho_U$. It follows that

$$\chi_\lambda^G = \frac{1}{e^{-\rho_U} \prod_{\alpha \in \Phi^+ - \Phi_U^+} (1 - e^\alpha)} \sum_{w \in \Xi} (-1)^{l(w)} \chi_{\lambda_w}^M.$$

This generalization of Lemma 1 could be used everywhere in this paper that the Laplace expansion is invoked.

5. Symplectic Group

A unitary matrix g is said to be *symplectic* if $gJ^tg = J$ where

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

A symplectic matrix has determinant equal to 1. The symplectic group $\mathrm{Sp}(2N)$ is the group of $2N \times 2N$ symplectic matrices. The eigenvalues of a symplectic matrix are

$$e^{\pm i\theta_1}, \cdots, e^{\pm i\theta_N}$$

with

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_N \leqslant \pi.$$

The Weyl integration formula [54] for integrating a symmetric function $f(A) = f(\theta_1, \cdots, \theta_N)$ over $\mathrm{Sp}(2N)$ with respect to Haar measure is

$$\mathbb{E}_{\mathrm{Sp}(2N)} f = \int_{\mathrm{Sp}(2N)} f(g)\, dg = \frac{2^{N^2}}{\pi^N N!} \times$$

$$\int_{[0,\pi]^N} f(\theta_1, \cdots, \theta_N) \prod_{1 \leqslant j < k \leq N} (\cos\ \theta_k - \cos\ \theta_j)^2 \prod_{n=1}^N \sin^2 \theta_n \, d\theta_1 \cdots d\theta_N.$$

Denoting the irreducible representation of $U(2n)$ with highest weight λ by $F_{(2n)}^\lambda$ and the irreducible representation of $\mathrm{Sp}(2n)$ with highest weight μ by $V_{(2n)}^\mu$ we have the following classical branching rule, due to Littlewood [45], [44]. Denoting by $[F_{(2n)}^\lambda, V_{(2n)}^\mu]$ the multiplicity of $V_{(2n)}^\lambda$ in the restriction to $\mathrm{Sp}(2n)$ of $F_{(2n)}^\lambda$, Littlewood [44] p. 295 gave the branching rule

$$[F_{(2n)}^\lambda, V_{(2n)}^\mu] = \sum_{\beta' \text{ even}} c_{\mu\beta}^\lambda \qquad (l(\lambda), l(\mu) < n). \tag{42}$$

Here a partition λ is called *even* if all its parts are even (and similarly, it is called *odd* if all its parts are odd. See also Howe, Tan and Willenbring [36] and King [41] for this branching rule.

Denoting the irreducible character of the symplectic group $\mathrm{Sp}(2n)$ labelled by partition λ by $\chi_\lambda^{\mathrm{Sp}_{2n}}$, the Weyl character formula [54] in the case of symplectic group can be written

$$\chi_\lambda^{\mathrm{Sp}_{2n}}(x_1^{\pm 1}, \cdots, x_n^{\pm 1}) = \frac{\det_{1 \leqslant i,j \leqslant n}(x_j^{\lambda_i + n - i + 1} - x_j^{-(\lambda_i + n - i + 1)})}{\det_{1 \leqslant i,j \leqslant n}(x_j^{n-i+1} - x_j^{-(n-i+1)})}. \tag{43}$$

The determinant in the denominator can be evaluated as by Weyl [54]:

$$\det_{1\leqslant i,j\leqslant n}\left(x_j^{n-i+1}-x_j^{-(n-i+1)}\right)=\frac{\prod_{i<j}(x_i-x_j)(x_ix_j-1)\prod_i(x_i^2-1)}{(x_1\cdots x_n)^n}. \tag{44}$$

The Weyl dimension formula for the the dimension of the irreducible representation of the symplectic group $\mathrm{Sp}(2k)$ labelled by partition λ is

$$\dim(\chi_\lambda^{\mathrm{Sp}_{2k}})=\frac{\prod_{i<j}(\mu_i-\mu_j)(\mu_i+\mu_j+2)\prod_i(\mu_i+1)}{(2k-1)!(2k-3)!\cdots 1!},$$

where $\mu_i=\lambda_i+k-i$.

An alternative expression is given by El-Samra and King [25], in the following analogue of the hook formula (19):

$$\dim(\chi_\lambda^{\mathrm{Sp}_{2k}})=\prod_{u\in\lambda}\frac{2k+c^{\mathrm{Sp}}(u)}{h(u)}, \tag{45}$$

where

$$c^{\mathrm{Sp}}(i,j)=\begin{cases} i+j-\lambda_i'-\lambda_j' & \text{if } i\leqslant j;\\ \lambda_i+\lambda_j+2-i-j & \text{if } i>j.\end{cases}$$

We will make crucial use of the following analog of Lemma 2:

Lemma 4. *For $\lambda\subseteq\langle N^k\rangle$ let $\tilde\lambda=(k-\lambda_N',\cdots,k-\lambda_1')$. Then we have*

$$\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i+x_i^{-1}-t_n-t_n^{-1}) = \sum_{\lambda\subseteq N^k}(-1)^{|\tilde\lambda|}\chi_\lambda^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1})\chi_{\tilde\lambda}^{\mathrm{Sp}(2N)}(t_1^{\pm1},\cdots,t_N^{\pm1}). \tag{46}$$

Proof Using Weyl character formula (43) and the Laplace expansion we can rewrite the right-hand side of (46) as follows:

$$\det\begin{vmatrix} x_1^{N+k}-x_1^{-(N+k)} & x_1^{N+k-1}-x_1^{-(N+k-1)} & \dots & x_1^1-x_1^{-1}\\ \vdots & \vdots & \ddots & \vdots\\ x_k^{N+k}-x_k^{-(N+k)} & x_k^{N+k-1}-x_k^{-(N+k-1)} & \dots & x_k^1-x_k^{-1}\\ t_1^{N+k}-t_1^{-(N+k)} & t_1^{N+k-1}-t_1^{-(N+k-1)} & \dots & t_1^1-t_1^{-1}\\ \vdots & \vdots & \ddots & \vdots\\ t_N^{N+k}-t_N^{-(N+k)} & t_N^{N+k-1}-t_N^{-(N+k-1)} & \dots & t_N^1-t_N^{-1} \end{vmatrix}$$
$$\times\frac{(x_1\dots x_k)^k}{\prod_{1\leqslant i<j\leq k}(x_i-x_j)(x_ix_j-1)\prod_{i=1}^k(x_i^2-1)}$$
$$\times\frac{(t_1\dots t_N)^N}{\prod_{1\leqslant i<j\leq N}(t_i-t_j)(t_it_j-1)\prod_{i=1}^N(t_i^2-1)}. \tag{47}$$

Now the determinant in the equation (47) can be evaluated using the Weyl denominator formula (44) to be equal to

(48)

$$\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)(x_ix_j-1)\prod_{i=1}^{k}(x_i^2-1)\prod_{1\leqslant i<j\leqslant N}(t_i-t_j)(t_it_j-1)\prod_{i=1}^{N}(t_i^2-1)\times$$
$$\frac{\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i-t_n)(x_it_n-1)}{(x_1\dots x_k)^{N+k}(t_1\dots t_N)^{N+k}}.$$

Finally combining (47) and (48) we get that the expression on the right-hand side of (46) equals to

$$\frac{\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i-t_n)(x_it_n-1)}{(x_1\dots x_k)^{N}(t_1\dots t_N)^{k}}=\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i+x_i^{-1}-t_n-t_n^{-1}),$$

completing the proof.

□

Remark 7. Lemma 4 is stated in slightly different form in Jimbo and Miwa [38]. Another proof can be found in Howe [37], Theorem 3.8.9.3, which we digress to briefly discuss. We will describe an action of a group on a space of functions (or tensor fields) on some space as *geometric* if it is induced by an action of the group on the underlying space. As with our discussion of the Cauchy identity, we may consider the action of

$$(g,h):X\longmapsto \det(h)^{-k}\cdot gX^th$$

of $\mathrm{Sp}(2k)\times\mathrm{GL}(N)$ on $\mathrm{Mat}_{2k\times N}(\mathbb{C})=\mathbb{C}^{2kN}$. This induces an action of $\mathrm{Sp}(2k)\times\mathrm{GL}(N)$ on the symmetric and exterior algebras over $\mathbb{C}^{2kN}$, which is geometric. The trace of $(g,h)\in\mathrm{Sp}(2k)\times\mathrm{GL}(N)$, where $t_i^{\pm1}$ are the eigenvalues of g and x_n are the eigenvalues of h is

$$\prod_{i=1}^{k}\prod_{n=1}^{N}x_n^{-1}(1+x_nt_i)(1+x_nt_i^{-1})=\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i+x_i^{-1}+t_n+t_n^{-1}).$$

Howe considers the centralizer of $\mathrm{Sp}(2k)$ in this geometric action on the exterior algebra $\Lambda(\mathbb{C}^{2kN})$ and observes that it is properly larger than $\mathrm{GL}(N)$; in fact, it is a group isomorphic to $\mathrm{Sp}(2N)$, though the second symplectic group does not act geometrically. He finds the complete isotypic decomposition of $\Lambda(\mathbb{C}^{2kN})$:

$$\Lambda(\mathbb{C}^{2kN})\cong\sum_{\lambda}V_{(2k)}^{\lambda}\otimes V_{(2N)}^{\tilde\lambda},\tag{49}$$

where $V_{(2k)}^{\lambda}$ is an irreducible representation of $\mathrm{Sp}(2k)$ with highest weight λ, and λ runs over all partitions fitting inside $k\times N$ rectangle. Taking traces gives

$$\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i + x_i^{-1} + t_n + t_n^{-1}) = \sum_{\lambda\subseteq N^k}\chi_{\lambda}^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1})\chi_{\tilde{\lambda}}^{\mathrm{Sp}(2N)}(t_1^{\pm1},\cdots,t_N^{\pm1}), \tag{50}$$

and replacing the second symplectic matrix with its negative gives (46).

5.1. **Products.** The goal of this section is to give simple proofs of the Proposition 11, first derived by Conrey, Farmer, Keating, Rubinstein, and Snaith in [17], and of Corollary 2, first derived by Keating and Snaith [39].

Proposition 11. *Notation being as above we have:*

$$\int_{\mathrm{Sp}(2N)}\prod_{j=1}^{k}\det(I + x_j g)\,dg = (x_1\dots x_k)^N\chi_{\langle N^k\rangle}^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1}) = \sum_{\varepsilon\in\{\pm1\}}\prod_{j=1}^{k}x_j^{N(1-\varepsilon_j)}\prod_{i\leqslant j}(1 - x_i^{\varepsilon_i}x_j^{\varepsilon_j})^{-1}. \tag{51}$$

Proof Denoting the eigenvalues of g in $\mathrm{Sp}(2N)$ by $t_1^{\pm1},\cdots,t_N^{\pm1}$ we have:

$$\prod_{i=1}^{k}\det(I + x_i g) = \prod_{i=1}^{k}\prod_{n=1}^{N}(1 + x_i t_n)(1 + x_i t_n^{-1}).$$

Using (50) we have:

$$\begin{aligned}
&(x_1\cdots x_k)^{-N}\mathbb{E}_{\mathrm{Sp}(2N)}\prod_{j=1}^{k}\det(I + x_j g) = \\
&\mathbb{E}_{\mathrm{Sp}(2N)}\prod_{n=1}^{N}\prod_{i=1}^{k}(x_i + x_i^{-1} + t_n + t_n^{-1}) = \\
&\mathbb{E}_{\mathrm{Sp}(2N)}\left(\sum_{\lambda\subseteq N^k}\chi_{\lambda}^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1})\chi_{\tilde{\lambda}}^{\mathrm{Sp}(2N)}(t_1^{\pm1},\cdots,t_N^{\pm1})\right) = \\
&\chi_{N^k}^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1}).
\end{aligned}$$

In the last line we used the fact that

$$\mathbb{E}_{\mathrm{Sp}(2N)}\chi_{\lambda}^{\mathrm{Sp}(2N)} = \begin{cases} 1 & \text{if } \lambda = \varnothing; \\ 0 & \text{otherwise.}\end{cases}$$

Consequently we obtain

$$\mathbb{E}_{\mathrm{Sp}(2N)}\prod_{j=1}^{k}\det(I + x_j g) = (x_1\dots x_k)^N\chi_{N^k}^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1}), \tag{52}$$

proving the first line of (51).

Now using the Weyl character formula (43) for symplectic group and the evaluation (44) of the denominator in (43) we have:

$$(x_1 \dots x_k)^N \chi_{N^k}^{\mathrm{Sp}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \quad =$$

$$\begin{vmatrix} x_1^{N+k} - x_1^{-(N+k)} & x_1^{N+k-1} - x_1^{-(N+k-1)} & \cdots & x_1^{N+1} - x_1^{-(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k} - x_k^{-(N+k)} & x_k^{N+k-1} - x_k^{-(N+k-1)} & \cdots & x_k^{N+1} - x_k^{-(N+1)} \end{vmatrix}$$

$$\times \frac{(x_1 \cdots x_k)^{k+N}}{\prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)(x_i x_j - 1) \prod_{i=1}^k (x_i^2 - 1)}. \tag{53}$$

Now splitting the determinant in (53) we can rewrite it as follows:

$$\sum_{\varepsilon \in \{\pm 1\}^k} \det \begin{vmatrix} \varepsilon_1 x_1^{\varepsilon_1(N+k)} & \varepsilon_1 x_1^{\varepsilon_1(N+k-1)} & \cdots & \varepsilon_1 x_1^{\varepsilon_1(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_k x_k^{\varepsilon_k(N+k)} & \varepsilon_k x_k^{\varepsilon_k(N+k-1)} & \cdots & \varepsilon_k x_k^{\varepsilon_k(N+1)} \end{vmatrix} =$$

$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{i=1}^k \varepsilon_i x_i^{\varepsilon_i(N+1)} \det \begin{vmatrix} x_1^{\varepsilon_1(k-1)} & x_1^{\varepsilon_1(k-2)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(k-1)} & x_k^{\varepsilon_k(k-2)} & \cdots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{i=1}^k \varepsilon_i x_i^{\varepsilon_i(N+1)} \prod_{i<j} (x_i^{\varepsilon_i} - x_j^{\varepsilon_j}),$$

where in the last line we have used the Vandermonde determinant evaluation.

Next, making use of the elementary identities

$$\frac{\prod_{i=1}^k \varepsilon_i x_i^{(1+\varepsilon_i)}}{\prod_{i=1}^k (x_i^2 - 1)} = \frac{1}{\prod_{i=1}^k (1 - x^{-2\varepsilon_i})}, \tag{54}$$

$$\frac{x_i^{\varepsilon_i} - x_j^{\varepsilon_j}}{(x_i - x_j)(x_i x_j - 1)} = \frac{x_i^{\varepsilon_i - 1} x_j^{\varepsilon - 1}}{x_i^{\varepsilon_i} x_j^{\varepsilon_j} - 1}, \tag{55}$$

and noting that

$$\prod_{1 \leqslant i < j \leqslant k} x_i^{\varepsilon_i - 1} x_j^{\varepsilon - 1} = \prod_{i=1}^k x_i^{(k-1)(\varepsilon_i - 1)}, \tag{56}$$

the right hand side of (53) is easily seen to be equal to

$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{j=1}^k x_j^{N(1+\varepsilon_j)} \prod_{i \leqslant j} (1 - x_i^{-\varepsilon_i} x_j^{-\varepsilon_j})^{-1} \quad =$$

$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{j=1}^k x_j^{N(1-\varepsilon_j)} \prod_{i \leqslant j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1},$$

completing the proof. $\square$

Corollary 2. *We have*

$$\int_{\mathrm{Sp}(2N)} \det(I-g)^k \, dg = \dim(\chi^{\mathrm{Sp}(2k)}_{\langle N^k \rangle}) =$$

$$\frac{(N+k)!}{N!k!} \prod_{i=1}^{k} \frac{(k+2N+i)!i!}{(2i+2N)!(2i-1)!} \tag{57}$$

The mean value $\mathbb{E}_{\mathrm{Sp}(2N)} \det(I-g)^k$ was computed by Keating and Snaith in [39] using the Selberg integral (without restriction that k be an integer). They found it to be

$$2^{2Nk} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(\frac{1}{2}+k+j)}{\Gamma(1+N+k+j)\Gamma(\frac{1}{2}+j)}. \tag{58}$$

For integer k we can rewrite (58) using the duplication formula

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$$

in the form

$$\prod_{j=1}^{N} \frac{(N+j)!(2k+2j-1)!(j-1)!}{(N+k+j)!(2j-1)!(k+j-1)!},$$

and the latter expression is equal to the one appearing on the right-hand side of (57) by an elementary computation.

Proof We have:

$$\int_{\mathrm{Sp}(2N)} \det(I-g)^k \, dg = \dim(\chi^{\mathrm{Sp}(2k)}_{\langle N^k \rangle}). \tag{59}$$

Applying (45) to the partition $\lambda = N^k$ and recalling that the product of hook numbers for partition N^k is given by (31), we obtain

$$\begin{aligned}
\dim(\chi^{\mathrm{Sp}(2k)}_{\langle N^k \rangle}) &= \prod_{j=1}^{N} \frac{(j-1)!}{(j+k-1)!} \prod_{j=1}^{k} \frac{(j+k)!}{j!} \prod_{i=1}^{k} \frac{(k+2N+i)!}{(2i+2N)!} \prod_{i=1}^{k} \frac{i!}{(2i-1)!} \\
&= \frac{(N+k)!}{N!k!} \prod_{i=1}^{k} \frac{(k+2N+i)!i!}{(2i+2N)!(2i-1)!}.
\end{aligned}$$

Thus we have established that

$$\mathbb{E}_{\mathrm{Sp}(2N)} \det(I-g)^k = \frac{(N+k)!}{N!k!} \prod_{i=1}^{k} \frac{(k+2N+i)!i!}{(2i+2N)!(2i-1)!},$$

completing the proof of Corollary 2. $\square$

5.2. **Ratios.** The goal of this section is to give a simple proof of Theorem 4, first established in Conrey, Farmer and Zirnbauer [15] and in Conrey, Forrester and Snaith [18].

Theorem 4. *Let y_j be complex numbers with $|y_j| < 1$. Suppose $N \geqslant l$. Then we have:*

$$\int_{\mathrm{Sp}(2N)} \frac{\prod_{j=1}^k \det(I + x_j g)}{\prod_{i=1}^l \det(I - y_i g)}\, dg \quad =$$
$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{j=1}^k x_j^{N(1-\varepsilon_j)} \frac{\prod_{i=1}^k \prod_{j=1}^l (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i \leqslant j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i < j \leqslant l}(1 - y_i y_j)}.$$

Proof: When $l(\lambda) \leqslant N$ we may write the branching rule (42) in the form

$$s_\lambda(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) = \sum_{\mu \subseteq \lambda} \chi_\mu^{\mathrm{Sp}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) \left(\sum_{\beta' \text{ even}} c_{\mu\beta}^\lambda \right). \tag{60}$$

Also we have the following identity of Littlewood [44]

$$\prod_{1 \leqslant i < j \leqslant l} (1 - y_i y_j)^{-1} = \sum_{\beta' \text{ even}} s_\beta(y_1, \cdots, y_l).$$

These two formulas, together with the Cauchy identity (8) yields the following Cauchy identity for $\mathrm{Sp}(2N)$, known to Weyl [54] and Littlewood [44]:

$$\frac{1}{\prod_{n=1}^N \prod_{j=1}^l (1 - y_j t_n)(1 - y_j t_n^{-1})} \quad =$$
$$\frac{1}{\prod_{i<j}(1 - y_i y_j)} \sum_\mu \chi_\mu^{\mathrm{Sp}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) s_\mu(y_1, \cdots, y_l).$$

Combining this identity with (50) we have, with $t_i^{\pm 1}$ the eigenvalues of $g \in \mathrm{Sp}(2N)$

$$\frac{\prod_{j=1}^k \det(I + x_j g)}{\prod_{i=1}^l \det(I - y_i g)} =$$
$$\frac{(x_1 \cdots x_k)^N}{\prod_{i<j}(1 - y_i y_j)} \sum_{\lambda \subseteq N^k} \chi_\lambda^{\mathrm{Sp}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \chi_{\tilde{\lambda}}^{\mathrm{Sp}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1})$$
$$\sum_\mu \chi_\mu^{\mathrm{Sp}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) s_\mu(y_1, \cdots, y_l).$$

Since

$$\mathbb{E}_{\mathrm{Sp}(2N)} \chi_\lambda^{\mathrm{Sp}(2N)}(g) \chi_\mu^{\mathrm{Sp}(2N)}(g) = \begin{cases} 1 & \text{if } \lambda = \mu,\ l(\lambda) \leqslant N; \\ 0 & \text{otherwise,} \end{cases}$$

the theorem follows from the following Proposition.

Proposition 12. *Notation being as in Lemma 4 we have*

$$\sum_{\lambda\subseteq\langle N^k\rangle} \chi_\lambda^{\mathrm{Sp}(2k)}(x_1^{\pm1},\cdots,x_k^{\pm1}) s_{\tilde\lambda}(y_1,\cdots,y_l) \quad = \tag{61}$$
$$\sum_{\varepsilon\in\{\pm1\}^k} \prod_{j=1}^k x_j^{-N\varepsilon_j} \frac{\prod_{i=1}^k\prod_{j=1}^l(1+x_i^{\varepsilon_i}y_j)}{\prod_{i\leqslant j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}.$$

Proof In order to yield a non-zero contribution to the sum on the left-hand side of (61) λ must be of the form $\lambda=(N-l)^k+\mu$ with $\mu\subseteq l^k$. Now keeping in mind the Weyl character formula for symplectic group (43) and the numerator evaluation (44), together with the definition (5) of the Schur function $s_{\tilde\lambda}$ and Vandermonde identity expressing the denominator in (5) as $\prod_{i<j}(x_i-x_j)$, we can rewrite the expression on the left-hand side of (61) as follows, using the Laplace expansion:

$$\det\begin{vmatrix} x_1^{N+k}-x_1^{-(N+k)} & x_1^{N+k-1}-x_1^{-(N+k-1)} & \cdots & x_1^{(N-l+1)}-x_1^{-(N-l+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k}-x_k^{-(N+k)} & x_k^{N+k-1}-x_k^{-(N+k-1)} & \cdots & x_k^{(N-l+1)}-x_k^{-(N-l+1)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix}$$

$$\times\frac{1}{\prod_{1\leqslant i<j\leqslant l}(y_j-y_i)}\frac{(x_1\cdots x_k)^k}{\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)(x_ix_j-1)\prod_{i=1}^k(x_i^2-1)}.$$

Now splitting the determinant in this expression we can rewrite it as follows:

$$\sum_{\varepsilon\in\{\pm1\}^k}\det\begin{vmatrix} \varepsilon_1x_1^{\varepsilon_1(N+k)} & \varepsilon_1x_1^{\varepsilon_1(N+k-1)} & \cdots & \varepsilon_1x_1^{\varepsilon_1(N-l+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_kx_k^{\varepsilon_k(N+k)} & \varepsilon_kx_k^{\varepsilon_k(N+k-1)} & \cdots & \varepsilon_kx_k^{\varepsilon_k(N-l+1)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix};$$

and this can be further expressed in the following way:

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^k \varepsilon_i x_i^{\varepsilon_i(N-l+1)} \det \begin{vmatrix} x_1^{\varepsilon_1(l+k-1)} & x_1^{\varepsilon_1(l+k-2)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_1(l+k-1)} & x_k^{\varepsilon_1(l+k-2)} & \cdots & 1 \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^k \varepsilon_i x_i^{\varepsilon_i(N-l+1)} \prod_{i<j}(x_i^{\varepsilon_i}-x_j^{\varepsilon_j}) \prod_{1\leqslant i<j\leqslant l}(y_j-y_i)\prod_{i=1}^k\prod_{j=1}^l(x_i^{\varepsilon_i}+y_j),$$

where in the last line we have used the Vandermonde determinant evaluation.

Next, making use of the elementary identities (54), (55), (56), the expression on the right-hand side is easily brought to the form

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{j=1}^k x_j^{-N\varepsilon_j}\frac{\prod_{i=1}^k\prod_{j=1}^l(1+x_i^{\varepsilon_i}y_j)}{\prod_{i\leqslant j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}.$$

□

6. Orthogonal group

A matrix g is said to be *orthogonal* if it is real and $g\,{}^tg = I$, where ${}^tg$ denotes the transpose of g. In particular, it is unitary. We let $O(N)$ denote the group of $N\times N$ orthogonal matrices. We let $\mathrm{SO}(N)$ denote the subgroup of $U(N)$ consisting of $N\times N$ orthogonal matrices with determinant equal to 1.

For any complex eigenvalue of an orthogonal matrix, its complex conjugate is also an eigenvalue. The eigenvalues of $g\in\mathrm{SO}(2N)$ can be written as

$$e^{\pm i\theta_1},\cdots,e^{\pm i\theta_N}$$

with

$$0\leqslant\theta_1\leqslant\theta_2\leqslant\cdots\leqslant\theta_N\leqslant\pi.$$

The Weyl integration formula [54] for integrating a symmetric function $f(g)=f(\theta_1,\cdots,\theta_N)$ over $\mathrm{SO}(2N)$ with respect to Haar measure is given by

$$\mathbb{E}_{\mathrm{SO}(2N)}f(g)=\int_{\mathrm{SO}(2N)}f(g)\;dg \quad =$$

$$\frac{2^{(N-1)^2}}{\pi^N N!}\int_{[0,\pi]^N}f(\theta_1,\cdots,\theta_N)\prod_{1\leqslant j<k\leqslant N}(\cos\theta_k-\cos\theta_j)^2 d\theta_1\cdots d\theta_N.$$

The eigenvalues of $g\in\mathrm{SO}(2N+1)$ can be written as

$$1,e^{\pm i\theta_1},\cdots,e^{\pm i\theta_N}$$

with

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_N \leqslant \pi.$$

The Weyl integration formula [54] for integrating a symmetric function $f(g) = f(\theta_1, \cdots, \theta_N)$ over $\mathrm{SO}(2N+1)$ with respect to Haar measure is given by

$$\mathbb{E}_{\mathrm{SO}(2N+1)} f = \int_{\mathrm{SO}(2N+1)} f(g)\, dg \quad = \\ \frac{2^{1+(N-1)^2}}{\pi^{N-1}(N-1)!} \int_{[0,\pi]^{N-1}} f(\theta_1, \cdots, \theta_{N-1}) \\ \prod_{1 \leqslant j < k \leqslant N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \cdots d\theta_{N-1},$$

where $\theta_N = 0$.

Denoting the irreducible representation of $U(2n)$ with highest weight λ by $F^\lambda_{(2n)}$ and the irreducible representation of $O(2n)$ with highest weight μ by $E^\mu_{(2n)}$ we have the following classical branching rule, due to Littlewood [45, 44]

$$[F^\lambda_{(2n)}, E^\mu_{(2n)}] = \sum_{2\delta} c^\lambda_{\mu(2\delta)}. \tag{62}$$

Denoting the irreducible character of $\mathrm{SO}(2n+1)$ labelled by partition λ by $\chi_\lambda^{\mathrm{SO}(2n+1)}$, the Weyl character formula [54] in the case of $\mathrm{SO}(2n+1)$ may be written

(63)

$$\chi_\lambda^{\mathrm{SO}(2n+1)}(x_1^{\pm 1}, \cdots, x_n^{\pm 1}, 1) = \frac{\det_{1\leqslant i,j\leqslant n}(x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)})}{\det_{1\leqslant i,j\leqslant n}(x_j^{n-i+1/2} - x_j^{-(n-i+1/2)})};$$

the determinant in the numerator can be evaluated as follows [54]:

$$\det_{1\leqslant i,j\leqslant n}(x_j^{n-i+1} - x_j^{-(n-i+1)}) = \frac{\prod_{i<j}(x_i - x_j)(x_i x_j - 1)\prod_{i=1}^k (1-x_i)}{(x_1\cdots x_n)^{n-1/2}}. \tag{64}$$

In case of $\mathrm{SO}(2n)$ the situation is more subtle. Denoting the irreducible character of $\mathrm{SO}(2n)$ labelled by partition λ by $\chi_\lambda^{\mathrm{SO}(2n)}$, the Weyl character formula [54] in the case of $\mathrm{SO}(2n)$ reads

$$\chi_\lambda^{\mathrm{SO}(2n)}(x^{\pm 1}) \quad = \\ \frac{\det_{1\leqslant i,j\leqslant n}(x_j^{\lambda_i+n-i} + x_j^{-(\lambda_i+n-i)}) + \det_{1\leqslant i,j\leqslant n}(x_j^{\lambda_i+n-i} - x_j^{-(\lambda_i+n-i)})}{\det_{1\leqslant i,j\leqslant n}(x_j^{n-i} + x_j^{-(n-i)})}. \tag{65}$$

Here $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_{n-1} \geqslant |\lambda_n|$, so for a partition λ we have two characters: λ_+ associated with $(\lambda_1, \cdots, \lambda_n)$ and λ_- associated with $(\lambda_1, \cdots, -\lambda_n)$ (this corresponds to the involution in the Dynkin diagram of type D_n). The second term in the numerator in (65) changes sign when λ_n is replaced by

$-\lambda_n$; in particular it vanished when $\lambda_n = 0$. When $\lambda_n = 0$, the character $\chi_\lambda^{\mathrm{SO}(2n)}$ also yields the character of the orthogonal group $O(2N)$. When $\lambda_n \neq 0$, the irreducible character of $O(2N)$ is given by $\chi_{\lambda_+}^{\mathrm{SO}(2n)} + \chi_{\lambda_-}^{\mathrm{SO}(2n)}$:

$$\chi_\lambda^{O(2n)}(x^{\pm 1}) = \frac{\det_{1\leqslant i,j\leqslant n}(x_j^{\lambda_i+n-i} + x_j^{-(\lambda_i+n-i)})}{\det_{1\leqslant i,j\leqslant n}(x_j^{n-i} + x_j^{-(n-i)})}. \tag{66}$$

The determinant in the numerator of (65) can be evaluated as follows [54]:

$$\det_{1\leqslant i,j\leqslant n}(x_j^{n-i} + x_j^{-(n-i)}) = \frac{\prod_{i<j}(x_i - x_j)(x_i x_j - 1)}{(x_1 \cdots x_n)^{n-1}}. \tag{67}$$

6.1. **Products and ratios for** SO$(2N)$.

Lemma 5. *For $\lambda \subseteq k^N$ let $\tilde{\lambda} = (k - \lambda'_N, \cdots, k - \lambda'_1)$. Then we have*

$$\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i + x_i^{-1} - t_n - t_n^{-1}) =$$
$$\sum_{\lambda\subseteq N^k}(-1)^{|\tilde{\lambda}|}(\chi_{\lambda_+}^{\mathrm{SO}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1})\chi_{\tilde{\lambda}_+}^{\mathrm{SO}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) +$$
$$\chi_{\lambda_-}^{\mathrm{SO}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1})\chi_{\tilde{\lambda}_-}^{\mathrm{SO}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1})). \tag{68}$$

Proof The lemma is stated in slightly different form in Jimbo and Miwa [38]. It may be proved by the method of Lemma 2 and Lemma 4. ☐

We begin by giving a simple proof of the following proposition, first established in [17].

Proposition 13. *Notation being as above we have*

$$\mathbb{E}_{g\in SO(2N)}\prod_{j=1}^{k}\det(I + x_j g) = (x_1 \ldots x_k)^N \chi_{\langle N^k\rangle}^{O_{2k}}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \quad =$$
$$\sum_{\varepsilon\in\{\pm 1\}}\prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)}\prod_{i<j}(1 - x_i^{\varepsilon_i}x_j^{\varepsilon_j})^{-1}. \tag{69}$$

Proof Denoting the eigenvalues of g in SO$(2N)$ by $t_1^{\pm 1}, \cdots, t_N^{\pm 1}$ we have:

$$\prod_{i=1}^{k}\det(I + x_i g) = \prod_{i=1}^{k}\prod_{n=1}^{N}(1 + x_i t_n)(1 + x_i t_n^{-1}).$$

Using (68) we have (where t_i are the eigenvalues of $g \in \mathrm{SO}(2N)$)

$$\begin{gathered}
(x_1 \cdots x_k)^{-N} \mathbb{E}_{\mathrm{SO}(2N)} \textstyle\prod_{j=1}^{k} \det(I + x_j g) = \\
\mathbb{E}_{\mathrm{SO}(2N)} \textstyle\prod_{n=1}^{N} \prod_{i=1}^{k} (x_i + x_i^{-1} + t_n + t_n^{-1}) = \\
\mathbb{E}_{\mathrm{SO}(2N)} (\textstyle\sum_{\lambda \subseteq N^k} \chi_{\lambda_+}^{\mathrm{SO}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \chi_{\widetilde{\lambda_+}}^{\mathrm{SO}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}) + \\
\chi_{\lambda_-}^{\mathrm{SO}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \chi_{\widetilde{\lambda_-}}^{\mathrm{SO}(2N)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1})) = \\
\chi_{\langle N^k \rangle}^{O(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}).
\end{gathered}$$

In the last line we used the fact that

$$\mathbb{E}_{\mathrm{SO}(2N)} \chi_\lambda^{\mathrm{SO}(2N)} = \begin{cases} 1 & \text{if } \lambda = \varnothing; \\ 0 & \text{otherwise} \end{cases}$$

and $\chi_\lambda^{O(2n)} = \chi_{\lambda_+}^{\mathrm{SO}(2n)} + \chi_{\lambda_-}^{\mathrm{SO}(2n)}$ (see discussion preceding (66)).

Consequently we obtain

$$\mathbb{E}_{\mathrm{SO}(2N)} \prod_{j=1}^{k} \det(I + x_j g) = (x_1 \ldots x_k)^N \chi_{\langle N^k \rangle}^{\mathrm{O}_{2k}}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}), \tag{70}$$

proving the first line of (69).

Now using the Weyl character formula for the orthogonal group (66) and numerator evaluation (67) we have:

(71)

$$\begin{aligned}
&(x_1 \ldots x_k)^N \chi_{N^k}^{O(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) = \\
&\det \begin{vmatrix} x_1^{N+k-1} + x_1^{-(N+k-1)} & x_1^{N+k-2} - x_1^{-(N+k-2)} & \cdots & x_1^N - x_1^{-(N)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k-1} - x_k^{-(N+k-1)} & x_k^{N+k-2} - x_k^{-(N+k-2)} & \cdots & x_k^N - x_k^{-(N)} \end{vmatrix} \\
&\times \frac{(x_1 \cdots x_k)^{k+N-1}}{\prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)(x_i x_j - 1)}.
\end{aligned}$$

Next, splitting the determinant in this expression we can rewrite it as follows:

$$\sum_{\varepsilon\in\{\pm1\}^k} \det\begin{vmatrix} x_1^{\varepsilon_1(N+k-1)} & x_1^{\varepsilon_1(N+k-2)} & \cdots & x_1^{\varepsilon_1(N)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(N+k-1)} & x_k^{\varepsilon_k(N+k-2)} & \cdots & x_k^{\varepsilon_k(N)} \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k} \prod_{i=1}^k x_i^{\varepsilon_i(N)} \det\begin{vmatrix} x_1^{\varepsilon_1(k-1)} & x_1^{\varepsilon_1(k-2)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(k-1)} & x_k^{\varepsilon_k(k-2)} & \cdots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k} \prod_{i=1}^k x_i^{\varepsilon_i N} \prod_{i<j}(x_i^{\varepsilon_i} - x_j^{\varepsilon_j}),$$

where in the last line we have used the Vandermonde determinant evaluation.

Now making use of the elementary identities (55) and (56) the right hand side of (71) is easily seen to be equal to

$$\sum_{\varepsilon\in\{\pm1\}} \prod_{j=1}^k x_j^{-N\varepsilon_j} \prod_{i<j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1},$$

completing the proof. □

We note that proceeding along exactly the same lines we easily establish the following result:

Proposition 14. *Notation being as above we have*

$$\begin{aligned} &\mathbb{E}_{M\in O(2N)} \prod_{j=1}^k \det(I + x_j M) = (x_1 \dots x_k)^N \chi^{\mathrm{SO}_{2k}}_{(N^k)_+}(x_1^{\pm1}, \dots, x_k^{\pm1}) = \\ &\sum_{\substack{\varepsilon\in\{\pm1\} \\ \mathrm{sgn}(\varepsilon)=1}} \prod_{j=1}^k x_j^{N(1-\varepsilon_j)} \prod_{i<j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1}. \end{aligned} \tag{72}$$

In particular, in analogy with the unitary and symplectic cases, we have:

$$\mathbb{E}_{g\in O(2N)} \ \det(I - g)^k = \dim(\chi^{\mathrm{SO}(2k)}_{\langle N^k\rangle}). \tag{73}$$

The Weyl dimension formula for the the dimension of the irreducible representation of the group $\mathrm{SO}(2k)$ labelled by partition λ is given by

$$\dim(\chi_\lambda^{\mathrm{SO}(2k)}) = 2^{k-1} \frac{\prod_{i<j}(\mu_i - \mu_j)(\mu_i + \mu_j)}{(2k-2)!(2k-4)! \cdots 2!},$$

where $\mu_i = \lambda_i + k - i$.

An alternative expression, given by El-Samra and King [25], is furnished by the following analogue of hook formula (19):

$$\dim(\chi_\lambda^{\mathrm{SO}_{2k}}) = \prod_{u\in\lambda} \frac{2k + c^{\mathrm{SO}}(u)}{h(u)}, \tag{74}$$

where

$$c^{\mathrm{SO}}(i,j) = \begin{cases} i+j-\lambda'_i-\lambda'_j-2 \text{ if } i<j \\ \lambda_i+\lambda_j-i-j \text{ if } i \geqslant j. \end{cases}$$

Applying (74) to the partition $\lambda = N^k$ and recalling that the product of hook numbers for partition N^k is given by (31), we obtain, in view of (73):

$$\mathbb{E}_{g\in O(2N)} \det(I-g)^k = 2^k \frac{(N+k-1)!}{(N-1)!(k-1)!} \times \prod_{i=1}^{k} \frac{(2i-3)!(i+k+2N-2)!(i+2N-2)!}{(i+N-1)!(2i+2N-2)!(i+k+N-2)!}.$$

We remark that in [39] Keating and Snaith derived the following expression using Selberg's integral (without restriction that k be an integer):

$$\mathbb{E}_{\mathrm{SO}(2N)} \det(I-g)^k = 2^{2Nk} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(k+j-\frac{1}{2})}{\Gamma(N+k+j-1)\Gamma(j-\frac{1}{2})}; \tag{75}$$

for integer k this is easily derived from (70) and the Weyl dimension formula.

We now turn to ratios.

Proposition 15. *Notation being as above we have*

$$\sum_{\lambda\subseteq N^k} \chi_\lambda^{\mathrm{O}_{2k}}(x_1^{\pm 1},\dots,x_k^{\pm 1}) s_{\tilde{\lambda}}(y_1,\dots,y_l) = \sum_{\varepsilon\in\{\pm 1\}^k} \prod_{j=1}^{k} x_j^{-N\varepsilon_j} \frac{\prod_{i=1}^k \prod_{j=1}^l (1+x_i^{\varepsilon_i} y_j)}{\prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}. \tag{76}$$

Proof In order to yield a non-zero contribution to the sum on the left-hand side of (76) λ must be of the form $\lambda = (N-l)^k + \mu$ with $\mu \subseteq l^k$. Now keeping in mind the Weyl character formula (66) and numerator evaluation (67), together with the definition of the Schur polynomial, we can rewrite the expression on the left-hand side of (76) as follows, using the Laplace expansion:

(77)

$$\det \begin{vmatrix} x_1^{N+k-1}+x_1^{-(N+k-1)} & x_1^{N+k-2}+x_1^{-(N+k-2)} & \dots & x_1^{N-l}+x_1^{-(N-l)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{N+k-1}+x_k^{-(N+k-1)} & x_k^{N+k-2}+x_k^{-(N+k-2)} & \dots & x_k^{N-l}+x_k^{-(N-l)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \dots & 1 \end{vmatrix}$$

$$\times \frac{1}{\prod_{1\leqslant i<j\leqslant l}(y_j-y_i)} \frac{(x_1\dots x_k)^{k-1}}{\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)(x_ix_j-1)}.$$

Now splitting the determinant in this expression we can rewrite it as follows:

$$\sum_{\varepsilon\in\{\pm1\}^k}\det\begin{vmatrix} x_1^{\varepsilon_1(N+k-1)} & x_1^{\varepsilon_1(N+k-2)} & \cdots & x_1^{\varepsilon_1(N-l)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(N+k-1)} & x_k^{\varepsilon_k(N+k-2)} & \cdots & x_k^{\varepsilon_k(N-l)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \dots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^k x_i^{\varepsilon_i(N-l)}\det\begin{vmatrix} x_1^{\varepsilon_1(l+k-1)} & x_1^{\varepsilon_1(l+k-2)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(l+k-1)} & x_k^{\varepsilon_k(l+k-2)} & \cdots & 1 \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \dots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^k x_i^{\varepsilon_i(N-l)}\prod_{i<j}(x_i^{\varepsilon_i}-x_j^{\varepsilon_j})\prod_{1\leqslant i<j\leqslant l}(y_j-y_i)\prod_{i=1}^k\prod_{j=1}^l(x_i^{\varepsilon_i}+y_j),$$

where in the last line we have used the Vandermonde determinant evaluation.

Next, making use of the elementary identities (55), (56), the expression on the right-hand side of (77) is easily brought to the form

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{j=1}^k x_j^{-N\varepsilon_j}\frac{\prod_{i=1}^k\prod_{j=1}^l(1+x_i^{\varepsilon_i}y_j)}{\prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})},$$

proving the proposition. □

Theorem 5. *Assume $N\geqslant l$ and $|y_i|<1$. Then we have:*

$$\mathbb{E}_{g\in SO(2N)}\frac{\prod_{j=1}^k\det(I+x_jg)}{\prod_{i=1}^l\det(I-y_ig)} =$$

$$\sum_{\varepsilon\in\{\pm1\}}\prod_{j=1}^k x_j^{N(1-\varepsilon_j)}\frac{\prod_{i=1}^k\prod_{j=1}^l(1+x_i^{\varepsilon_i}y_j)}{\prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})\prod_{1\leqslant i\leqslant j\leqslant l}(1-y_iy_j)}.$$

Proof The Cauchy identity for SO$(2N)$ has the following form:

$$\frac{1}{\prod_{n=1}^N\prod_{j=1}^l(1-y_jt_n)(1-y_jt_n^{-1})} =$$

$$\frac{1}{\prod_{i\leqslant j}(1-y_iy_j)}\sum_\mu\chi_\mu^{O(2N)}(t_1^{\pm1},\cdots,t_N^{\pm1})s_\mu(y_1,\cdots,y_l). \tag{78}$$

Combining (78) and (68) we have:

$$\frac{\prod_{j=1}^{k}\det(I+x_jM)}{\prod_{i=1}^{l}\det(I-y_iM)} = \frac{(x_1\dots x_k)^N}{\prod_{i\leqslant j}(1-y_iy_j)}\times$$
$$\sum_{\mu}\chi_{\mu}^{\mathrm{O}_{2N}}(t_1^{\pm 1},\dots,t_N^{\pm 1})s_{\mu}(y_1,\dots,y_l)\times$$
$$\sum_{\lambda\subseteq N^k}(\chi_{\lambda_+}^{\mathrm{SO}_{2k}}(x_1^{\pm 1},\dots,x_k^{\pm 1})\chi_{\tilde{\lambda}_+}^{\mathrm{SO}_{2N}}(t_1^{\pm 1},\dots,t_N^{\pm 1})+$$
$$\chi_{\lambda_-}^{\mathrm{SO}_{2k}}(x_1^{\pm 1},\dots,x_k^{\pm 1})\chi_{\tilde{\lambda}_-}^{\mathrm{SO}_{2N}}(t_1^{\pm 1},\dots,t_N^{\pm 1}))$$

Since

$$\mathbb{E}_{\mathrm{SO}(2N)}\chi_{\lambda_\pm}^{\mathrm{SO}_{2N}}(M)\chi_{\mu}^{\mathrm{O}_{2N}}(M) = \begin{cases}1, & \text{if } \lambda=\mu,\ l(\lambda)\leqslant N;\\ 0, & \text{otherwise,}\end{cases} \tag{79}$$

the result now follows from Proposition 15. □

We note that proceeding along the same lines we easily establish the following result.

Theorem 6. *Assume $N\geqslant l$ and $|y_i|<1$. Then we have:*

$$\mathbb{E}_{O(2n)}\frac{\prod_{j=1}^{k}\det(I+x_jg)}{\prod_{i=1}^{l}\det(I-y_ig)} =$$
$$\sum_{\substack{\varepsilon\in\{\pm 1\}\\ \mathrm{sgn}(\varepsilon)=1}}\prod_{j=1}^{k}x_j^{N(1-\varepsilon_j)}\frac{\prod_{i=1}^{k}\prod_{j=1}^{l}(1+x_i^{\varepsilon_i}y_j)}{\prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})\prod_{1\leqslant i\leqslant j\leqslant l}(1-y_iy_j)}.$$

6.2. **Products and ratios for** $\mathrm{SO}(2N+1)$**.**

Lemma 6. *For $\lambda\subseteq N^k$ let $\tilde{\lambda}=(k-\lambda_N',\cdots,k-\lambda_1')$. Then we have*

$$\begin{aligned}&\prod_{i=1}^{k}\prod_{n=1}^{N}(x_i+x_i^{-1}-t_n-t_n^{-1}) =\\ &\sum_{\lambda\subseteq N^k}(-1)^{|\tilde{\lambda}|}\chi_{\lambda}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1},\cdots,x_k^{\pm 1},1)\chi_{\tilde{\lambda}}^{\mathrm{SO}(2N+1)}(t_1^{\pm 1},\cdots,t_N^{\pm 1},1).\end{aligned} \tag{80}$$

Proof The lemma is stated in slightly different form in Jimbo and Miwa [38]. It may be proved by the method of Lemma 2 and Lemma 4. □

Proposition 16. *Notation being as above we have*

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \prod_{j=1}^{k} \det(I - x_j g) =$$

$$(x_1 \dots x_k)^N \prod_{i=1}^{k} (1 - x_i) \chi_{\langle N^k \rangle}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1) =$$

$$\sum_{\varepsilon \in \{\pm 1\}} \mathrm{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{(N+1/2)(1-\varepsilon_j)} \prod_{i<j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1}. \tag{81}$$

Proof Using (80) we have:

$$(x_1 \cdots x_k)^{-N} \mathbb{E}_{\mathrm{SO}(2N+1)} \prod_{j=1}^{k} \det(I - x_j g) =$$

$$\prod_{i=1}^{k} (1 - x_i) \mathbb{E}_{\mathrm{SO}(2N+1)} \prod_{n=1}^{N} \prod_{i=1}^{k} (x_i + x_i^{-1} - t_n - t_n^{-1}) =$$

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \sum_{\lambda \subseteq N^k} (-1)^{|\tilde{\lambda}|} \chi_{\lambda}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1) \chi_{\tilde{\lambda}}^{\mathrm{SO}(2N+1)}(t_1^{\pm 1}, \cdots, t_N^{\pm 1}, 1) =$$

$$\chi_{\langle N^k \rangle}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1).$$

In the last line we used the fact that

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \chi_{\lambda}^{\mathrm{SO}(2N+1)} = \begin{cases} 1 & \text{if } \lambda = \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently we obtain

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \prod_{i=1}^{k} \det(I - x_i g) = (x_1 \dots x_k)^N \prod_{i=1}^{k} (1 - x_i) \chi_{\langle N^k \rangle}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1), \tag{82}$$

proving the first line of (81).

Next, using the Weyl character formula for $\mathrm{SO}(2n+1)$ together with the Weyl denominator formula for $\mathrm{SO}(2n+1)$ we can rewrite the right-hand side of (82) as follows:

$$(x_1 \dots x_k)^N \prod_{i=1}^{k} (1 - x_i) \chi_{\langle N^k \rangle}^{\mathrm{SO}(2k+1)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1) = \tag{83}$$

$$\det(x_i^{N+k+\frac{1}{2}-j} - x_i^{-(N+k+\frac{1}{2}-j)})_{1 \leqslant i,j \leqslant k} \frac{(x_1 \cdots x_k)^{k+N-1/2}}{\prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)(x_i x_j - 1)}.$$

Splitting the determinant in this expression we can rewrite it as follows:

$$\sum_{\varepsilon\in\{\pm1\}^k} \det\begin{vmatrix} \varepsilon_1 x_1^{\varepsilon_1(N+k-1/2)} & \varepsilon_1 x_1^{\varepsilon_1(N+k-3/2)} & \cdots & \varepsilon_1 x_1^{\varepsilon_1(N+1/2)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_k x_k^{\varepsilon_k(N+k-1/2)} & \varepsilon_k x_k^{\varepsilon_k(N+k-3/2)} & \cdots & \varepsilon_k x_k^{\varepsilon_k(N+1/2)} \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k} \prod_{i=1}^{k} \varepsilon_i x_i^{\varepsilon_i(N+1/2)} \det\begin{vmatrix} x_1^{\varepsilon_1(k-1)} & x_1^{\varepsilon_1(k-2)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{\varepsilon_k(k-1)} & x_k^{\varepsilon_k(k-2)} & \cdots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k} \prod_{i=1}^{k} \varepsilon_i x_i^{\varepsilon_i(N+1/2)} \prod_{i<j}(x_i^{\varepsilon_i} - x_j^{\varepsilon_j}),$$

where in the last line we have used the Vandermonde determinant evaluation.

Now making use of the elementary identities (55) and (56) the right hand side of (83) is easily seen to be equal to

$$\sum_{\varepsilon\in\{\pm1\}^k} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{(N+1/2)(1+\varepsilon_j)} \prod_{i<j}(1 - x_i^{-\varepsilon_i} x_j^{-\varepsilon_j})^{-1} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{(N+1/2)(1-\varepsilon_j)} \prod_{i<j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1},$$

completing the proof. $\square$

We remark that proceeding along similar lines we easily establish the following result.

Proposition 17. *Notation being as above we have*

$$\mathbb{E}_{g\in O^-(2N)} \prod_{j=1}^{k} \det(I + x_j g) = (x_1\cdots x_k)^N \chi_{(N^k)_-}^{\mathrm{SO}(2k)}(x_1^{\pm1}, \cdots, x_k^{\pm1}) =$$

$$\sum_{\varepsilon\in\{\pm1\}} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \prod_{i<j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1}.$$

We now turn to ratios.

Proposition 18. *Notation being as above we have*

$$\sum_{\lambda\subseteq N^k} (-1)^{|\tilde\lambda|} \chi_\lambda^{\mathrm{SO}(2k+1)}(x_1^{\pm1}, \cdots, x_k^{\pm1}, 1) s_{\tilde\lambda}(y_1, \cdots, y_l) = \tag{84}$$

$$\frac{(x_1\cdots x_k)^{1/2}}{\prod_{i=1}^{k}(1-x_i)} \sum_{\varepsilon\in\{\pm1\}^k} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{-(N+1/2)\varepsilon_j} \frac{\prod_{i=1}^{k}\prod_{j=1}^{l}(1 + x_i^{\varepsilon_i} y_j)}{\prod_{i<j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})}.$$

Proof In order to yield a non-zero contribution to the sum on the left-hand side of (84), λ must be of the form $\lambda = (N-l)^k+\mu$ with $\mu \subseteq l^k$. Now keeping in mind the Weyl character formula (63) for $\mathrm{SO}(2n+1)$ and the numerator evaluation (64), together with the definition (5) of the Schur polynomial, and the Vandermonde determinant evaluation of the denominator in that formula, we can rewrite the expression on the left-hand side of (84) as follows, using the Laplace expansion:

$$\det\begin{vmatrix} X_{1,k+l} & X_{1,k+l-1} & \cdots & X_{11} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k,k+l} & X_{k,k+l-1} & \cdots & X_{k1} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix} \times$$

$$\prod_{1\leqslant i<j\leqslant l}\frac{1}{y_j-y_i}\prod_{1\leqslant i<j\leqslant k}\frac{(x_1\cdots x_k)^{k-1/2}}{(x_i-x_j)(x_ix_j-1)}\prod_{i=1}^{k}\frac{1}{1-x_i}, \tag{85}$$

where $X_{ij} = x_i^{N-l-\frac{1}{2}+j} - x_i^{-(N-l-\frac{1}{2}+j)}$. Splitting the determinant in this expression we can rewrite it as follows:

$$\sum_{\varepsilon\in\{\pm1\}^k}\det\begin{vmatrix} \varepsilon_1 x_1^{\varepsilon_1(N+k-1/2)} & \varepsilon_1 x_1^{\varepsilon_1(N+k-3/2)} & \cdots & \varepsilon_1 x_1^{\varepsilon_1(N-l+1/2)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_k x_k^{\varepsilon_k(N+k-1/2)} & \varepsilon_k x_k^{\varepsilon_k(N+k-3/2)} & \cdots & \varepsilon_k x_k^{\varepsilon_k(N-l+1/2)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix} =$$

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^{k}\varepsilon_i x_i^{\varepsilon_i(N-l+1/2)}\det\begin{vmatrix} 1 & x_1^{\varepsilon_1} & \cdots & x_1^{\varepsilon_1(l+k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k^{\varepsilon_k} & \cdots & x_k^{\varepsilon_k(l+k-1)} \\ (-y_1)^{l+k-1} & (-y_1)^{l+k-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (-y_l)^{l+k-1} & (-y_l)^{l+k-2} & \cdots & 1 \end{vmatrix}$$

or, using the Vandermonde determinant evaluation,

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{i=1}^{k}\varepsilon_i x_i^{\varepsilon_i(N-l+1/2)}\prod_{i<j}(x_i^{\varepsilon_i}-x_j^{\varepsilon_j})\prod_{1\leqslant i<j\leqslant l}(y_j-y_i)\prod_{i=1}^{k}\prod_{j=1}^{l}(x_i^{\varepsilon_i}+y_j),$$

Now making use of the elementary identities (55), (56), the expression on the right-hand side of (85) is easily brought to the form expressed on the right-hand of (84). $\square$

Theorem 7. *Assume $N \geqslant l$ and $|y_i| < 1$. Then we have:*

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \frac{\prod_{j=1}^{k} \det(I - x_j g)}{\prod_{i=1}^{l} \det(I - y_i g)} =$$

$$\sum_{\varepsilon \in \{\pm 1\}} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{(N+1/2)(1-\varepsilon_j)} \frac{\prod_{i=1}^{k} \prod_{j=1}^{l} (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i<j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i \leqslant j \leqslant l} (1 - y_i y_j)}.$$

Proof The Cauchy identity for $\mathrm{SO}(2N+1)$ has the following form:

$$\prod_{n=1}^{N} \prod_{j=1}^{l} \frac{1}{(1 - y_j t_n)(1 - y_j t_n^{-1})} \prod_{j=1}^{l} \frac{1}{1 - y_j} =$$

$$\left(\prod_{i \leqslant j} \frac{1}{1 - y_i y_j} \right) \sum_{\mu} \chi_{\mu}^{\mathrm{SO}(2N+1)} (t_1^{\pm 1}, \cdots, t_N^{\pm 1}, 1) s_{\mu}(y_1, \cdots, y_l). \tag{86}$$

Consequently, using (86) and (80) we have

$$\frac{\prod_{j=1}^{k} \det(I - x_j g)}{\prod_{i=1}^{l} \det(I - y_i g)} = \frac{(x_1 \cdots x_k)^N \prod_{i=1}^{k} (1 - x_i)}{\prod_{i \leqslant j} (1 - y_i y_j)} \times$$

$$\sum_{\lambda \subseteq N^k} (-1)^{|\tilde{\lambda}|} \chi_{\lambda}^{\mathrm{SO}(2k+1)} (x_1^{\pm 1}, \cdots, x_k^{\pm 1}, 1) \chi_{\tilde{\lambda}}^{\mathrm{SO}(2N+1)} (t_1^{\pm 1}, \cdots, t_N^{\pm 1}, 1) \times$$

$$\sum_{\mu} \chi_{\mu}^{\mathrm{SO}(2N+1)} (t_1^{\pm 1}, \cdots, t_N^{\pm 1}, 1) s_{\mu}(y_1, \cdots, y_l).$$

Since

$$\mathbb{E}_{\mathrm{SO}(2N+1)} \chi_{\lambda}^{\mathrm{SO}(2N+1)}(g) \chi_{\mu}^{\mathrm{SO}(2N+1)}(g) = \begin{cases} 1 & \text{if } \lambda = \mu, l(\lambda) \leqslant N; \\ 0 & \text{otherwise,} \end{cases}$$

The theorem now follows from Proposition 18. □

We note that proceeding along similar lines we easily establish the following result.

Theorem 8. *Suppose $N \geqslant l$ and $|y_i| < 1$. Then we have:*

$$\mathbb{E}_{g \in O^{-}(2N)} \frac{\prod_{j=1}^{k} \det(I + x_j g)}{\prod_{i=1}^{l} \det(I - y_i g)} =$$

$$\sum_{\varepsilon \in \{\pm 1\}} \operatorname{sgn}(\varepsilon) \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \frac{\prod_{i=1}^{k} \prod_{j=1}^{l} (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i<j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i \leqslant j \leqslant l} (1 - y_i y_j)}.$$

7. Classical group characters of rectangular shape

In this section using results of Sections 5 and 6 we give simple proofs of identities of Okada [48] and Krattenthaler [42] and derive generalizations of their results for Littlewood-Schur functions.

7.1. **Symplectic Group.** The branching rule (42) implies

$$\mathbb{E}_{\mathrm{Sp}(2N)} s_\lambda(g) = \begin{cases} 1 & \text{if } \lambda' \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \tag{87}$$

The importance of (87) in random matrix theory context was emphasized by Rains [49] and Baik and Rains [3].

Using (87) we obtain:

$$\mathbb{E}_{Sp(2N)} \prod_{j=1}^{k} \det(I + x_j g) = \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda \text{ even}}} s_\lambda(x_1, \dots x_k), \tag{88}$$

an identity first noted by Baik and Rains [3].

Combining this with Proposition 11, we have

$$\sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda \text{ even}}} s_\lambda(x_1, \dots x_k) = (x_1 \dots x_k)^N \chi^{\mathrm{Sp}_{2k}}_{\langle N^k \rangle}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) =$$

$$\sum_{\varepsilon \in \{\pm 1\}} \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \prod_{i \leqslant j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1}, \tag{89}$$

an identity first derived by Okada [48] and Krattenthaler [42].

Using Theorem 4 we now prove the following generalization of (89):

Proposition 19. *Assume $N \geqslant l$. Then we have:*

$$\sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda \text{ even}}} \mathrm{LS}_\nu(x_1, \dots, x_k; y_1, \dots, y_l) =$$

$$\sum_{\varepsilon \in \{\pm 1\}^k} \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \frac{\prod_{i=1}^{k} \prod_{j=1}^{l} (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i \leqslant j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i < j \leqslant l} (1 - y_i y_j)}.$$

Proof By the dual Cauchy identity

$$\prod_{j=1}^{k} \det(I + x_j g) = \sum_\eta s_\eta(x_1, \dots, x_k) s_{\eta'}(t_1^{\pm 1}, \dots, t_N^{\pm 1}).$$

On the other hand, by the Cauchy identity

$$\frac{1}{\prod_{i=1}^{l} \det(I - y_i g)} = \sum_\mu s_\mu(y_1, \dots, y_l) s_\mu(t_1^{\pm 1}, \dots, t_N^{\pm 1}).$$

Consequently, using (87), we have

(90)

$$\mathbb{E}_{M\in Sp(2N)}\frac{\prod_{j=1}^{k}\det(I+x_jM)}{\prod_{i=1}^{l}\det(I-y_iM)}=\sum_{\substack{l(\nu)\leq 2N\\ \nu'\text{ even}}}c^{\nu}_{\eta'\mu}s_\eta(x_1,\ldots,x_k)s_\mu(y_1,\ldots,y_l)=$$

$$\sum_{\substack{l(\nu)\leqslant 2N\\ \nu'\text{ even}}}\mathrm{LS}_\nu(y_1,\ldots,y_l;x_1,\ldots,x_k)=\sum_{\substack{\nu_1\leqslant 2N\\ \nu\text{even}}}\mathrm{LS}_\nu(x_1,\ldots,x_k;y_1,\ldots,y_l),$$

an identity first noted by Baik and Rains [3]. Combining (90) with Theorem 4 completes the proof. □

We remark that Krattenthaler [42] (3.6) proved the following generalization of (89):

$$\sum_{\substack{\lambda\subseteq(2N)^k\\ r(\lambda)=r}}s_\lambda(x_1,\ldots,x_k)=(x_1\cdots x_k)^N\chi^{\mathrm{Sp}(2k)}_{N^{k-r}\cup(N-1)^r}(x_1^{\pm1},\cdots,x_k^{\pm1}) \tag{91}$$

where $r(\lambda)$ denotes the number of odd rows of λ.

We now show how (91) can be used to give an alternative proof of Theorem 4. In (90) write $\nu=\lambda\cup\tau$ with $\lambda\in(2N)^k$. By the Berele-Regev factorization (36) we have

$$\mathrm{LS}_\nu(x_1,\ldots,x_k;y_1,\ldots,y_l)=\mathrm{LS}_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_l)s_{\tau'}(y_1,\ldots,y_l).$$

Consequently,

$$\sum_{\substack{\nu_1\leqslant 2N\\ \nu\text{ even}}}\mathrm{LS}_\nu(x_1,\ldots,x_k;y_1,\ldots,y_l)=\sum_{\tau'\text{even}}s_\tau(y_1,\ldots,y_l)\times$$

$$\sum_{\substack{\lambda\subseteq(2N)^k\\ \lambda\text{ even}}}\mathrm{LS}_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_l).$$

Since

$$\sum_{\tau'\text{ even}}s_\tau(y_1,\ldots,y_l)=\frac{1}{\prod_{1\leqslant i<j\leqslant l}(1-y_iy_j)}, \tag{92}$$

Theorem 4 follows from the following Proposition (applied with $u=0$).

Proposition 20. *Let $r(\lambda)$ denote the number of odd rows of λ. Then*

$$\sum_{\lambda\subseteq(2N)^k}u^{r(\lambda)}\,\mathrm{LS}_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_l)\quad=$$

$$\sum_{\varepsilon\in\{\pm1\}^k}\prod_{j=1}^{k}x_j^{N(1-\varepsilon_j)}\frac{\prod_{i=1}^{k}\prod_{j=1}^{l}(1+x_i^{\varepsilon_i}y_j)\prod_{i=1}^{k}(1+x_i^{\varepsilon_i}u)}{\prod_{i\leqslant j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}$$

Proposition 20 in turn follows from the following lemma by induction on the number of variables y using the generalized Pieri formula given in Proposition 7.

Lemma 7. *Let $r(\lambda)$ denote the number of odd rows of λ. Then*

$$\sum_{\lambda\subseteq(2N)^k} u^{r(\lambda)}s_\lambda(x_1,\ldots,x_k) = \sum_{\varepsilon\in\{\pm1\}^k}\prod_{j=1}^k x_j^{N(1-\varepsilon_j)}\frac{\prod_{i=1}^k(1+x_i^{\varepsilon_i}u)}{\prod_{i\leqslant j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}. \tag{93}$$

We remark that (93) generalizes the classical formula due to Littlewood [44]

$$\sum_\lambda u^{r(\lambda)}s_\lambda(x_1,\ldots,x_k) = \prod_{i=1}^k\frac{1+ux_i}{1-x_i^2}\prod_{i<j}\frac{1}{1-x_ix_j}. \tag{94}$$

We now give a proof of Lemma 7 using Krattenthaler's formula (91).

Let

$$\varphi_{k,N}(r) = (x_1\ldots x_k)^N\chi^{\mathrm{Sp}_{2k}}_{N^{k-r}\cup(N-1)^r}(x_1^{\pm1},\ldots,x_k^{\pm1}).$$

We have

$$\sum_{\lambda\subseteq(2N)^k} u^{r(\lambda)}s_\lambda(x_1,\ldots,x_k) = \sum_{r=0}^k u^r\varphi_{k,N}(r). \tag{95}$$

Let

$$a_{ij}^{(r)} = \begin{cases} x_j^{i-1}-x_j^{2N+2k+1-i} & \text{if } i\leqslant k-r,\\ x_j^i - x_j^{2N+2k-i} & \text{if } i>k-r.\end{cases}$$

In light of the Weyl character formula (43) applied to partition $\lambda = N^{k-r}\cup(N-1)^r$, combined with (44) we have

$$\varphi_{k,N}(r) = \frac{\det(a_{ij}^{(r)})}{\prod_{i=1}^k(1-x_i^2)\prod_{i<j}(x_i-x_j)(x_ix_j-1)}. \tag{96}$$

Now using (89) we have

$$\varphi_{k,N}(0) = \frac{\det(a_{ij}^{(0)})}{\prod_{i=1}^k(1-x_i^2)\prod_{i<j}(x_i-x_j)(x_ix_j-1)} = \tag{97}$$

$$\frac{\det(x_i^{j-1}-x_i^{2N+2k+1-j})}{\prod_{i=1}^k(1-x_i^2)\prod_{i<j}(x_i-x_j)(x_ix_j-1)} = \sum_{\varepsilon\in\{\pm1\}}\frac{\prod_{j=1}^k x_j^{N(1-\varepsilon_j)}}{\prod_{i\leqslant j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})}$$

Next, by the definition of the determinant,

$$\det(a_{ij}^{(0)}) = \det(x_i^{j-1}-x_i^{2N+2k+1-j}) =$$

$$\sum_{w\in S_k}\mathrm{sgn}(w)\sum_\varepsilon\prod_{\varepsilon_{w_j}=1}x_{w_j}^{j-1}\prod_{\varepsilon_{w_j}=-1}(-x_{w_j}^{2N+2k-j+1}), \tag{98}$$

and
(99)

$$\det(a_{ij}^{(r)}) = \sum_{w\in S_k} \mathrm{sgn}(w) \sum_{\varepsilon} \prod_{j=1}^{k-r} \left(\prod_{\varepsilon_{w_j}=1} x_{w_j}^{j-1} \prod_{\varepsilon_{w_j}=-1} (-x_{w_j}^{2N+2k-j+1}) \right) \times$$

$$\prod_{j=k-r+1}^{k} \left(\prod_{\varepsilon_{w_j}=1} x_{w_j}^{j} \prod_{\varepsilon_{w_j}=-1} (-x_{w_j}^{2N+2k-j}) \right) =$$

$$\sum_{w\in S_k} \mathrm{sgn}(w) \sum_{\varepsilon} \prod_{j=1}^{k-r} \left(\prod_{\varepsilon_{w_j}=1} x_{w_j}^{j-1} \prod_{\varepsilon_{w_j}=-1} (-x_{w_j}^{2N+2k-j+1}) \right) \times$$

$$\prod_{j=k-r+1}^{k} \left(\prod_{\varepsilon_{w_j}=1} x_{w_j}^{j-1} \prod_{\varepsilon_{w_j}=-1} (-x_{w_j}^{2N+2k-j+1}) \right) \prod_{j=k-r+1}^{k} x_{w(j)}^{\varepsilon_{w(j)}} =$$

$$\sum_{\varepsilon} \left(\sum_{w\in S_k} \mathrm{sgn}(w) \prod_{\varepsilon_{w_j}=1} x_{w_j}^{j-1} \prod_{\varepsilon_{w_j}=-1} (-x_{w_j}^{2N+2k-j+1}) \right) \sum_{i_1<\cdots<i_r} x_{i_1}^{\varepsilon_{i_1}} \dots x_{i_r}^{\varepsilon_{i_r}}.$$

Consequently, using (96), (97), (98) and (99) we have

$$\varphi_{k,N}(r) = \sum_{\varepsilon\in\{\pm1\}} \frac{\prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)}}{\prod_{i\leqslant j}(1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})} e_r(x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}), \tag{100}$$

and this combined with (95) and

$$\sum_{j=0}^{k} e_j(x_1, \cdots, x_k) y^j = \prod_{i=1}^{k} (1 + x_i y)$$

completes the proof of Lemma 7, and consequently Proposition 20 and Theorem 4.

7.2. **Orthogonal group.** Branching rule (62) implies

$$\mathbb{E}_{O(2N)} s_\lambda(g) = \begin{cases} 1 & \text{if } \lambda \text{ is even, } l(\lambda) \leqslant 2N; \\ 0 & \text{otherwise.} \end{cases} \tag{101}$$

where 2ν represents the partition produced by doubling each elements of ν; note that $\lambda = 2\nu$ means that λ is even. The importance of (101) in random matrix theory context was emphasized by Rains [49] and Baik and Rains [3].

Proceeding exactly as in the case of symplectic group, we have, using (101)

$$\mathbb{E}_{O(2N)} \prod_{j=1}^{k} \det(I + x_j g) = \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ even}}} s_\lambda(x_1, \dots x_k), \tag{102}$$

an identity first noted by Baik and Rains [3]. Combining this with Proposition 14, we recover the following formula due to Okada [48] and Krattenthaler [42].

$$\sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ even}}} s_\lambda(x_1,\dots x_k) = (x_1\dots x_k)^N \chi^{\mathrm{SO}_{2k}}_{(N^k)_+}(x_1^{\pm1},\dots,x_k^{\pm1}) =$$

$$\sum_{\substack{\varepsilon\in\{\pm1\}\\ \mathrm{sgn}(\varepsilon)=1}} \prod_{j=1}^k x_j^{N(1-\varepsilon_j)} \prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})^{-1}. \tag{103}$$

Next, considering separately the subgroup $\mathrm{SO}(2N)$ and its coset $O^-(2N)$, we have

$$\mathbb{E}_{SO(2N)} f(g) = \mathbb{E}_{g\in O(2N)}(1+\det(g))f(g), \tag{104}$$

and

$$\mathbb{E}_{O^-(2N)} f(g) = \mathbb{E}_{O(2N)}(1-\det(g))f(g). \tag{105}$$

Now, denoting the eigenvalues of M by $(t_1,\dots,t_{2N})$, we have

$$(1+\det(g))\prod_{j=1}^k \det(I+x_jg) =$$

$$(1+e_{2N}(t_1,\dots,t_{2N}))\sum_\lambda s_\lambda(x_1,\dots,x_k)s_{\lambda'}(t_1,\dots,t_{2N}),$$

and, therefore using (101) and Pieri's formula (14), we obtain

$$\mathbb{E}_{SO(2N)}\prod_{j=1}^k \det(I+x_jg) =$$

$$\sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ even}}} s_\lambda(x_1,\dots x_k) + \sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ odd}}} s_\lambda(x_1,\dots x_k)$$

and

$$\mathbb{E}_{O^-(2N)}\prod_{j=1}^k \det(I+x_jg) =$$

$$\sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ even}}} s_\lambda(x_1,\dots,x_k) - \sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ odd}}} s_\lambda(x_1,\dots,x_k).$$

Using Proposition 17 and Proposition 13 we therefore recover the following formula due to Okada [48] and Krattenthaler [42]:

$$\sum_{\substack{\lambda_1\leqslant 2N\\ \lambda'\text{ odd}}} s_\lambda(x_1,\dots x_k) = \sum_{\substack{\varepsilon\in\{\pm1\}\\ \mathrm{sgn}(\varepsilon)=-1}} \prod_{j=1}^k x_j^{N(1-\varepsilon_j)} \prod_{i<j}(1-x_i^{\varepsilon_i}x_j^{\varepsilon_j})^{-1}. \tag{106}$$

Finally, proceeding exactly as in the case of symplectic group we obtain using Proposition 5 and (101):

$$\mathbb{E}_{O(2N)} \frac{\prod_{j=1}^{k} \det(I + x_j g)}{\prod_{i=1}^{l} \det(I - y_i g)} = \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ even}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l);$$

$$\mathbb{E}_{M \in SO(2N)} \frac{\prod_{j=1}^{k} \det(I + x_j M)}{\prod_{i=1}^{l} \det(I - y_i M)} = \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ even}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l) +$$
$$\sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ odd}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l);$$

and

$$\begin{aligned} \mathbb{E}_{O^-(2N)} \frac{\prod_{j=1}^{k} \det(I + x_j g)}{\prod_{i=1}^{l} \det(I - y_i g)} &= \\ \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ even}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l) & \\ - \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ odd}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l), & \end{aligned}$$

Consequently, using Theorem 6, Theorem 5, and Theorem 8, we obtain the following generalizations of Okada-Krattenthaler formulae (103) and (106):

Proposition 21. *Suppose $N \geqslant l$. Then we have:*

$$\begin{aligned} \sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \textit{ even}}} \mathrm{LS}_\lambda(x_1, \dots x_k; y_1, \dots, y_l) &= \\ \sum_{\substack{\varepsilon \in \{\pm 1\} \\ \mathrm{sgn}(\varepsilon) = 1}} \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \frac{\prod_{i=1}^{k} \prod_{j=1}^{l} (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i<j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i \leqslant j \leqslant l} (1 - y_i y_j)}, & \end{aligned}$$

and

$$\sum_{\substack{\lambda_1 \leqslant 2N \\ \lambda' \text{ odd}}} \mathrm{LS}_\lambda(x_1, \ldots x_k; y_1, \ldots, y_l) =$$

$$\sum_{\substack{\varepsilon \in \{\pm 1\} \\ \mathrm{sgn}(\varepsilon) = -1}} \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \frac{\prod_{i=1}^{k} \prod_{j=1}^{l} (1 + x_i^{\varepsilon_i} y_j)}{\prod_{i<j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j}) \prod_{1 \leqslant i \leqslant j \leqslant l} (1 - y_i y_j)}.$$

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