Note Title 11/4/2015

$y' = ky \Rightarrow y(t) = Ce^{kt} \to \infty$ as $t \to \infty$

1. Simplex Method Review

a) Find $\det[A]$, A is 2×2, 3×3 or such

b) Find EV, Eigenvectors

Logistic Model

DE: $y' = Ky(L-y)$ carrying capacity

$$\frac{dy}{dt} = Ky(L-y)$$

$$\frac{dy}{y(L-y)} = K\,dt$$

$$\int K\,dt = Kt + c$$

$$\int \frac{dy}{y(L-y)} = \frac{1}{L}\int \frac{1}{y} + \frac{1}{L-y}\,dy$$

$$= \frac{1}{L}\left[\ln|y| - \ln|L-y|\right] + c$$

$$= \frac{1}{L}\left[\ln\left(\frac{y}{L-y}\right)\right] + c$$

$$L\left[\frac{1}{L}\ln\left(\frac{y}{L-y}\right) = Kt + c\right]$$

$$e^{\left[\ln\left(\frac{y}{L-y}\right)\right]} = e^{\left[LKt + c\right]}$$

$$\frac{y}{L-y} = e^{LKt}\left(e^{c}\right)$$

$$\left[\frac{y}{L-y} = ce^{LKt}\right] L-y$$

$$y = (L-y)ce^{LKt}$$

$$y = Lce^{LKt} - yce^{LKt}$$

$$y + yce^{LKt} = Lce^{LKt}$$

$$y(1 + ce^{LKt}) = Lce^{LKt}$$

$$y = \frac{Lce^{LKt}}{1+ce^{LKt}}$$

$$y = \frac{L}{1+ce^{-LKt}}$$

$$y(t) = \frac{L}{1 + ce^{-LKt}}$$

$$\lim_{t \to \infty} \left[\frac{L}{1 + ce^{-LKt}} \right] = L$$

$A = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$

E Values: $\det(A - \lambda I_d) = 0$ $\quad \det \begin{pmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = -\lambda(4-\lambda) + 5 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\underline{\lambda_1 = 2+i}$, $\underline{\lambda_2 = 2-i}$

E Vectors: $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I_d)\vec{v} = 0$

$\lambda_1 = 2+i$ $\quad \begin{pmatrix} -2-i & 1 \\ -5 & \underset{2-i}{4-(2+i)} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{matrix} (-2-i)v_1 + v_2 = 0 \\ -5v_1 + (2-i)v_2 = 0 \end{matrix}$

$v_2 = v_1(2+i)$

$-5v_1 + (2-i)(2+i)v_1 =$

$-5v_1 + 5v_1 = 0 = 0$

$\begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

Any multiple of $\begin{pmatrix} 1 \\ 2+i \end{pmatrix}$ is eigenvector, i.e. $(2-i)\begin{pmatrix} 1 \\ 2+i \end{pmatrix} = \begin{pmatrix} 2-i \\ 5 \end{pmatrix}$

Both right!

Solve $X' = AX$, $A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$

1. A has 2 real EValues
2. A has 2 complex EValues
3. A has 1 EV with mult. 2

$\det(A - \lambda Id) = 0$ is 2nd -degree polynomial

A has $\lambda = 6$, $v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\lambda = -1$, $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Case 1

$\boxed{X(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \text{ is general solution}}$

Ex $x(t) = c_1 \begin{pmatrix}3\\4\end{pmatrix} e^{6t} + c_2 \begin{pmatrix}-1\\1\end{pmatrix} e^{-t}$, i.e.

$x_1(t) = c_1 3 e^{6t} - c_2 e^{-t}$

$x_2(t) = c_1 4 e^{6t} + c_2 e^{-t}$

Check!

$X' = AX$

$X' = \begin{pmatrix}2 & 3\\ 4 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$. $2x_1 + 3x_2$

$x_1'(t) = c_1 18 e^{6t} + c_2 e^{-t} \overset{?}{=} 2x_1(t) + 3x_2(t) = c_1 6 e^{6t} - 2c_2 e^{-t} + c_1 12 e^{6t} + 3c_2 e^{-t}$

$x_2'(t) = c_1 24 e^{6t} - c_2 e^{-t} = 4x_1(t) + 3x_2(t)$

<u>Case 2:</u> Two Evalues are complex conjugates: $\mu + i\lambda$ and $\mu - i\lambda$

$\lambda + i\mu$, $A + Bi : \vec{v} = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix}$

$e^{it} = \cos(t) + i\sin(t)$

$$v_2 e^{(\lambda + i\mu)t} = (A + Bi)\, e^{\lambda t}\, e^{i\mu t}$$

$$= (A + Bi)\, e^{\lambda t} \left(\cos(\mu t) + i \sin(\mu t)\right) =$$

$$= \left(A \cos(\mu t) - B \sin(\mu t) + i\left(B \cos(\mu t) + A \sin(\mu t)\right)\right) e^{\lambda t}$$

$$X(t) = c_1 e^{\lambda t}\left(a \cos(\mu t) - b \sin(\mu t)\right) + c_2 e^{\lambda t}\left(b \cos(\mu t) + a \sin(\mu t)\right)$$ <u>is solution</u>

$X' = \begin{pmatrix} -1 & -6 \\ 3 & 5 \end{pmatrix} X$. Evalues $2+3i \Rightarrow \lambda = 2, \mu = 3$

Evect. $\begin{pmatrix} -1+i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a + ib$, $a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$X(t) = c_1 e^{2t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos(3t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(3t) \right) + c_2 e^{2t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(3t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(3t) \right)$$

$$x_1(t) = c_1 e^{2t} \left(-\cos(3t) - \sin(3t) \right) + c_2 e^{2t} \left(\cos(3t) - \sin(3t) \right)$$

$$x_2(t) = c_1 e^{2t} \left(\cos(3t) \right) + c_2 e^{2t} \left(\sin(3t) \right)$$

Check it!!!

$x_1'(t) = \underline{\quad} \overset{?}{=} -x_1(t) - 6x_2(t)$

$x_2'(t) = \underline{\quad} \overset{?}{=} 3x_1(t) + 5x_2(t)$

$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad x_1(0) = 1 = -c_1 + c_2 1$ $\quad \rightarrow c_1 = 1,\ c_2 = 2$

$\quad x_2(0) = 1 = c_1 \cdot 1$

$\boxed{\text{plot solution}}$