Note Title 10/28/2015

PDE's vs ODE 1-variable DE

$y' = ky$ $\qquad$ $\underline{1^{st}\text{ order}}$, $\underline{\text{linear}}$, $\underline{\text{homogeneous}}$ EQ.

In general,

$$y' = f(t)y + g(t)$$

If $g(t) = 0$ then homogeneous

Solve $y' = f(t) \cdot y$ $\qquad$ Think of f as a coefficient

$$\frac{dy}{y} = f(t)dt \quad \Big| \int$$

$$\ln|y| = \int f(t)\,dt + c$$

$$\underline{y(t)} = e^{\int f(t)dt + c} = \underline{C e^{\int f(t)dt}}$$

If $f(t) = r$ (constant)

$$= \underline{C e^{rt}}$$

<u>Systems of n 1st order linear (homogeneous) DE</u>

$$x_1' = a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + \cancel{g_1}(t)$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n + \cancel{g_2}(t)$$
$$\vdots$$
$$x_n' = a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n + \cancel{g_n}(t)$$

All a_{jk} are functions of t.

If $g_j(t) = 0$ then system is homogeneous

$$\Leftrightarrow \quad X' = A \cdot X, \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & \\ \vdots & \ddots & \\ & & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$y' = k y$$

Thm: Every n-th order linear DE has an equivalent system of n 1^{st}-order equations.

Ex: $mu'' + \gamma u' + ku = F(t)$ is 2nd-order DE (mechanical vibration eqn.)

then an equiv. system of 2 1st-order DE's

$x_1 = u =$

$x_2 = u' = x_1'$

$x_3 = u'' = x_2'$

$u'' = -\frac{\gamma}{m} u' - \frac{k}{m} u + F(t)$

$x_1' = x_2 + 0$

$x_2' = -\frac{\gamma}{m} x_2 - \frac{k}{m} x_1 + F(t)$

$X' = A \cdot X + G$, where $X' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix}$, $G(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}$

Ex $y''' - 2y'' + 3y' - 4y = 0 \Leftrightarrow X' = AX + G$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$, $A = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$

$y = x_1$ $\qquad x_1' = \qquad x_2$

$\underline{x_1}' = y' = \underline{x_2}$ $\qquad x_2' = \qquad x_3$ $\qquad - \; X' = AX$, where

$\underline{x_2'} = y'' = \underline{x_3}$

$\underline{x_3'} = y''' = 2y'' - 3y' + 4y$

$x_3' = 2x_3 - 3x_2 + 4x_1$

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix}$

Want to solve $X' = AX$ (*)

Know: solve $y' = ky \Rightarrow y = Ce^{kt}$

<u>Guess</u>: $X = C \cdot V e^{kt}$ solves (*) ? $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$X' = \cancel{C} \cdot V \cancel{e^{kt}} \cdot k = A\, \cancel{C} \cdot V \cancel{e^{kt}} \iff V \cdot k = A \cdot V$$

That means: k is an Eigenvalue of A

V is an Eigenvector of A

Solve $X' = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} \cdot X$

characteristic equ. of A

① Find EV: $\det(A - \lambda \, Id) = 0$

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 4 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 12 =$$

$$= 6 - 5\lambda + \lambda^2 - 12 =$$

$$= \lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

EV's are: $\lambda_1 = 6$

$\lambda_2 = -1$

Eigenvector for $\lambda_1 = 6$: $V \cdot 6 = A \cdot V$

$$6v_1 = 2v_1 + 3v_2 \iff 4v_1 = 3v_2$$

$$6v_2 = 4v_1 + 3v_2 \iff 4v_1 = 3v_2$$

$$V = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \lambda_1 = 6$$

$\lambda_1 = -1$. $\quad W \cdot (-1) = A \cdot W$

$$-v_1 = 2w_1 + 3w_2 \iff -3w_1 = 3w_2$$

$$-v_2 = 4w_1 + 3w_2 \qquad 4w_1 = -4w_2$$

$$W = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -1$$

Solve $X' = A \cdot X$, $A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$

General solution:

$$X(t) = C \cdot V e^{6t} + D W e^{-t}$$

HW check this out!!!

Ex: Solve $y'' - y - 2y' = 0$. Convert this to $X' = A \cdot X$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$y = x_1$

$x_1' = y' = \boxed{x_2}$

$x_2' = y'' = y + 2y' = x_1 + 2x_2$

$X' = A \cdot X$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda_1 = 1 + \sqrt{2}$, $V =$

$\lambda_2 = 1 - \sqrt{2}$; $W =$

$x(t) = A \cdot V e^{\lambda_1 t} + B W e^{\lambda_2 t}$

Solve above DE!

Check using Mathematica.