CE503 Rotation Matrices

Derivation of 2D Rotation Matrix

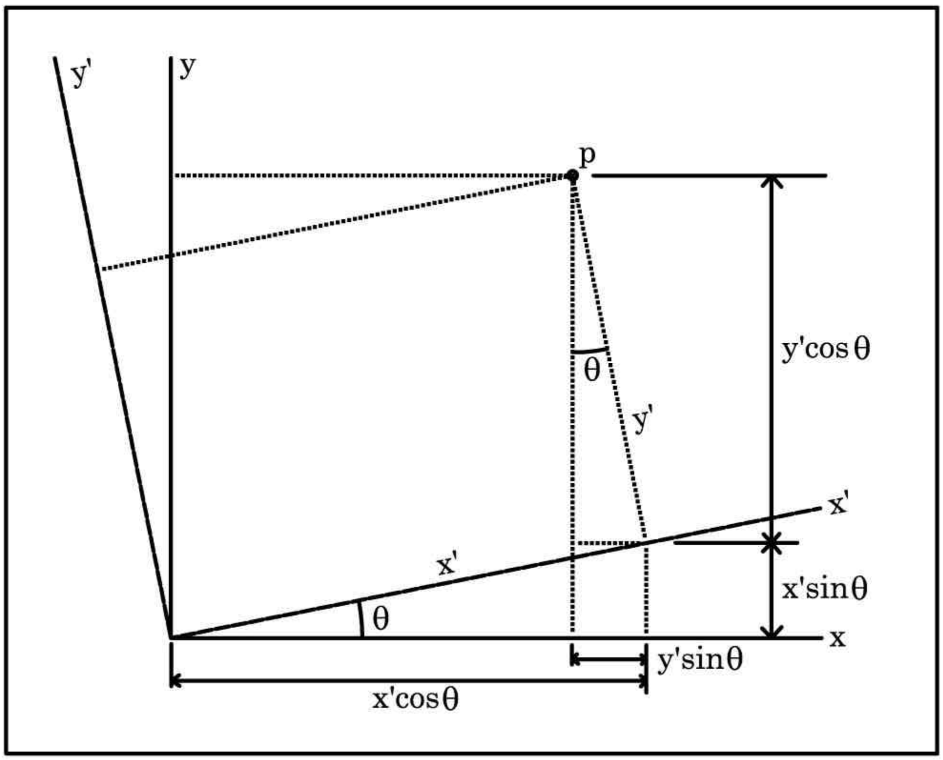

Figure 1. Coordinates of point p in two systems

Write the (x,y) coordinates in terms of the (x',y') coordinates by inspection,

$$
\begin{aligned}
x &= x'\cos q - y'\sin q \\
y &= x'\sin q + y'\cos q
\end{aligned}
$$

In matrix form,

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}
$$

Multiplying on the left by the transpose of the matrix (it is orthogonal so transpose equals inverse),

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

This represents the basic equation describing 2D rotations. Note that the sense of the angle θ is defined by the right hand rule. A positive rotation means that if the thumb of the right hand is pointed along the positive direction of the rotation axis (z), then the fingers curl in the positive direction, i.e. counterclockwise. We will adopt the convention that rotation means a rotation of the *coordinate axes*, not the *point*. If the axes are rotated counterclockwise, then the point itself appears to rotate clockwise, with respect to fixed axes. See figures below.

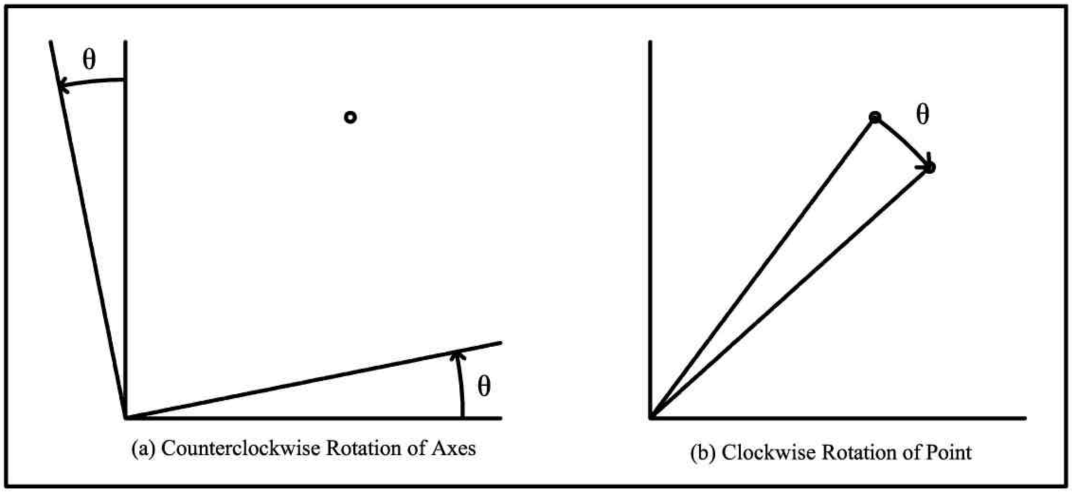

Figure 2. Equivalence of rotating axes in one direction, and a point in the opposite direction

Derivation of 3D Elementary Rotation Matrices

We can extend the prior development into 3D rotations by constructing elementary 3D rotation matrices. The elementary 3D rotation matrices are constructed to perform rotations individually about the three coordinate axes. We begin with the rotation about the z-axis (photogrammetrists call it, κ, or kappa), since it is virtually identical to what was just developed. We keep the same xy transformation but add an identity transformation for the z-coordinate, since it will not change during a rotation about the z-axis. See figure 3.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M_k \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

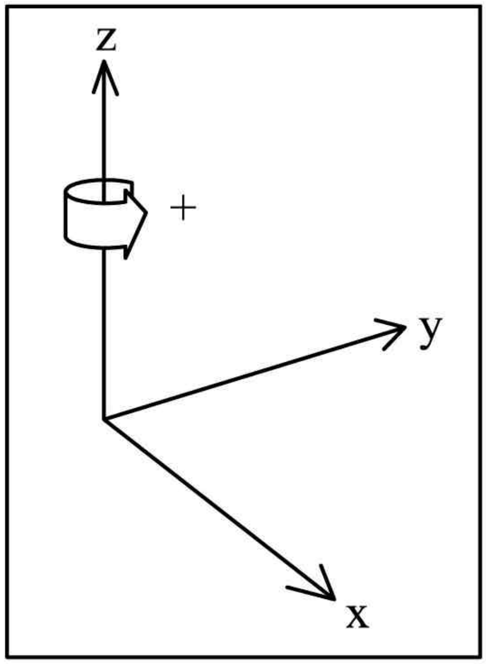

Figure 3. Rotation about the z-axis

Next let us consider a rotation about the x-axis. Photogrammetrists call this rotation ω, or omega. See the drawing in figure 4. We can relate this back to our prior derivation by letting the y-axis *play the role of x,* and letting the z-axis *play the role of y*. If we do that then we can write the 3D elementary rotation matrix directly by inspection, albeit with a coordinate component order that is not conventional. Then we can rearrange the order and thereby obtain the conventional elementary matrices.

The equation, written by inspection,

$$\begin{bmatrix} y' \\ z' \\ x' \end{bmatrix} = \begin{bmatrix} \cos w & \sin w & 0 \\ -\sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

For the vector on the right we want to move the first two elements down, and the third element we want to move to the first position. That corresponds to moving the first two columns of the matrix to the right, and moving the third column to the first column position.

$$\begin{bmatrix} y' \\ z' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & \cos w & \sin w \\ 0 & -\sin w & \cos w \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

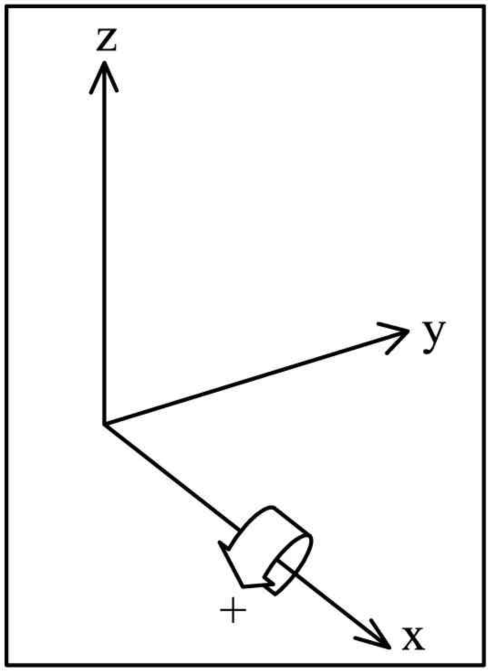

Figure 4. Rotation about the x-axis

For the vector on the left we want to move the two top elements down, and we want to move the third element up to the top. This corresponds to moving the corresponding matrix rows in the same way. This completes the elementary rotation about x.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos w & \sin w \\ 0 & -\sin w & \cos w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M_w \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Figure 5 shows a rotation about the y-axis. In order to be able to write the rotation matrix directly, imagine that the the z-axis is *playing the role of* the x-axis, and the x-axis is *playing the role of* the y-axis. With that coordinate order, we write the matrix directly, in terms of the angle, φ (phi),

$$\begin{bmatrix} z' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos j & \sin j & 0 \\ -\sin j & \cos j & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

In order to rearrange the order of the vector on the right, we must slide the last two matrix columns left, and move the leftmost column over to the right.

$$\begin{bmatrix} z' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} \sin j & 0 & \cos j \\ \cos j & 0 & -\sin j \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

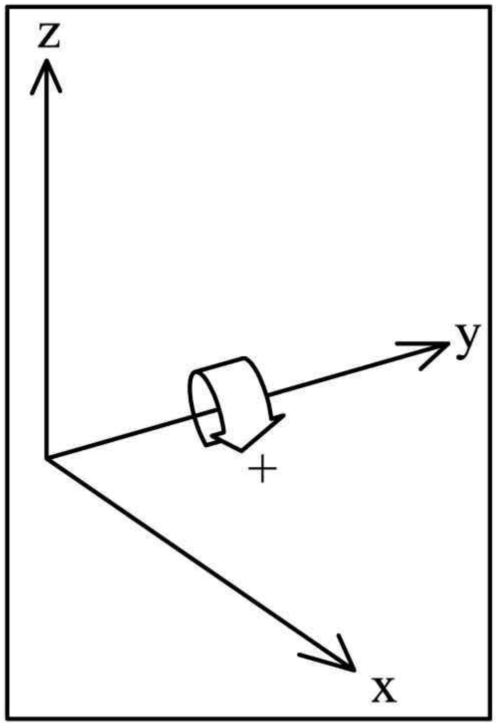

Figure 5. Rotation about the y-axis

In order to put the elements of the vector on the left into the conventional xyz order, we must slide the bottom two matrix rows up, and move the top row down to the bottom.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos j & 0 & -\sin j \\ 0 & 1 & 0 \\ \sin j & 0 & \cos j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M_j \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This completes the elementary rotation about y. These elementary matrices can be combined to create any 3D rotation. In photogrammetry the usual order of the rotations is omega (x) first, then phi (y), and lastly kappa (z). A matrix applied first is on the right, therefore the general composite rotation is,

$$M = M_k M_j M_w$$

Writing out all of the elements of the composite rotation we get,

$$M = \begin{bmatrix} \cos j \cos k & \cos w \sin k + \sin w \sin j \cos k & \sin w \sin k - \cos w \sin j \cos k \\ -\cos j \sin k & \cos w \cos k - \sin w \sin j \sin k & \sin w \cos k + \cos w \sin j \sin k \\ \sin j & -\sin w \cos j & \cos w \cos j \end{bmatrix}$$