PATTERNS IN DIFFERENCES BETWEEN ROWS IN k-ZECKENDORF ARRAYS

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ABSTRACT. For a fixed integer $k \geq 2$, we study the k-Zeckendorf array, $\mathcal{X}_k$, based upon the k-th order recurrence $u_n = u_{n-1} + u_{n-k}$. We prove that the pattern of differences between successive rows is a k-letter infinite word generalizing the infinite Fibonacci.

1. INTRODUCTION AND BACKGROUND

Definition 1. The *k-Zeckendorf array* [3], $\mathcal{X}_k = \{x_{r,c} \mid r, c \geq 0\}$, is a doubly subscripted array of positive integers. The first row begins with $x_{0,c} = c + 1$ for $0 \leq c < k$. For $i \geq k$,

$$x_{0,i} = x_{0,i-1} + x_{0,i-k}. \tag{1}$$

Subsequent rows are specified inductively as follows. For $r > 0$, $x_{r,0}$ is the smallest integer not in previous rows. Let the k-Zeckendorf representation (see Definition 2 below) of $x_{r,0}$ be $\sum_{i=0}^{m} d_i x_{0,i}$. Then for $c > 0$,

$$x_{r,c} = \sum_{i=0}^{m} d_i x_{0,i+c}. \tag{2}$$

Definition 2. The *k-Zeckendorf representation of n* is $\sum_{i=0}^{m} d_i x_{0,i}$, where for all i, $d_i \in \{0, 1\}$ and every sequence $\{d_i, d_{i+1}, \ldots, d_{i+k-1}\}$ contains at most one 1. The upper limit m in the sum is the largest integer such that $x_{0,m} \leq n$.

The well-known Zeckendorf theorem is for $k = 2$, and the sequence $\{x_{0,c}\}$ is the Fibonacci sequence $\{F_{c+2}\}$ (see [5, 8]). This generalizes easily to the k-Zeckendorf representation, which is also unique.

Definition 3. If the k-Zeckendorf representation of n is $\sum_{i=0}^{m} d_i x_{0,i}$, *the k-shift of n* is $S(n) = \sum_{i=0}^{m} d_i x_{0,i+1}$.

Taking $n = x_{r,0}$, we can write equation (2) as

$$x_{r,c} = S^c(x_{r,0}). \tag{3}$$

The arrays $\mathcal{X}_k$ have several well-known properties [2, 3, 4]:

(1) Every row of $\mathcal{X}_k$ satisfies the recurrence $x_{r,c} = x_{r,c-1} + x_{r,c-k}$.
(2) $\mathcal{X}_k$ contains every positive integer exactly once.
(3) $\mathcal{X}_k$ is an interspersion [3]. If $x_{r,c} < x_{r',c'} < x_{r,c+1}$, then $x_{r,c+1} < x_{r',c'+1} < x_{r,c+2}$.

Portions of $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$ are displayed below. $\mathcal{X}_2$ is also known as the Wythoff array [3] and is given as OEIS # A035513 in [7]. We use *precursion*:[1] $x_{r,n-k} = x_{r,n} - x_{r,n-1}$, to prepend k columns to each $\mathcal{X}_k$. We will establish later in Theorem 2 that $x_{r,-k} = r$ for $r \geq 0$ in all $\mathcal{X}_k$, as shown by column $c = -k$ in these tables.

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[1]Clark Kimberling suggested the terminology at the 2010 Fibonacci Association conference in Mexico.

$\mathcal{X}_2$	c : -2	-1	0	1	2	3	4	5	6	7
r : 0	0	1	1	2	3	5	8	13	21	34
1	1	3	4	7	11	18	29	47	76	123
2	2	4	6	10	16	26	42	68	110	178
3	3	6	9	15	24	39	63	102	165	267
4	4	8	12	20	32	52	84	136	220	356
5	5	9	14	23	37	60	97	157	254	411
6	6	11	17	28	45	73	118	191	309	500

$\mathcal{X}_3$	c : -3	-2	-1	0	1	2	3	4	5	6	7
r : 0	0	1	1	1	2	3	4	6	9	13	19
1	1	3	4	5	8	12	17	25	37	54	79
2	2	4	5	7	11	16	23	34	50	73	107
3	3	5	7	10	15	22	32	47	69	101	148
4	4	7	10	14	21	31	45	66	97	142	208
5	5	9	13	18	27	40	58	85	125	183	268
6	6	10	14	20	30	44	64	94	138	202	296

$\mathcal{X}_4$	c : -4	-3	-2	-1	0	1	2	3	4	5	6	7
r : 0	0	1	1	1	1	2	3	4	5	7	10	14
1	1	3	4	5	6	9	13	18	24	33	46	64
2	2	4	5	6	8	12	17	23	31	43	60	83
3	3	5	6	8	11	16	22	30	41	57	79	109
4	4	6	8	11	15	21	29	40	55	76	105	145
5	5	8	11	15	20	28	39	54	74	102	141	195
6	6	10	14	19	25	35	49	68	93	128	177	245

Sequences in $\mathcal{X}_k$ for rows 0 and columns 0 that are recorded in [7] are referenced here.

	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
Row 0	A000045	A000930	A003269
Column 0	A003622	A020942	

2. Preliminaries

We focus our attention now on column zero of $\mathcal{X}_k$, $\{x_{r,0} \mid r \geq 0\}$. The elements of column zero of $\mathcal{X}_k$ are those numbers whose k-Zeckendorf representation ends with the least significant portion (abbreviated LSP, with its complement MSP as the most significant portion) given by

$$(d_{k-1}, d_{k-2}, \ldots, d_1, d_0) = (0, 0, \ldots, 0, 1). \tag{4}$$

In this notation we mimic the usual binary number representation with bit strings. The rightmost bit is the coefficient of $x_{0,0}$.

Lemma 1. $x_{r,0} = S^k(r) + 1 = S \cdots S(r) + 1$.

Proof. This follows from the definition of $\mathcal{X}_k$ and equation (4). $\square$

The shift function preserves order, so $x_{r,0} < x_{r+1,0}$ for all $r \geq 0$.

In Theorem 1 below, we examine the pattern of the differences $x_{r+1,0} - x_{r,0}$ and then the differences between successive rows of $\mathcal{X}_k$. This pattern is captured in the sequence of words $\mathcal{W}_k$, which for $k = 2$ is the simplest Sturmian word (the Fibonacci word, see [1]), with its higher order generalizations.

Definition 4. $\mathcal{W}_k = \{w_i\}_{i\geq 0}$ is an infinite sequence of words over the k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. w_i is a word of length $x_{0,i}$, as follows:

$$
\begin{aligned}
w_0 &= a_0, \\
w_1 &= a_0 a_1, \\
w_2 &= a_0 a_1 a_2, \\
&\vdots \quad\quad \vdots \\
w_{k-1} &= a_0 a_1 a_2 \cdots a_{k-1}, \\
w_i &= w_{i-1} w_{i-k} \quad \text{for } i \geq k.
\end{aligned}
$$

We define another infinite sequence of words $\mathcal{W}'_k$, which is related to $\mathcal{W}_k$. Each derivation has useful word properties, which will be highlighted in the table in Lemma 2.

Definition 5. $\mathcal{W}'_k = \{w'_i\}_{i\geq 1}$ is an infinite sequence of words over the same k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$, determined by the iterative algorithm:

$$
w'_i = \begin{cases} a_i & \text{for} \quad 1 \leq i < k, \\ a_0 & \text{at} \quad i = k, \end{cases}
$$

$$
w'_i = w'_{i-1} \, w'_{i-k} \quad \text{for } i > k.
$$

w'_i is a word of length $x_{0,i-k}$, where the word length is one for $0 < i < k$ according to precursion in row 0 of $\mathcal{X}_k$. The shifted relationships between w_i and w'_i and between their respective lengths $|w_i|$ and $|w'_i|$ are given by

$$
w_i = w'_{i+k} \quad \text{and} \quad |w_i| = |w'_{i+k}| = x_{0,i} \quad \text{for } i \geq 0.
$$

The Fibonacci case ($k = 2$) has $|w_i| = x_{0,i} = F_{i+2}$ for $i \geq 0$, and $|w'_i| = x_{0,i-2} = F_i$ for $i \geq 1$.

For $i \geq k$, w_{i-1} is a prefix of w_i and $w_{i-k} = w'_i$ is a suffix of w_i, yielding the infinite word

$$
w = \lim_{i \to \infty} w_i = \lim_{i \to \infty} w'_i = \{w(n)\}_{n\geq 0} = w(0)w(1)w(2)\cdots, \tag{5}
$$

where $w(n) \in \Sigma$. At $k = 2$, the infinite Fibonacci word w begins $1011010110110\ldots$ with $a_0 = 1, a_1 = 0$.

Lemma 2. *The least significant portion LSP of $k - 1$ coefficients of the k-Zeckendorf representation of the sequence of nonnegative integers follows the pattern of $\mathcal{W}_k$.*

Proof. We express the k-Zeckendorf representation using at least $k-1$ bits. The LSP of length $k-1$ may take on k different patterns (composed of either all zeros or zeros with a single one). From these k patterns, we create our k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. The k-Zeckendorf representation of 0 is 0, and we assign a_0 to instances when the LSP is 0^{k-1} ($k-1$ 0's). For $1 \leq i \leq k-1$, we have by Definition 1 that $i = x_{0,i-1}$. The k-Zeckendorf representation of i consists of a single term $x_{0,i-1}$ with LSP of $0^{k-1-i}10^{i-1}$, to which we assign the letter a_i.

The following table for $k = 3$ illustrates the principle of this proof. In this example, we use $\{c, a, b\}$ for $\{a_0, a_1, a_2\}$.

A table of k-Zeckendorf representations (k = 3).

n	MSP of k-Zeck.	LSP of k-Zeck.	letters	i
0		0 0	c	0
1		0 1	a	1
2		1 0	b	2
3	1	0 0	c	0
4	1 0	0 0	c	0
5	1 0	0 1	a	1
6	1 0 0	0 0	c	0
7	1 0 0	0 1	a	1
8	1 0 0	1 0	b	2
9	1 0 0 0	0 0	c	0
10	1 0 0 0	0 1	a	1
11	1 0 0 0	1 0	b	2
12	1 0 0 1	0 0	c	0

The horizontal lines in the above table are placed over the values n when $n = x_{0,c}$, the entries of $\mathcal{X}_3$ in row 0 at $c \geq 0$ beginning with $1, 2, 3, 4, 6, 9, 13$. The sequence of words in the letters column from the top of the table down to each horizontal line is the following:

$$c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc. \ldots.$$

Thus from Definition 4, each successive word sequence w_i from $i \geq 0$ gives the letter sequence from the table top to the next horizontal line, and each word length is $|w_i| = x_{0,i}$ as required.

The sequence of words in the letters column between consecutive horizontal lines in the table are the words w_i' for $i \geq 1$ listed in sequence as

$$a, b, c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc, \ldots.$$

At $k = 3$, row 0 of $\mathcal{X}_3$ for $x_{0,c-k}$ at $c \geq 1$ beginning with $1, 1, 1, 2, 3, 4, 6, 9, 13$ gives $|w_c'|$ as the number of terms between the horizontal lines in the table.

To form such a table, we initialize the first k lines ($k = 3$ in the illustration) straightforwardly: for $0 \leq i < k$, $x_{0,i} = i + 1$, so the k-Zeckendorf representation of i corresponds to the letter a_i. Then proceed inductively. Having constructed the first $x_{0,n-1}$ lines, the next $x_{0,n-k}$ lines are built as follows. If $x_{0,n-1} \leq m < x_{0,n}$, the k-Zeckendorf representation of m is the k-Zeckendorf representation of $m - x_{0,n-k}$ augmented by $x_{0,n-1}$. As bit strings, take the initial $x_{0,n-k}$ lines of the table, pad their k-Zeckendorf representations with zeros on the left to make them all bit strings of length $n - k$. Then prepend each of these strings with 10^{k-1}, giving the new $x_{0,n-k}$ lines, extending the table to $x_{0,n}$ lines.

The list of LSPs follows the same pattern used to construct $\mathcal{W}_k$. $\qquad\square$

In the above, in the words in $\{a, b, c\}$, a is always followed by b or c, b is always followed by c, and c is always followed by a or c. For general k, $\Sigma = \{a_i \mid 0 \leq i < k\}$, a_i will always be followed by either a_{i+1} or a_0, with the latter occurring when there is a *carry*, a replacement of $x_i + x_{i+k-1}$ with x_{i+k}. a_{k-1} is always followed by a_0.

The subscript i of a_i is simply the value of the length $k - 1$ LSP of the Zeckendorf representation of i. In other words, if the k-Zeckendorf representation of r is $\sum_{h=0}^{m} d_h x_{0,h}$, then $i = \sum_{h=0}^{k-2} d_h x_{0,h}$.

3. MAIN THEOREMS

Theorem 1. *The difference between two adjacent rows of $\mathcal{X}_k$ is a shift of row 0, where the difference $\delta_r(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)}$ with $1 \leq j(r) \leq k$. Specifically the shift index $j(r)$ is*

$$j(r) = \begin{cases} i & \text{if } w(r) = a_i \quad \text{for} \quad 0 < i < k, \\ k & \text{if } w(r) = a_0. \end{cases} \tag{6}$$

$w(r)$ is the letter of the infinite word w of equation (5), located at position r.

Proof. By Lemma 1, $x_{r,0} = S^k(r) + 1$, so $x_{r+1,0} - x_{r,0} = S^k(r+1) - S^k(r)$.

Suppose $w(r) = a_i$. Then $w(r+1) = a_{i+1}$ or $w(r+1) = a_0$, with the latter case occurring when there was one or more carries.

Case 1: $w(r) = a_i$ with $0 < i < k$. Express the k-Zeckendorf representation of r as $\sum d_n x_{0,n}$. $i - 1$ is the smallest subscript such that $d_{i-1} = 1$, and $i - 1 + k$ is the smallest subscript such that $d'_{i-1+k} = 1$ in the k-Zeckendorf representation of $S^k(r)$ as $\sum d'_n x_{0,n}$.

Case 1.1: Suppose $w(r+1) = a_{i+1}$ and the k-Zeckendorf representation of $S^k(r+1)$ is $\sum d''_n x_{0,n}$. We have $d''_n = d'_n$ for $n > i + k$, so $S^k(r+1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i}$.

Case 1.2: If $w(r+1) = a_0$, we may execute a sequence of *borrows* reversing the above mentioned sequence of carries, to write $S^k(r+1) = \sum d''_n x_{0,n}$ (which, after borrowing is *not k-Zeckendorf*). We have $d''_n = d'_n$ for $n > i+1$, so again $S^k(r+1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i}$.

Case 2: $w(r) = a_0$. The two sub-cases, $w(r+1) = a_1$ and $w(r+1) = a_0$ follow a very similar argument to that given in Case 1. In both these sub-cases, $S^k(r+1) - S^k(r) = x_{0,k} - 0 = x_{0,k}$.

The above argument carries through identically for the general case

$$\begin{aligned} \delta_r(c) = x_{r+1,c} - x_{r,c} &= S^{c+k}(r+1) - S^{c+k}(r) \qquad (7) \\ &= S^c(S^k(r+1) - S^k(r)) \\ &= S^c(x_{0,j(r)}) \\ &= x_{0,j(r)+c} \end{aligned}$$

where $j(r)$ is given in equation (6). $\qquad \square$

For illustration, we show that the first 13 differences in column 0 of $\mathcal{X}_3$ have the same pattern as the 3-letter word of $\mathcal{W}_3$. The first 13 terms of the sequences $\{\delta_r(0)\}$, $\{j(r)\}$ and $\{w(r)\}$ for $0 \leq r \leq 12$ are shown below. The substitution of $\{c, a, b\}$ for $\{a_0, a_1, a_2\}$ is made in the word $\{w(r)\}_{r \geq 0}$.

$\delta_{0,\ldots,12}(0)$	$j(0,\ldots,12)$	$w(0,\ldots,12)$
4,2,3,4,4,2,4,2,3,4,2,3,4	3,1,2,3,3,1,2,3,1,2,3	cabccacabcabc

Theorem 2. *For all r, $x_{r,-k} = r$.*

Proof. $x_{r+1,-k} - x_{r,-k} = x_{0,-k+j(r)} = x_{0,n}$. Because $1 \leq j(r) \leq k$, we have $1 - k \leq n \leq 0$. It follows from Definition 1 and a simple precursion argument that $x_{0,n}$ is always 1, and that $x_{0,-k} = 0$. $\qquad \square$

Theorem 3. *For all $r \geq 0$, $w(x_{r,c}) = a_{c+1}$ if $0 \leq c \leq k - 2$, and $w(x_{r,c}) = a_0$ if $c > k - 2$.*

Proof. This follows from: the definition of $\mathcal{X}_k$, Definition 3, equation (3), Lemma 1, equation (4), and Lemma 2.

By Lemma 2, the LSP of $0^{k-1-i}10^{i-1}$ assigns $w(n) = a_i$ if $1 \leq i \leq k - 1$ and $w(n) = a_0$ if $i = 0$ or $i \geq k$. Likewise by Lemma 1, column 0 has LSP of $0^{k-1-i}1$; and by Definition 3 (equation (3)), column c has LSP of $0^{k-1-c}10^{c-1}$. $\qquad \square$

Some examples of Theorem 3 include:

(1) $w(x_{r,0}) = a_1$ for $r \geq 0, k \geq 2$,
(2) $w(x_{r,1}) = a_2$ for $r \geq 0, k \geq 3$,
(3) $w(x_{r,k-1}) = a_0$ for $r \geq 0$.

The following results deal with how the numbers in a column c of $\mathcal{X}_k$ punctuate (i.e., break into factors) the infinite word w. By Theorem 1, we know there are k different intervals $[x_{r,c}, x_{r+1,c})$, and the lengths of these intervals are k successive numbers of the sequence $\{x_{0,i}\}$.

Definition 6. Let the sequence

$$v_{r,c} = \langle w(i) \mid x_{r,c} \leq i < x_{r+1,c} \rangle$$

be a factor of the infinite word w. Its length is

$$|v_{r,c}| = \delta_r(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)}.$$

Definition 7. Let $\sigma : \Sigma^* \to \Sigma^*$ be a homomorphism defined on the letters of Σ by

$$
\begin{aligned}
\sigma(a_0) &= a_0 a_1, \\
\sigma(a_i) &= a_{i+1} \quad \text{for } 1 \leq i \leq k-1.
\end{aligned}
$$

As above, take $a_k = a_0$.

Lemma 3. *The infinite word w is the fixed point of σ.*

Proof. A straightforward inductive argument shows $\sigma(w_i) = w_{i+1}$ for all $i \geq 0$. $\qquad \square$

Definition 8. Denote by $\psi_\lambda(c)$, for $0 \leq \lambda < k$ words, as follows. For $c = 0$ we have the following

$$
\begin{aligned}
\psi_0(0) &= a_1\, a_0, \\
\psi_\lambda(0) &= a_1 \cdots a_{\lambda+1}\, a_0 \quad \text{for } 1 \leq \lambda \leq k-2, \\
\psi_{k-1}(0) &= a_1 \cdots a_{k-1}\, a_k\, a_0.
\end{aligned}
$$

For $c > 0$, let $\psi_\lambda(c) = \sigma(\psi_\lambda(c-1))$.

Lemma 4. $|\psi_\lambda(c)| = x_{0,c+\lambda+1}$ *for $c \geq 0$.*

Proof. For $0 \leq \lambda \leq k-1$, the word $\psi_\lambda(0)$ has length $x_{0,\lambda+1}$ and is a permutation of the word $w_{\lambda+1}$. The result then follows from the observation that $\sigma(w_i) = w_{i+1}$ for all $i \geq 0$ (see Lemma 3) and that σ is a homomorphism. $\qquad \square$

Theorem 4. *For $c \geq 0$, the $\psi_\lambda(c)$ are the only strings among the words $v_{r,c}$.*

Proof. From Definition 8, each string $\psi_\lambda(0)$ begins with the letter a_1, and it contains no other instance of a_1. In the infinite word w, every factor $\psi_\lambda(0)$ is followed by the letter a_1, beginning the next factor. The infinite string w, therefore, has the following descriptions:

$$
\begin{aligned}
w &= w_0, \langle \psi_{j(r)+1}(0) \mid r = 0, 1, 2, \ldots \rangle, \\
\sigma(w) &= w_1, \langle \psi_{j(r)+1}(1) \mid r = 0, 1, 2, \ldots \rangle, \\
&\ \ \vdots \qquad \vdots \\
\sigma^c(w) &= w_c, \langle \psi_{j(r)+1}(c) \mid r = 0, 1, 2, \ldots \rangle.
\end{aligned}
\tag{8}
$$

Because w is the fixed point of σ, the result follows. $\qquad \square$

We illustrate the first few iterations from (8) using $k = 3$ with $\{a_0, a_1, a_2\} = \{c, a, b\}$.

$$w \;=\; c, abcc, ac, abc, abcc, abcc, ac, abcc, ac, abc, abcc, ac, abc, abcc, abcc, ac, abc, \ldots$$
$$\sigma(w) \;=\; ca, bccaca, bca, bcca, bccaca, bccaca, bca, bccaca, bca, bcca, bccaca, bca, bcca, \ldots$$
$$\sigma^2(w) \;=\; cab, ccacabcab, ccab, ccacab, ccacab, cab, ccacab, cab, ccab, ccacab, cab, ccab, \ldots$$

Finally for $k = 3$, we display the k distinct strings $\psi_\lambda(c)$ in the first few columns $c \geq 0$.

$\psi_\lambda(c)$	$c:0$	1	2	3	4
w_c	c	ca	cab	cabc	cabcca
$\lambda:0$	ac	bca	ccab	cacabc	cabcabcca
1	abc	bcca	ccacab	cacabcabc	cabcabccabcca
2	abcc	bccaca	ccacabcab	cacabcabccabc	cabcabccabccacabcca

We note that the strings, $\psi_\lambda(c)$ and $\psi_{\lambda-1}(c+1)$, have the same length and are a rotation permutation of each other. We thus define a rotation operator ρ such that $\rho(a_i x) = x a_i$ for $a_i \in \Sigma$ and $a_i x \in \Sigma^*$, and we present without proof:

Proposition 1. $\psi_\lambda(c) = \rho \cdots \rho(w_{c+\lambda+1}) = \rho^{|w_c|}(w_{c+\lambda+1})$.

Taking $k = 3$ and $c = 1$, Proposition 1 gives $\psi_\lambda(1) = \rho^{|w_1|}(w_{2+\lambda})$ with $\psi_0(1) = \rho^2(cab) = bca$, $\psi_1(1) = \rho^2(cabc) = bcca$, and $\psi_2(1) = \rho^2(cabcca) = bccaca$. As noted for diagonals, at $\lambda + c = 3$ we get $\psi_\lambda(c) = \rho^{|w_c|}(w_4)$ with $\psi_2(1) = \rho^2(cabcca) = bccaca$, $\psi_1(2) = \rho^3(cabcca) = ccacab$, and $\psi_0(3) = \rho^4(cabcca) = cacabc$; having equal lengths $|\psi_\lambda(c)| = |w_4| = x_{0,4} = 6$ as required by Proposition 1, Theorem 1, and Lemma 4.

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