Enumeration of Domino Tilings on the Projective Grid Graph

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Abstract

We provide an overview of Tesler's [8] findings on the enumeration of perfect matchings on nonorientable surfaces, and discuss the following theorems and conjectures.

Theorem. *The number of domino tilings of the $2 \times 2k$ projective plane equals*

1. $C_k + C_{k-1}$, *where C_k is the number of tilings of a $2 \times (k+1)$ projective grid with the center four tiles covered.*
2. $\left(3 + \frac{5\sqrt{3}}{3}\right)\left(2+\sqrt{3}\right)^k + \left(3 - \frac{5\sqrt{3}}{3}\right)\left(2-\sqrt{3}\right)^k$.
3. $2\,U_k(2) + 2\,U_{k-2}(2)$, *where $U_n(x)$ is the nth Chebyshev polynomial of the second kind.*

Theorem. *The number of domino tilings of the $2 \times (2k+1)$ projective plane equals*

1. $4\,U_k(2)$.
2. $\left(2 + \frac{4\sqrt{3}}{3}\right)(2+\sqrt{3})^k + \left(2 - \frac{4\sqrt{3}}{3}\right)(2-\sqrt{3})^k$.

Theorem. *[10] The number of domino tilings of the $2M \times 2N$ Möbius strip equals*

$$\prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\cos^2\left(\frac{m\pi}{2M+1}\right) + 4\sin^2\left(\frac{(4n-1)\pi}{4N}\right)\right]$$

Theorem. *The number of domino tilings of the $2M \times 2N$ projective plane can be reduced to the computation of a skew-Hermitian $M \times N$ matrix.*

Conjecture. *Every $2M \times 2N$ projective grid can be associated with a sequence of rational numbers that multiplies to the number of domino tilings. This sequence in turn can be associated with an integer sequence terminating in the number of tilings.*

Conjecture. *In the case of the $2 \times N$ projective plane, with N even,*

1. *If the Aitken sequence is taken to start at the final element of the first chunk, the Nth element of the sequence is $U_{n-1}(2)$.*
2. *With the exception of the final element, the second chunk of the Aitken list converges to $2+\sqrt{3}$.*
3. *As N increases, the final element of the $2 \times 2N$ Aitken list converges to $2+2\sqrt{3}$.*
4. *As N increases, the final element of the $2n \times N$ Aitken list converges to $2+2\sqrt{3}$.*
5. *The kth element of the $2n \times 2$ Aitken sequence is the kth Fibbonacci number.*
6. *The final element of the Aitken sequence is equal to $2\,U_{n-1}(2) - 2\,U_{n-2}(2)$.*

Conjecture. *In the case of the $4 \times N$ projective plane, when N is even,*

1. *The $\frac{N}{2}$ through Nth elements of the Aitken sequence are given by*

   $$U_k\left(\frac{3}{2}\right) - U_{k-1}\left(\frac{3}{2}\right)$$

   for $k \geq 1$.
2. *The kth element of the first chunk of a $4 \times N$ Aitken sequence is the number of domino tilings of the $2 \times 2k$ rectangular grid.*

Introduction

Problems related to enumerating domino tilings of grids arose in the field of physical chemistry in the investigation of arrangements of dimer molecules over crystal lattices. In 1961, P.W. Kastelyn derived

$$Z_{Rect}^{2M,2N} = \prod_{m=1}^{M} \prod_{n=1}^{N} \left[4\cos^2\left(\frac{m\pi}{2M+1}\right) + 4\cos^2\left(\frac{n\pi}{2N+1}\right) \right]$$

for the number of domino tilings of the $2M \times 2N$ rectangular grid [3]. Tesler [8] made significant contributions to the theory for graphs embedded in more general surfaces in 1998, and Lu and Wu found solutions for the Möbius strip and Klein bottle in 1999 and 2002 [10, 9]. We attempt to do the same for the projective plane.

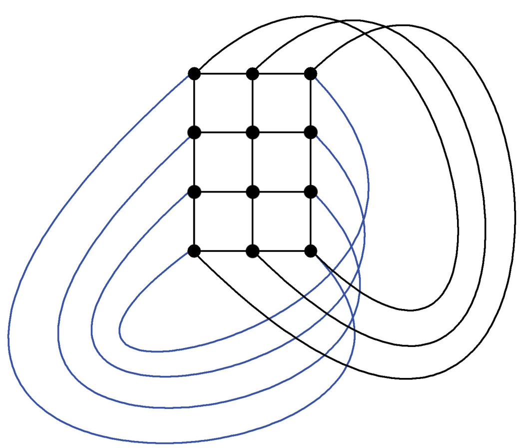


Figure 1: The 4×3 projective plane grid graph.

The graph in Figure 1 is the 4×3 *projective grid*. We refer to the intersecting edges outside the ordinary grid as *crossing edges*. Removing the blue set of crossing edges from the graph leaves a 3×4 *Möbius strip*, and removing the black set leaves a 4×3 Möbius strip. Removing both sets yields a *rectangular grid*. We treat the double edges on the corners as single edges of weight two, so an alternative representation of the graph is shown in Figure 2.

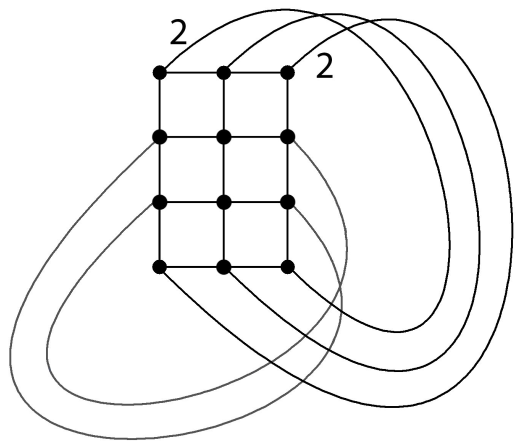


Figure 2: 4×3 projective plane with double edges

Figure 3 contains *pasting maps* for the projective plane and Möbius strip, respectively.

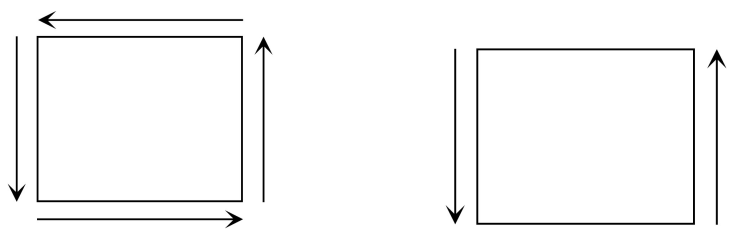


Figure 3: Pasting maps for projective plane (left) and Möbius strip (right).

If the sides of the inner rectangle associated with arrows are glued to their opposites so the arrows match in direction, the corresponding surface is produced. More generally, pasting maps may be formed from polygons with any number of sides [8].

If the vertices of the graph in Figure 2 were replaced with checkerboard squares, a *domino tiling* would be an arrangement of dominoes over the resulting checkerboard that covers every square and leaves no dominoes overlapping. Naturally, dominoes would be allowed to straddle the crossing edges.

Domino tilings are also known as *perfect matchings*. A perfect matching of a graph is a set of edges that collectively touch every vertex but have no vertices in common.

To be explicit, suppose we wish to describe the perfect matching σ drawn in Figure 4. The vertices are numbered left-to-right and top-to-bottom, and the four tiles in the picture are really two "dominoes" wrapping around crossing edges.

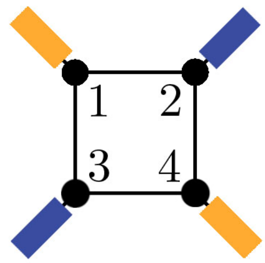


Figure 4: A domino tiling on the 2×2 projective plane grid graph.

The perfect matching σ can be associated with a permutation in S_4:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23).$$

By representing pairs of "covered" vertices as transpositions, it is apparent that perfect matchings on an $M \times N$ graph are products of disjoint transpositions of adjacent vertices of G in S_{MN}, with no fixed points.

As an aside, since both edges straddled by dominoes in Figure 4 are double edges, which each represent a choice of two possible edges, this tiling would have weight $2 \cdot 2 = 4$ in a full enumeration.

A statement of the problem should now be clear: we wish to enumerate weighted perfect matchings of the $M \times N$ projective plane grid graph.

Chapter 1

$2 \times n$ Recurrence Solution

We determine the number of domino tilings of a $2 \times n$ projective grid graph by evaluating a mutual recurrence relation. The cases of even and odd n are treated separately in Theorems 1 and 2, respectively.

Definition 1.

The Chebyshev polynomials of the second kind *are defined by the recurrence*

$$
\begin{aligned}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_{n+1}(x) &= 2x\, U_n(x) - U_{n-1}(x).
\end{aligned}
$$

Theorem 1. *The number of domino tilings of the $2 \times 2k$ projective plane equals*

1. $C_k + C_{k-1}$, *where C_k is the number of tilings of a $2 \times (k+1)$ projective grid with the center four tiles covered.*
2. $\left(3 + \frac{5\sqrt{3}}{3}\right)\left(2+\sqrt{3}\right)^k + \left(3 - \frac{5\sqrt{3}}{3}\right)\left(2-\sqrt{3}\right)^k$.
3. $2\, U_k(2) + 2\, U_{k-2}(2)$, *where $U_n(x)$ is the nth Chebyshev polynomial of the second kind.*

Proof of Theorem 1.1. Let T_n equal the number of tilings of a $2 \times n$ projective plane, where n is even, and define recurrence relations C, D, and F for the possible placements of tiles over the middle four squares, as in Figure 1.1.

In reference to Figure 1.1, it is easy to reason that the fifth case leads to no viable tilings, so by assigning the labels C_{n-2}, D_{n-2}, and F_{n-3} to the number of tilings of the first three cases, we write

$$T_n = 3\, C_{n-2} + 2\, D_{n-2} + 2\, F_{n-3} + C_{n-4} \tag{1.1}$$

with T_n standing for the number of tilings of the even $2 \times n$ projective grid.

Similarly, we derive an expanded recurrence for C_{n-k}. We start by conditioning on the placement of a tile over the square marked with an orange dot in the grid

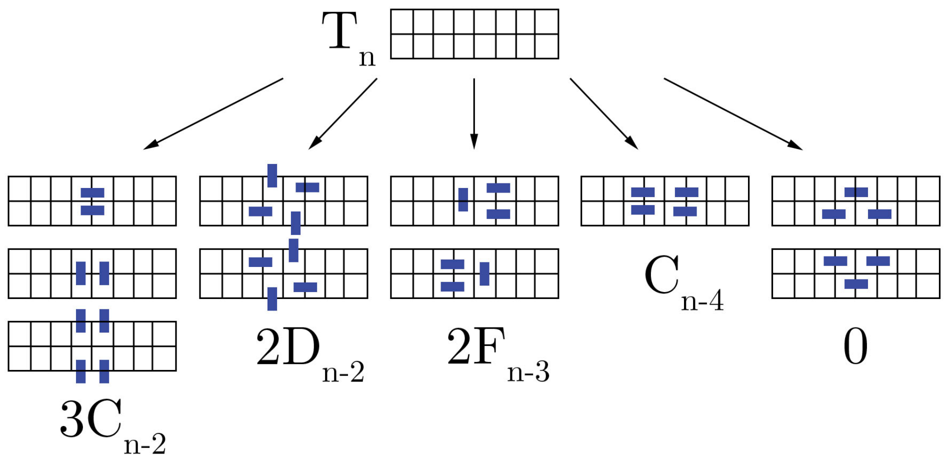


Figure 1.1: Possible arrangements of dominoes over the center of an even $2 \times n$ projective plane.

labeled C_{n-k} in Figure 1.2. The recurrences we have already defined are sufficient to describe C_{n-k}.

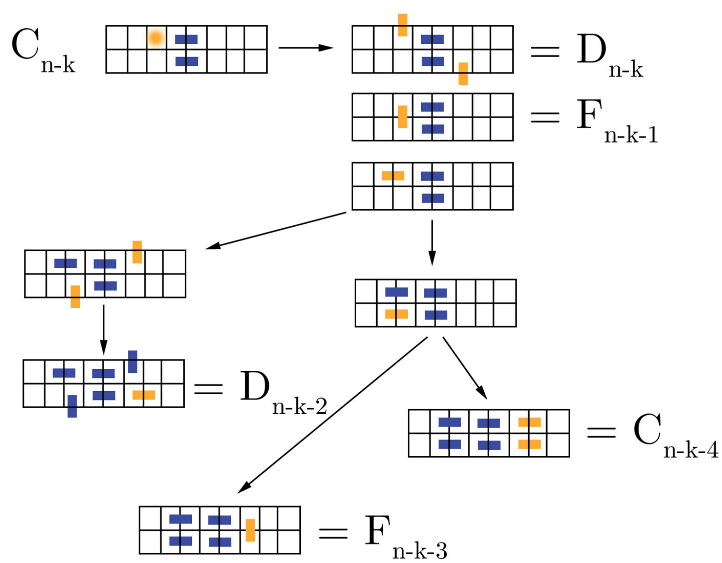


Figure 1.2: Possible initial tilings of C_{n-k}.

From this we conclude that

$$C_{n-k} = D_{n-k} + F_{n-k-1} + D_{n-k-2} + F_{n-k-3} + C_{n-k-4}\,. \tag{1.2}$$

We examine D_{n-k} and F_{n-k-1} by conditioning on the tile placed over the orange dot in the grids labeled D_{n-k} and F_{n-k-1}, and find that both D and F are mutually recursive with C.

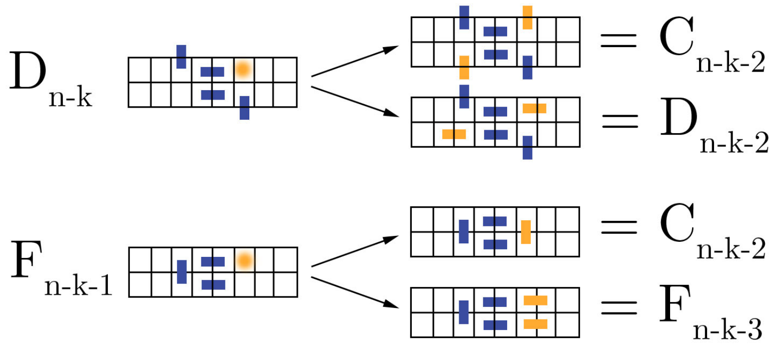


Figure 1.3: Possible initial tilings of D_{n-k} and F_{n-k-1}

Reading from Figure 1.3,

$$D_{n-k} = C_{n-k-2} + D_{n-k-2}$$

and

$$F_{n-k-1} = C_{n-k-2} + F_{n-k-3}.$$

We define $D_0 = C_0 = 1$, as these configurations are already tiled. Solving the two equations above for C_{n-k-2} yields

$$\begin{aligned} C_{n-k-2} &= D_{n-k} - D_{n-k-2} \\ &= F_{n-k-1} - F_{n-k-3}, \end{aligned} \tag{1.3}$$

indicating that the differences between successive terms of D and F are equal.

Now, $D_2 = 2$:

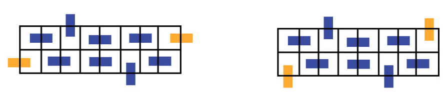


Figure 1.4: Configurations of D_2

and $F_1 = 1$:

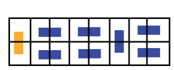


Figure 1.5: Single configuration of F_1

Since $D_0 = 1$, (1.3) gives us $D_k = F_{k-1} + 1$ for all $k > 0$. Furthermore,

$$\begin{aligned} D_{n-k} &= C_{n-k-2} + D_{n-k-2} \\ &= C_{n-k-2} + (C_{n-k-4} + (C_{n-k-6} + (\ldots C_2 + D_2))\ldots)) \\ &= 1 + C_{n-k-2} + (C_{n-k-4} + (C_{n-k-6} + (\ldots C_2 + C_0))\ldots)) \\ &= 1 + \sum_{j=\frac{k}{2}+1}^{\frac{n}{2}} C_{n-2j}. \end{aligned} \tag{1.4}$$

Turning to F_{n-k-1},

$$\begin{aligned} F_{n-k-1} &= C_{n-k-2} + F_{n-k-3} \\ &= C_{n-k-2} + (C_{n-k-4} + (C_{n-k-6} + (\ldots C_2 + F_1))\ldots)). \end{aligned}$$

Since $F_1 = C_0$,

$$\begin{aligned} F_{n-k-1} &= C_{n-k-2} + (C_{n-k-4} + (C_{n-k-6} + (\ldots C_2 + C_0))\ldots)) \\ &= \sum_{j=\frac{k}{2}+1}^{\frac{n}{2}} C_{n-2j}, \end{aligned} \tag{1.5}$$

as expected.

Making use of the relationship between F and D, we revisit C_{n-k}:

$$\begin{aligned} C_{n-k} &= D_{n-k} + F_{n-k-1} + D_{n-k-2} + F_{n-k-3} + C_{n-k-4} \\ &= 2\,D_{n-k} + 2\,D_{n-k-2} + C_{n-k-4} - 2. \end{aligned}$$

Removing the F terms leaves only even indices, so we adjust by letting $k = \frac{n-k}{2}$:

$$\begin{aligned} C_k &= 2\,D_k + 2\,D_{k-1} + C_{k-2} - 2 \\ &= 2\sum_{j=0}^{k-1} C_j + 2\sum_{j=0}^{k-2} C_j + C_{k-2} + 2 \\ &= 4\sum_{j=0}^{k-2} C_j + 2\,C_{k-1} + C_{k-2} + 2. \end{aligned} \tag{1.6}$$

Note that $C_0 = 1$ and $C_1 = 5$:

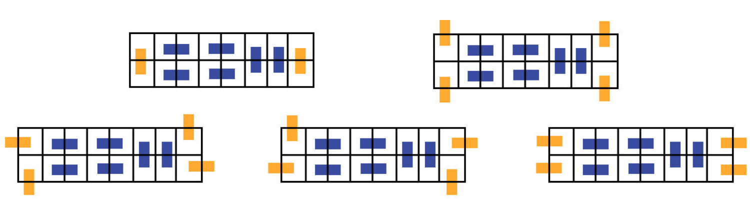


Figure 1.6: Possible configurations of C_1

Given these starting values, the first ten values in this sequence are

$$1, 5, 17, 65, 241, 901, 3361, 12545, 46817, 174725.$$

Searching the Online Encyclopedia of Integer Sequences (OEIS) for this sequence suggests only that it satisfies the recurrence

$$a_n = 3\, a_{n-1} + 3\, a_{n-2} - a_{n-3}. \tag{1.7}$$

To prove that C_k satisfies the recurrence (1.7), we evaluate the recurrence on (1.6).

$$\begin{aligned}
K &:= 3\, C_{n-1} + 3\, C_{n-2} - C_{n-3} \\
&= 3\left(4\sum_{j=0}^{k-3} C_j + 2\, C_{k-2} + C_{k-3} + 2\right) + 3\left(4\sum_{j=0}^{k-4} C_j + 2\, C_{k-3} + C_{k-4} + 2\right) \\
&\quad - 4\sum_{j=0}^{k-5} C_j - 2\, C_{k-4} - C_{k-5} - 2 \\
&= 12\sum_{j=0}^{k-3} C_j + 6\, C_{k-2} + 9\, C_{k-3} + 12\sum_{j=0}^{k-4} C_j + C_{k-4} \\
&\quad - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 10 \\
&= \left(4\sum_{j=0}^{k-2} C_j + C_{k-2} + 2\right) + 8\sum_{j=0}^{k-3} C_j + C_{k-2} + 9\, C_{k-3} + 12\sum_{j=0}^{k-4} C_j + C_{k-4} \\
&\quad - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 8.
\end{aligned}$$

To maintain readability, we note that the parenthesized portion of the expression above is equal to $C_k - 2\, C_{k-1}$ and proceed to evaluate the non-parenthesized portion to arrive at $2C_{k-1}$. Let the non-parenthesized portion be called **NP**:

$$\begin{aligned}
\mathbf{NP} &= 8\sum_{j=0}^{k-3} C_j + C_{k-2} + 9\, C_{k-3} + 12\sum_{j=0}^{k-4} C_j + C_{k-4} - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 8 \\
&= \left(4\sum_{j=0}^{k-4} C_j + C_{k-3} + C_{k-4} + 2\right) + 8\sum_{j=0}^{k-3} C_j + C_{k-2} + 7\, C_{k-3} \\
&\quad + 8\sum_{j=0}^{k-4} C_j - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 6.
\end{aligned}$$

The parenthesized expression above is C_{k-2}, therefore,

$$\begin{aligned}
\mathbf{NP} &= C_{k-2} + 8\sum_{j=0}^{k-3} C_j + C_{k-2} + 7\,C_{k-3} + 8\sum_{j=0}^{k-4} C_j - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 6 \\
&= \left(4\sum_{j=0}^{k-3} C_j + 2\,C_{k-2} + C_{k-3} + 2\right) + 6\,C_{k-3} + 8\sum_{j=0}^{k-4} C_j + 4\sum_{j=0}^{k-3} C_j \\
&\qquad - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 4.
\end{aligned}$$

The parenthesized expression above is C_{k-1}, so

$$\begin{aligned}
\mathbf{NP} &= C_{k-1} + 6\,C_{k-3} + 8\sum_{j=0}^{k-4} C_j + 4\sum_{j=0}^{k-3} C_j - 4\sum_{j=0}^{k-5} C_j - C_{k-5} + 4 \\
&= C_{k-1} + \left(4\sum_{j=0}^{k-3} C_j\right) + \left(8\sum_{j=0}^{k-4} C_j + 4\,C_{k-3} + 4\right) + 2\,C_{k-3} - 4\sum_{j=0}^{k-5} C_j - C_{k-5}.
\end{aligned}$$

Expanding $2\,C_{k-3}$,

$$\begin{aligned}
&= C_{k-1} + \left(4\sum_{j=0}^{k-3} C_j\right) + \left(8\sum_{j=0}^{k-4} C_j + 4\,C_{k-3} + 4\right) + 2\left(4\sum_{j=0}^{k-5} C_j + 2\,C_{k-4} + C_{k-5} + 2\right) \\
&\quad - 4\sum_{j=0}^{k-5} C_j - C_{k-5} \\
&= C_{k-1} + \left(4\sum_{j=0}^{k-3} C_j\right) + \left(8\sum_{j=0}^{k-4} C_j + 4\,C_{k-3} + 4\right) + 4\sum_{j=0}^{k-5} C_j + 4\,C_{k-4} + C_{k-5} + 4 \\
&= C_{k-1} + \left(4\sum_{j=0}^{k-3} C_j\right) + \left(8\sum_{j=0}^{k-4} C_j + 4\,C_{k-3} + 2\,C_{k-4} + 4\right) + C_{k-3} + 2 \\
&= C_{k-1} + \left(4\sum_{j=0}^{k-3} C_j + 2\,C_{k-2} + C_{k-3} + 2\right) \\
&= 2\,C_{k-1}.
\end{aligned}$$

Since $\mathbf{NP}$ is equal to $2C_{k-1}$, K is equal to C_k, and so C_k satisfies (1.7).

Going back to (1.1), keep in mind that the equation applies to a $2 \times n$ projective grid for even n:

$$\begin{aligned}
T_n &= 3\,C_{n-2} + 2\,D_{n-2} + 2\,F_{n-3} + C_{n-4} \\
&= 3\,C_{n-2} + 4\,D_{n-2} + C_{n-4} - 2,
\end{aligned}$$

by (1.3). With the elimination of F, we reindex for a $2 \times 2k$ grid:

$$\begin{aligned} T_k &= 3\,C_{k-1} + 4\,D_{k-1} + C_{k-2} - 2 \\ &= 3\,C_{k-1} + 4\left(\sum_{j=0}^{k-2} C_j + 1\right) + C_{k-2} - 2 \\ &= 3\,C_{k-1} + 4\sum_{j=0}^{k-2} C_j + C_{k-2} + 2. \end{aligned} \tag{1.8}$$

This leaves us with an expression for the number of tilings of the projective plane grid in terms of only C_k.

Recall that by (1.6),

$$C_k = 4\sum_{j=0}^{k-2} C_j + 2\,C_{k-1} + C_{k-2} + 2,$$

which is similar to (1.8). Combining the two yields a cleaner recurrence:

$$T_k = C_k + C_{k-1}. \tag{1.9}$$

□

Proof of Theorem 1.2. We have already proven that C_k satisfies (1.7), and a simple argument will show that given (1.9), T_k must also follow (1.7). Rather than deriving a generating function directly, however, we do some intermediate simplification.

Using (1.5) for computation of C_k, the first 10 terms of T_k are computed by (1.8) to be

$$1, 6, 22, 82, 306, 1142, 4262, 15906, 59362, 221542$$

A linear recurrence finder suggests that in addition to satisfying the original recurrence for C, these satisfy the simpler recurrence

$$a_n = 4\,a_{n-1} - a_{n-2}.$$

To prove that this is so, we define a sequence

$$t_n = 3\,t_{n-1} + 3\,t_{n-2} - t_{n-3}$$

with $t_0 = 6, t_1 = 22$ and check to see if $t_n = 4\,t_{n-1} - t_{n-2}$. As a base case, note that $4 \cdot 22 - 8 = 82$, as required by the sequence above. Applying induction, suppose $t_k = 4\,t_{k-1} - t_{k-2}$ for all k up to some n. Then

$$\begin{aligned} t_n &= 3\,t_{n-1} + 3\,t_{n-2} - t_{n-3} \\ &= 3\,t_{n-1} + (t_{n-1} - t_{n-2}) \\ &= 4\,t_{n-1} - t_{n-2}, \end{aligned}$$

and so the sequences are equal. Thus, T_k is described by the simpler recurrence.

We next solve the two-term recurrence

$$t_n = 4\,t_{n-1} - t_{n-2},$$

with $t_0 = 6, t_1 = 22$.

Let $T(x)$ be an ordinary generating function with form

$$T(x) = \sum_{n=0}^{\infty} t_n x^n.$$

From the conditions, we have $t_n x^n = 4\,t_{n-1}x^n - t_{n-2}x^n$, so naturally,

$$\sum_{n=2}^{\infty} t_n x^n = 4\sum_{n=2}^{\infty} t_{n-1}x^n - \sum_{n=2}^{\infty} t_{n-2}x^n.$$

Working with these sums individually to get them in terms of $T(x)$, we find that

$$\begin{aligned}
\sum_{n=2}^{\infty} t_n x^n &= T(x) - t_1 x - t_0 \\
&= T(x) - 22x - 6, \\
4\sum_{n=2}^{\infty} t_{n-1}x^n &= 4x\sum_{n=2}^{\infty} t_{n-1}x^{n-1} \\
&= 4x(T(x) - t_0) \\
&= 4xT(x) - 24x, \\
\sum_{n=2}^{\infty} t_{n-2}x^n &= x^2\sum_{n=2}^{\infty} t_{n-2}x^{n-2} \\
&= x^2 T(x).
\end{aligned}$$

Recombining as indicated by the conditions,

$$T(x) - 22x - 6 = 4x\,T(x) - 24x - x^2 T(x).$$

Thus,

$$T(x) = \frac{-2x+6}{x^2 - 4x + 1}.$$

The number of tilings of the $2 \times 2k$ projective plane is the kth coefficient of this generating function. To find it, we find the roots of the denominator:

$$r = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

So,
$$x^2 - 4x + 1 = (1 - (2+\sqrt{3})x)(1 - (2-\sqrt{3})x),$$
and by partial fraction decomposition, we find A and B such that
$$\frac{-2x+6}{x^2-4x+1} = \frac{A}{1-(2+\sqrt{3})x} + \frac{B}{1-(2-\sqrt{3})x}.$$
By straightforward calculation, these are
$$A = 3 + \frac{5\sqrt{3}}{3}$$
$$B = 3 - \frac{5\sqrt{3}}{3},$$
and by geometric series, the number of tilings of the $2 \times 2k$ projective plane is
$$T_k = \left(3 + \frac{5\sqrt{3}}{3}\right)\left(2+\sqrt{3}\right)^k + \left(3 - \frac{5\sqrt{3}}{3}\right)\left(2-\sqrt{3}\right)^k. \tag{1.10}$$

□

Proof of Theorem 1.3. A Chebyshev recurrence finder developed in Sage [7] suggests the values of T_k may also be generated by the relation
$$T_k = 2\,U_k(2) - 2\,U_{k-1}(2)$$
where $U_n(x)$ is the nth Chebyshev polynomial of the second kind, evaluated at x.

It follows from the base cases in Definition 1 that
$$2\,U_1(2) - 2\,U_0(2) = 6 = T_1.$$
By induction, assume $T_k = 2\,U_k(2) - 2\,U_{k-1}(2)$ for all $k \le n$. Then
$$\begin{aligned} T_n &= 4\,T_{n-1} - T_{n-2} \\ &= 8\,U_{n-1}(2) - 8\,U_{n-2}(2) - 2\,U_{n-2}(2) + 2\,U_{n-3}(2) \\ &= 2(2\cdot 2\,U_{n-1}(2) - 2\,U_{n-2}(2)) - 2(2\cdot 2\,U_{n-2}(2) - 2\,U_{n-3}(2)) \\ &= 2\,U_n(2) - 2\,U_{n-1}(2), \end{aligned}$$
and so another expression for the number of domino tilings of the $2 \times 2n$ projective plane is
$$T_n = 2\,U_n(2) - 2\,U_{n-1}(2).$$

□

This concludes the proof of Theorem 1. We move to the $2 \times (2k+1)$ case.

Theorem 2. *The number of domino tilings of the $2\times(2k+1)$ projective plane equals*

1. $\left(2+\frac{4\sqrt{3}}{3}\right)(2+\sqrt{3})^k+\left(2-\frac{4\sqrt{3}}{3}\right)(2-\sqrt{3})^k$.
2. $4\,U_k(2)$.

Proof of Theorem 2.1. For the odd case, let T_n be the number of tilings of the $2\times(n-1)$ projective grid, and start similarly by examining placements of tiles over the middle two squares:

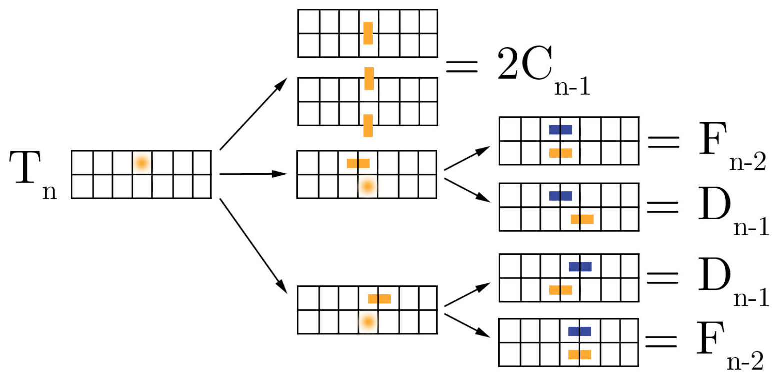


Figure 1.7: Initial configurations of the $2\times(n-1)$ projective grid

This indicates that
$$T_n = 2\,C_{n-1} + 2\,D_{n-1} + 2\,F_{n-2},$$
and since $D_{n-1} = F_{n-2} + 1$, we have

$$T_n = 2\,C_{n-1} + 4\,D_{n-1} - 2$$

for odd n, and so

$$T_{n-1} = 2\,C_{n-2} + 4\,D_{n-2} - 2$$

for even n.

Recall that by (1.1),
$$T_n = 3\,C_{n-2} + 4\,D_{n-2} + C_{n-4} - 2$$
for even n. Combining this with the relation for T_{n-1}, we find that

$$\begin{aligned} T_{n-1} &= T_n - C_{n-2} - C_{n-4} \\ &= (C_n + C_{n-2}) - (C_{n-2} + C_{n-4}) \\ &= C_n - C_{n-4}. \end{aligned} \tag{1.11}$$

Switching indices as before,

$$T_{k-1} = C_k - C_{k-2}$$

for a $2 \times (2k-1)$ projective plane. From the previous section, we know that for a $2 \times 2k$ plane,

$$T_k = C_k + C_{k-1},$$

which tells us that

$$\begin{aligned} T_k - T_{k-1} &= (C_k + C_{k-1}) - (C_k - C_{k-2}) \\ &= C_{k-1} + C_{k-2}, \end{aligned}$$

and so,

$$T_n = T_{n-1} + T_{n-2}$$

for even n.

Using (1.11), and reindexing the right side for consistency with the previous section, we can also write that

$$\begin{aligned} T_{n-1} &= T_n - C_{n-2} - C_{n-4} \\ &= (3\,C_{n-2} + 4\,D_{n-2} + C_{n-4} - 2) - C_{n-2} - C_{n-4} \\ &= 2\,C_{n-2} + 4\,D_{n-2} - 2 \\ T_k &= 2\,C_{k-1} + 4\sum_{j=0}^{k-2} C_j + 2 \end{aligned}$$

for a $2 \times (2k-1)$ projective grid.
The first ten elements of this sequences are

$$4, 16, 60, 224, 836, 3120, 11644, 43456, 162180, 605264$$

with 4 corresponding to the 2×1 plane, 16 to the 2×3 plane, and so on. A linear recurrence finder suggests that this sequence obeys the same recurrences as the even case:

$$a_n = 4\,a_{n-1} - a_{n-2}$$

with $a_0 = 4, a_1 = 16, a_2 = 60$. As before, assume $T(x) = \sum_{n=0}^{\infty} a_n x^n$ is a generating function for the sequence. Then $a_n = 4\,a_{n-1} - a_{n-2}$, and so

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-1} - \sum_{n=2}^{\infty} a_{n-2},$$

from which we find

$$T(x) - a_1 x - a_0 = 4x\,(T(x) - a_0) - x^2 T(x)$$

and

$$T(x) = \frac{4}{1 - 4x + x^2}.$$

The roots of the denominator are $2 \pm \sqrt{3}$, so applying partial fractions,

$$\frac{4}{1 - 4x + x^2} = \frac{A}{1 - (2 + \sqrt{3})x} + \frac{B}{1 - (2 - \sqrt{3})x}$$

and thus,

$$4 = A(1 - (2 - \sqrt{3})x) + B(1 - (2 + \sqrt{3})x),$$

leads to the conclusion that

$$\begin{aligned} A &= 2 + \frac{4\sqrt{3}}{3} \\ B &= 2 - \frac{4\sqrt{3}}{3}. \end{aligned}$$

By geometric series, the number of tilings of the $2 \times (2k - 1)$ projective plane is

$$T_k = \left(2 + \frac{4\sqrt{3}}{3}\right)(2 + \sqrt{3})^k + \left(2 - \frac{4\sqrt{3}}{3}\right)(2 - \sqrt{3})^k.$$

$\square$

Proof of Theorem 2.2. The Chebyshev recurrence finder suggests that the sequence T_k is generated by

$$T_k = 4\,U_k(2).$$

We will prove by induction that this is so. Since $U_1(2) = 1$ is a base case for the Chebyshev polynomials, this relationship holds for $k = 1$. Supposing it holds for all $k < n$, i.e

$$4\,U_k(2) = 2\,C_{k-1} + 4\sum_{j=0}^{k-2} C_j + 2$$

for all $k < n$, and keeping in mind that

$$C_k = 4\sum_{j=0}^{k-2} C_j + 2\,C_{k-1} + C_{k-2} + 2$$

and

$$U_n(x) = 2x\,U_{n-1}(2) - U_{n-1}$$

we evaluate

$$\begin{aligned}
4\,U_n(2) &= 4\,(4\,U_{n-1}(2) - U_{n-2}(2)) \\
&= 16\,U_{n-1}(2) - 4\,U_{n-2}(2) \\
&= 4\,(2\,C_{n-2} + 4\sum_{j=0}^{n-3} C_j + 2) - 2\,C_{n-3} - 4\sum_{j=0}^{n-4} C_j - 2 \\
&= 8\,C_{n-2} + 16\sum_{j=0}^{n-3} C_j - 2\,C_{n-3} - 4\sum_{j=0}^{n-4} C_j + 6 \\
&= 8\,C_{n-2} + 12\sum_{j=0}^{n-3} C_j + 2\,C_{n-3} + 6 \\
&= 2\,C_{n-1} + 4\,C_{n-2} + 4\sum_{j=0}^{n-3} C_j + 2 \\
&= 2\,C_{n-1} + 4\sum_{j=0}^{n-2} C_j + 2,
\end{aligned}$$

as desired. Thus the number of domino tilings of a $2 \times (2k-1)$ projective plane is

$$T_k = 4\,U_k(2).$$

□

Since both the even and odd cases follow the recurrence

$$t_n = 4t_{n-1} - t_{n-2},$$

it is a corollary that the interlaced sequences follow the recurrence

$$t_n = 4t_{n-2} - t_{n-4}.$$

The initial conditions are $t_0 = 4$, $t_1 = 6$, $t_2 = 16$. The generating function for this sequence is

$$T(x) = \frac{-2x^3 + 6x + 4}{1 - 4x^2 + x^4}.$$

Chapter 2

Classical Approach

The classical approach to enumerating domino tilings on grids, conceived by Kastelyn [3], hinges on some observations about the *Pfaffian* of a graph's adjacency matrix.

Definition 2. *Let A be a $2N \times 2N$ skew-symmetric matrix, let the set of transposition products*

$$\mathbf{\Pi} = \{(u_1, v_1) \cdots (u_N, v_N)\}$$

range over the partitions of $\{1, \ldots, 2N\}$ into N pairs (u_k, v_k) without regard to order, and define the signed weight *of a permutation $\sigma \in \mathbf{\Pi}$ with respect to A as*

$$w_\sigma = sgn \begin{bmatrix} 1 & 2 & 3 & 4 & \ldots & 2N \\ u_1 & v_1 & u_2 & v_2 & \ldots & v_N \end{bmatrix} \cdot A_{u_1,v_1} \cdot A_{u_2,v_2} \cdots A_{u_N,v_N}. \tag{2.1}$$

Then,

$$Pf(A) = \sum_{\sigma \in \mathbf{\Pi}} w_\sigma. \tag{2.2}$$

Additionally, $(Pf\, A)^2 = \det A$ and if A is not $2N \times 2N$ skew-symmetric, $Pf\, A$ is not defined.

Definition (2) makes the connection to perfect matchings fairly explicit. Let an *orientation* of a graph be an assignment of direction to every edge. Let w_{uv} be the weight of an edge between u and v. Define the *signed adjacency matrix* by

$$A_{u,v} = \begin{cases} w_{uv} & \text{if an edge is directed from } u \text{ to } v; \\ -w_{uv} & \text{if an edge is directed from } v \text{ to } u; \\ 0 & \text{if } u \text{ and } v \text{ are not connected.} \end{cases}$$

The signed adjacency matrix of an oriented graph is skew-symmetric, since an edge connecting vertex a to b must connect b to a in the opposite direction. Accordingly, suppose A is the signed adjacency matrix of an oriented graph G with numbered vertices, and let $\sigma \in \mathbf{\Pi}$. Then if for some vertex v, v and $\sigma(v)$ are not adjacent,

$A_{v,\sigma v} = 0$ and thus $w_\sigma = 0$. This means the only permutations in $\mathbf{\Pi}$ with nonzero weight are products of transpositions of adjacent vertices. In other words,

$$w_\sigma \neq 0 \iff \sigma \text{ is a perfect matching.}$$

As Kastelyn first noticed, if the edges of G can be oriented so that all nonzero summands of the Pfaffian have the same sign, then Pf A will directly enumerate perfect matchings of G. Let the edges of a graph covered by a particular tiling be referred to as *matched edges*. Since each matching is counted with weight equal to the product of the weights of the matched edges, enumeration of matchings on weighted graphs requires no adjustment to the method.

Kastelyn discovered a way to orient planar graphs in such a way, and derived

$$Z_{Rect}^{2M,2N} = \prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\cos^2\left(\frac{m\pi}{2M+1}\right) + 4\cos^2\left(\frac{n\pi}{2N+1}\right)\right]$$

for the number of tilings of the $2M \times 2N$ rectangular grid. He conjectured, and Tesler later confirmed [8], that surfaces of higher genus could be approached with a linear combination of Pfaffians.

Tesler extended Kastelyn's regular orientation to a more general *crossing orientation*, which serves a similar purpose for non-planar graphs [8, 3]. While a regular orientation is characterized by a Pfaffian with the same sign for each summand, thus allowing the Pfaffian to count perfect matchings of a graph directly, in a crossing orientation, the sign of each perfect matching depends on the number of crossing edges the matching contains. Ultimately, this allows the number of perfect matchings to be expressed as a linear combination of Pfaffians of modified adjacency matrices, as we will describe in Chapter 3.

The influence of Kastelyn's method is evident in the resulting expressions for the number of domino tilings for the $2M \times 2N$ Möbius strip and Klein bottle, and cylindical grid graphs:

$$\begin{aligned}
Z_{Mob}^{2M,2N} &= \prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\sin^2\left(\frac{(4n-1)\pi}{4N}\right) + 4\cos^2\left(\frac{m\pi}{2M+1}\right)\right] \\
Z_{Klein}^{2M,2N} &= \prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\sin^2\left(\frac{(4n-1)\pi}{4N}\right) + 4\sin^2\left(\frac{(2m-1)\pi}{2M}\right)\right]. \\
Z_{Cyl}^{2M,2N} &= \prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\sin^2\left(\frac{(2n-1)\pi}{2N}\right) + 4\cos^2\left(\frac{m\pi}{2M+1}\right)\right]
\end{aligned}$$

These formulas work only for even-by-even grids. The even-by-odd Möbius and Klein grids require the addition of imaginary edge weights, but the results for all three are similar:

$$Z_{Mob}^{M,N} = \mathrm{Re}\left[(1-i)\prod_{m=1}^{M/2}\prod_{n=1}^{N}\left(2i\,(-1)^{\frac{M}{2}+m+1}\sin\left(\frac{(4n-1)\pi}{2N}\right)+2\cos\left(\frac{m\pi}{M+1}\right)\right)\right]$$

$$Z_{Klein}^{M,N} = \mathrm{Re}\left[(1-i)\prod_{m=1}^{M/2}\prod_{n=1}^{N}\left(2i\,(-1)^{\frac{M}{2}+m+1}\sin\left(\frac{(4n-1)\pi}{2N}\right)+2\sin\left(\frac{(2m-1)\pi}{M}\right)\right)\right]$$

$$Z_{Cyl}^{M,N} = \frac{1}{2}\prod_{m=1}^{M/2}\prod_{n=1}^{(N+1)/2}\left[4\sin^2\left(\frac{(2m-1)\pi}{M}\right)+4\cos^2\left(\frac{n\pi}{N+1}\right)\right]$$

where M and N are assumed to be even and odd, respectively [10, 9].

The double product structure results from a similar step in the derivations. Since familiarity with the general method is useful to understanding the projective plane case, we now demonstrate with a presentation of Lu and Wu's proof for the Möbius strip [10].

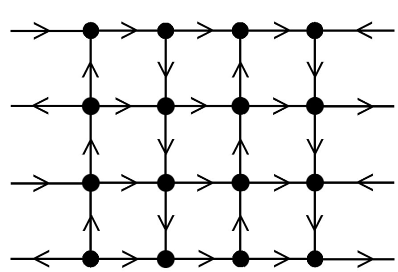
2.1 Classical Approach Applied to the Möbius Grid

Theorem 3. *[10]The number of domino tilings of a $2M \times 2N$ Möbius strip is equal to*

$$\prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\cos^2\left(\frac{m\pi}{2M+1}\right)+4\sin^2\left(\frac{(4n-1)\pi}{4N}\right)\right]. \tag{2.3}$$

Proof. Let G be a $2M \times 2N$ Möbius grid graph, drawn in the plane. The first step in the enumeration of domino tilings is to find an orientation of the edges of G that causes all summands of $\mathrm{Pf}\,(G)$ have the same sign.

There is a more detailed discussion of the identification of crossing orientations in Chapter 3. For now, we simply note that Möbius strips oriented in the following way have consistent signs across all summands of the Pfaffian:



This orientation corresponds to signed adjacency matrices taking the form of that in Figure 2.1:

$$
\left(\begin{array}{cccccccccccccccccccccccc}
0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
-1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0&0\\
1&0&0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&1&0&0&0&0&0&0\\
0&-1&0&0&0&0&-1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&-1&0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&-1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&-1&0&1&0&0&0&0&-1&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1&0
\end{array}\right)
$$

Figure 2.1: The 4×6 Möbius adjacency matrix

For general M and N, define the 2×2 matrices

$$
a(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a(0,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
a(1,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a(-1,0) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},
$$

the $N \times N$ analogs of the following matrices,

$$
F_N = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \qquad K_N = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix},
$$

and the $2M \times 2M$ analog of the matrix

$$
J_{2M} = \begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&1&0&0\\ -1&0&0&0 \end{pmatrix}.
$$

Let A be the signed adjacency matrix of G. Then we can write

$$
A = I_{2M} \otimes A_{2N} + \left[F_{2M} - F_{2M}^t\right] \otimes I_N \otimes a(0,1) + J_{2M} \otimes B_{2N}
$$

where

$$
A_{2N} = I_N \otimes a(0,0) + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0),
$$

$$
B_{2N} = -K_N \otimes a(1,0) + K_N^t \otimes a(-1,0),
$$

and to be explicit,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

We are interested in Pf A, but since $(\text{Pf}\, A)^2 = \det A$, the Pfaffian can easily be found from the determinant. The matrices J_{2M}, $F_{2M} - F_{2M}^t$, and I_{2M} commute and are simultaneously diagonalized by the $2M \times 2M$ matrix U:

$$\begin{aligned} U_{m,m'} &= i^m \sqrt{\frac{2}{2M+1}} \sin\left(\frac{mm'\pi}{2M+1}\right) \\ U^{-1}_{m,m'} &= (-i)^{m'} \sqrt{\frac{2}{2M+1}} \sin\left(\frac{mm'\pi}{2M+1}\right) \end{aligned}$$

for $m, m' = 1, 2, \ldots, 2M$ [10].

Conjugation by U yields diagonalizations

$$\begin{aligned} (U^{-1} I_{2M} U)_{m,m'} &= \delta_{m,m'} \\ (U^{-1} J_{2M} U)_{m,m'} &= i\,(-1)^{M+m} \delta_{m,m'} \\ (U(F_{2M} - F_{2M}^t) U^{-1})_{m,m'} &= (2i \cos\phi_m) \delta_{m,m'} \end{aligned}$$

for $m, m' = 1, 2, \ldots 2M$, where

$$\phi_m = \frac{m\pi}{2M+1}.$$

Conjugating A by $U_{2M} \otimes I_{2N}$, we find

$$\begin{aligned} \det A &= \left| I_{2M} \otimes A_{2N} + \left[F_{2M} - F_{2M}^t \right] \otimes I_N \otimes a(0,1) + J_{2M} \otimes B_{2N} \right| \\ &= \prod_{m=1}^{2M} \left| A_{2N} + 2i \cos\phi_m I_N \otimes a(0,1) + i\,(-1)^{M+m} B_{2N} \right|. \end{aligned} \tag{2.4}$$

Let T_N be the $N \times N$ matrix defined

$$T_N = F_N + i\,(-1)^{M+m+1} K_N = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ i(-1)^{M+m+1} & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Starting with the expression,

$$R_N := A_{2N} + 2i\cos\phi_m I_N \otimes a(0,1) + i\,(-1)^{M+m} B_{2N},$$

we substitute the definition of A_N to find

$$\begin{aligned}
R_N &= \Big[I_N \otimes a(0,0) + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \Big] + 2i\cos\phi_m I_N \otimes a(0,1) \\
&\quad + i\,(-1)^{M+m} B_{2N} \\
&= I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \\
&\quad + i\,(-1)^{M+m} B_{2N}.
\end{aligned}$$

Expanding B_{2N}, and continuing,

$$\begin{aligned}
R_N &= I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \otimes F_N^t \\
&\quad + i\,(-1)^{M+m} \Big[-K_N \otimes a(1,0) + K_N^t \otimes a(-1,0) \Big] \\
&= I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + \Big[F_N + (-1)^{M+m+1} i K_N \Big] \otimes a(1,0) \\
&\quad + \Big[F_N^t - (-1)^{M+m+1} i K_n^t \Big] \otimes a(-1,0) \\
&= I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + T_N \otimes a(1,0) + T_N^\dagger \otimes a(-1,0)
\end{aligned}$$

where $T_N^\dagger$ denotes the conjugate transpose of T_N.

Therefore, we rewrite (2.4) as

$$\det A = \prod_{m=1}^{2M} \left| I_N \otimes \Big[a(0,0) + (2i\cos\phi_m) a(0,1) \Big] + T_N \otimes a(1,0) + T_N^\dagger \otimes a(-1,0) \right|. \quad (2.5)$$

The matrices T_N and $T_N^\dagger$ commute and can be simultaneously diagonalized with eigenvalues $e^{i\theta_n}$ and $e^{-i\theta_n}$, where

$$\theta_n = (-1)^{M+m+1} \left[\frac{(4n-1)\pi}{2N} \right]$$

for $n = 1, 2 \ldots, N$.

Lu and Wu do not provide the matrices that diagonalize T and $T^\dagger$, but we will require them later so we derive them now. To find the eigenvectors of T, we identify the kernel of $T_N - tI_N$:

$$\begin{pmatrix} -t & 1 & 0 & \dots & 0 \\ 0 & -t & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ i(-1)^{M+m+1} & 0 & 0 & \dots & -t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

from which it easily follows that $x_i = t^{i-1}x_1$ for $i = 2, 3 \dots N$. A general eigenvector is therefore

$$\begin{pmatrix} 1 & t & t^2 & \dots & \dots & t^{n-1} \end{pmatrix}$$

and furthermore, any eigenvalue t will satisfy the equation

$$i(-1)^{M+m+1} - t^n = 0.$$

Let X_N be the $N \times N$ matrix that simultaneously diagonalizes T and $T^\dagger$. From the general eigenvector and the eigenvalues provided in [10],

$$X_{n,n'} = \left[e^{-i(n-1)\theta_{n'}} \right]$$

for $n, n' = 1 \dots N$. Since θ_n depends on m, we refer to $\theta_{n,m}$ when specificity is required. Similarly, we may refer to X_m, which will mean X_N evaluated at m, and will be understood to be $N \times N$ in dimension.

We can get X^{-1} from the left-eigenvectors of T. These are the vectors $\mathbf{v}$ that satisfy

$$\mathbf{v}(T - tI) = 0.$$

Proceeding in the same manner, we find that a general left-eigenvector takes form

$$\begin{pmatrix} \frac{1}{N} & \frac{t^{-1}}{N} & \dots & \dots & \frac{t^{-(n-1)}}{N} \end{pmatrix}.$$

The left-eigenvectors of T are the rows of X^{-1}, so using the same eigenvalues, we write

$$X^{-1}_{n,n'} = \left[\frac{1}{N} e^{i(n'-1)\theta_n} \right]$$

for $n, n' = 1 \dots N$. Conjugating A by

$$\bigoplus_{m=1}^{2M} X_m \otimes I_2,$$

we find

$$\begin{aligned} Z^{2M,2N}_{\text{Mob}} &= \sqrt{\det A} \\ &= \prod_{m=1}^{2M} \prod_{n=1}^{N} \left(\left| a(0,0) + 2i\, \cos\phi_m\, a(0,1) + a(1,0)\, e^{i\theta_n} + a(-1,0)\, e^{-i\theta_n} \right| \right)^{1/2} \\ &= \prod_{m=1}^{2M} \prod_{n=1}^{N} \left(\left| \begin{pmatrix} -2i\cos\phi_m & 1 - e^{-i\theta_n} \\ e^{i\theta_n} - 1 & 2i\cos\phi_m \end{pmatrix} \right| \right)^{1/2}. \end{aligned} \tag{2.6}$$

Evaluating the determinant,

$$\begin{aligned}\left|\begin{pmatrix}-2i\cos\phi_m & 1-e^{-i\theta_n}\\ e^{i\theta_n}-1 & 2i\cos\phi_m\end{pmatrix}\right| &= 4\cos^2\phi_m - (1-e^{-i\theta_n})(e^{i\theta_n}-1)\\ &= 4\cos^2\phi_m - e^{i\theta_n} + 2 - e^{i\theta_n}\\ &= 4\cos^2\phi_m - 2\cos\theta_n + 2\\ &= 4\cos^2\phi_m + 2\left(1-\cos\theta_n\right)\\ &= 4\cos^2\phi_m + 4\sin^2\frac{\theta_n}{2}.\end{aligned}$$

Accordingly, we rewrite (2.6):

$$Z_{\text{Mob}}^{2M,2N} = \prod_{m=1}^{2M}\prod_{n=1}^{N}\left(4\cos^2\phi_m + 4\sin^2\frac{\theta_n}{2}\right)^{1/2}. \tag{2.7}$$

To eliminate the square root, observe that

$$\begin{aligned}\cos^2\phi_m &= \cos^2\left(\frac{m\pi}{2M+1}\right)\\ &= \cos^2\left(\pi - \frac{m\pi}{2M+1}\right)\\ &= \cos^2\left(\frac{(2M-m+1)\pi}{2M+1}\right)\\ &= \cos^2\phi_{2M-m+1}.\end{aligned}$$

Since $\cos^2\phi_m = \cos^2\phi_{2M-m+1}$,

$$\begin{aligned}Z_{\text{Mob}}^{2M,2N} &= \prod_{m=1}^{2M}\prod_{n=1}^{N}\left(4\cos^2\phi_m + 4\sin^2\frac{\theta_n}{2}\right)^{1/2}\\ &= \prod_{m=1}^{M}\prod_{n=1}^{N}\left(4\cos^2\phi_m + 4\sin^2\frac{\theta_n}{2}\right)\\ &= \prod_{m=1}^{M}\prod_{n=1}^{N}\left[4\cos^2\left(\frac{m\pi}{2M+1}\right) + 4\sin^2\left(\frac{(4n-1)\pi}{4N}\right)\right].\end{aligned} \tag{2.8}$$

$\square$

Chapter 3

General Method for Non-Orientable Surfaces

The method used in the previous section requires adjustment for an even-by-odd Möbius strip, involving the introduction of imaginary weights on the horizontal edges of the base grid. Lu and Wu detail these steps in [9], but here we will describe a more general method for the enumeration of tilings on graphs embedded in compact boundaryless 2-manifolds, produced by Tesler [8].

Suppose we wish to enumerate perfect matchings on the graph G, which embeds in the compact 2-dimensional surface S. On a sheet of paper, draw the *pasting map* of S with the following modifications: label each arrowed side of the central polygon with the letter a, and affix matching subscripts to pasted sides. For each pair of pasted sides, if one arrow points clockwise and the other counterclockwise, give the counterclockwise-pointing side an exponent of -1. Figure 3.1 contains an illustration of these modifications on the Klein bottle.

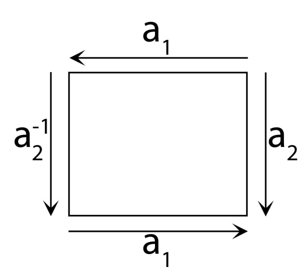


Figure 3.1: Pasting map for the Klein bottle with subscripts attached.

The surface S can now be expressed by a *pasting word*. If S is the Klein bottle, the word is $a_1a_2a_1a_2^{-1}$.

Let the sides of the pasting map associated with subscript j be called *j-sides*. Let S be *j-nonoriented* if the arrows on the j-sides point in the same direction (clockwise or counterclockwise) and *j-oriented* otherwise. Let S be *j,k-alternating* if the occurrences of a_j and a_k in the pasting word are interleaved, at any separation and without regard to superscript, and j, k-*nonalternating* otherwise. The Klein bottle above is $1, 2$-alternating, 1-nonoriented, and 2-oriented, as indicated by its pasting word.

Draw a representation of G on the pasting map for S, with all vertices lying in the polygon, and any crossings between edges at nonvertex points occurring outside the polygon. In this drawing, we refer to the edges protruding from a j-side as j-*edges*, and to the edges inside the rectangle as 0-*edges*. If S is j-oriented, j-edges of G are drawn without crossing each other. If S is j-nonoriented, the j-edges are drawn so that each pair of j-edges intersects once. Henceforth, "G" refers specifically to G drawn this way. For an example, see Figure 3.2. With respect Figure 3.1, the 1-edges are in black and the 2-edges in blue.

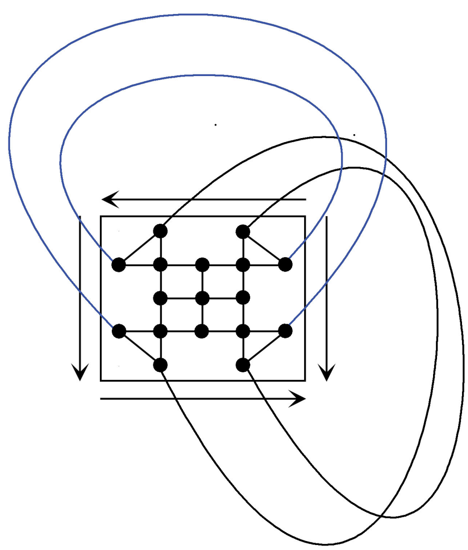


Figure 3.2: A graph on the Klein bottle.

Let a *superposition cycle* be a loop of alternating dominoes formed by the superposition of two domino tilings of G. For example, the superposition of the tilings in Figure 3.3 creates two superposition cycles.

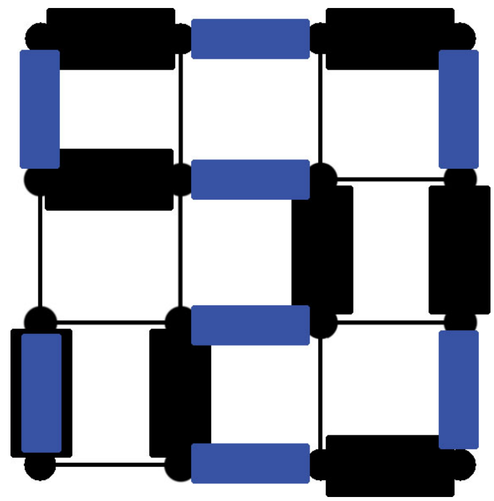


Figure 3.3: Two cycles resulting from superposed tilings of the 4×4 grid.

By Tesler's definition,

Definition 3. *A crossing orientation of a graph is an orientation in which for every superposition cycle C, $r(C) + \kappa_e(C) + \iota(C)$ is odd, where r, κ_e and ι respectively stand for the routing number, number of monochromatic crossings, and the number of vertices enclosed by C, to be defined below.*

The routing number is the number of edges directed in opposition to a clockwise traversal of C. The only requirement for a crossing orientation of a nonplanar graph is that the routing number be odd for every superposition cycle.

If the dominoes in C are colored in alternation, the number of monochromatic crossings is the number of intersections between dominoes of the same color. The graph above is planar; it has no crossing edges or monochromatic crossings. For an example of a superposition cycle with a monochromatic crossing, see Figure 3.4.

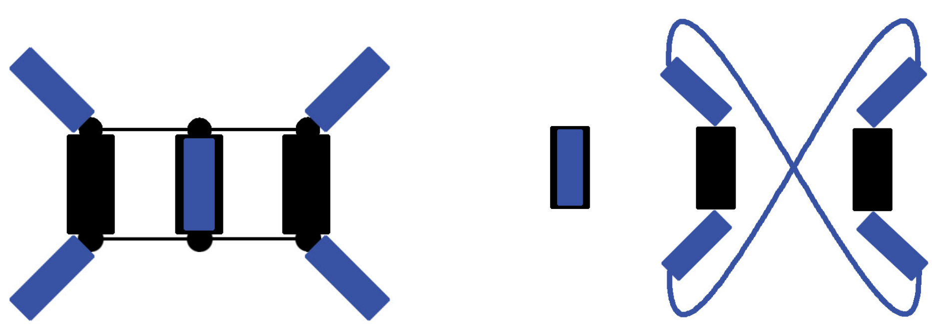


Figure 3.4: Left, a superposition of tilings on the 2×3 Möbius strip. Right, the two (separated) resulting superposition cycles. The cycle on the right has a monochromatic crossing.

Finally, let a vertex v be *inside* a cycle C if the winding number of C about v is odd, and *outside* C otherwise. In Figure 3.3, $\iota(C) = 0$ for both cycles.

Tesler carries out a proof of the following theorem in his work [8]:

Theorem 4. *(a) A graph can be oriented so that every perfect matching has sign*

$$\epsilon_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{\kappa(\mathbf{m})} W_{\mathbf{m}} \tag{3.1}$$

where $\epsilon_0 = \pm 1$ is constant, $\kappa(\mathbf{m})$ is the number of intersections between matched crossing edges in $\mathbf{m}$, and $W_{\mathbf{m}}$ is the product of the weights of the matched edges.

(b) An orientation of a graph satisfies (a) if and only if it is a crossing orientation.

Given the existence of crossing orientations, Tesler presents the following:

Theorem 5. *The number of perfect matchings of a crossing-oriented graph can be computed as a linear combination of Pfaffians of modified signed adjacency matrices.*

Proof. Let A be the signed adjacency matrix of the crossing-oriented graph G. Consider a perfect matching $\mathbf{m}$ of G, and let $N_{\mathbf{m}}(j)$ be the number of j-edges in $\mathbf{m}$. Let $C_m(j,k)$ be the number of crossings formed by a j-edge with a k-edge. Then,

$$C_{\mathbf{m}}(j,j) = \begin{cases} \binom{N_{\mathbf{m}}(j)}{2} & \text{if } S \text{ is } j\text{-nonoriented} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

$$C_{\mathbf{m}}(j,k) = \begin{cases} N_{\mathbf{m}}(j) \cdot N_{\mathbf{m}}(k) & \text{if } S \text{ is } j,k\text{-alternating} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Let n be the number of pairs of pasted sides in the pasting map of S. The total number of intersections between crossing edges of such a graph is

$$C_{\mathbf{m}} = \sum_{1 \le j \le k \le n} C_{\mathbf{m}}(j,k) \tag{3.4}$$

As a consequence of Theorem 4, every perfect matching has weight

$$w_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{C_{\mathbf{m}}} W_{\mathbf{m}} \tag{3.5}$$

in Pf A, where $W_{\mathbf{m}}$ is the unsigned product of the weights of all edges in $\mathbf{m}$. The approach will be to find a linear combination of Pfaffians of re-weightings of A, yielding collective weight

$$\epsilon_0 W_{\mathbf{m}}$$

for each perfect matching.

Form the *x-adjacency matrix* $B(x_1 \dots x_n)$ by multiplying the weight of each j-edge in G by the indeterminate x_j and forming the signed adjacency matrix of the resulting re-weighted graph using the same orientation as for A. Let $f(\omega_1, \dots, \omega_n) \in \mathbb{C}[\omega_1, \dots, \omega_n]/(1-\omega_1^4, \dots 1-\omega_n^4)$ be defined

$$f(\omega_1, \dots, \omega_n) = \sum_{1 \le s_1 \dots s_n \le 4} \alpha_s \omega_1^{s_1} \cdots \omega_n^{s_n}.$$

In other words, f is a multivariate polynomial in $\omega_1, \dots, \omega_n$ in which all exponents have been reduced modulo 4.

Define the f-weight of $\mathbf{m}$ to be

$$w_{\mathbf{m}}(f) = f(i^{N_{\mathbf{m}}(1)}, \dots, i^{N_{\mathbf{m}}(n)}) \cdot w_{\mathbf{m}} \tag{3.6}$$

and the f-weight of G to be

$$\begin{aligned} W_G(f) &= \sum_{s \in K} \alpha_s \mathrm{Pf}\, B(i^{s_1}, \dots, i^{s_n}) \\ &= \sum_{s \in K} \alpha_s \sum_{\mathbf{m}} w_{\mathbf{m}} \cdot i^{s_1 N_m(1)} \dots i^{s_n N_{\mathbf{m}}(n)} \\ &= \sum_{\mathbf{m}} w_{\mathbf{m}}(f). \end{aligned}$$

Lemma 1. *For $0 < j < k \le n$, let*

$$L_{jj} = \begin{cases} \frac{1-i}{2}(\omega_j + i\omega_j^{-1}) & \text{if } S \text{ is } j\text{-nonoriented} \\ 1 & \text{otherwise} \end{cases} \tag{3.7}$$

$$L_{jk} = \begin{cases} \frac{1}{2}(1 + \omega_j^2 + \omega_k^2 - \omega_j^2\omega_k^2) & \text{if } S \text{ is } j,k\text{-alternating} \\ 1 & \text{otherwise.} \end{cases} \tag{3.8}$$

Then

$$\begin{aligned} &(a) \quad w_{\mathbf{m}}(L_{jj} \cdot f) = (-1)^{C_{\mathbf{m}}(j,j)} w_{\mathbf{m}}(f). \\ &(b) \quad w_{\mathbf{m}}(L_{jk} \cdot f) = (-1)^{C_{\mathbf{m}}(j,k)} w_{\mathbf{m}}(f). \end{aligned}$$

Proof. [8]
(a) If S is j-oriented, then $C_{\mathbf{m}}(j,j) = 0$, and $L_{jj} = 1$, so (a) reduces to $w_{\mathbf{m}}(f) = w_{\mathbf{m}}(f)$. Suppose then that S is j-nonoriented. Since the unsigned weight of a perfect matching is the product of the weights of its edges, multiplying the weights of the j-edges by a number α multiplies $w_{\mathbf{m}}$ by $\alpha^{N_{\mathbf{m}}(j)}$. As can be seen from (3.6),

$$w_{\mathbf{m}}(\omega_j^s f) = i^{N_{\mathbf{m}}(j)s} w_{\mathbf{m}}(f)$$

and likewise,

$$w_{\mathbf{m}}\left(\left(\frac{1-i}{2}\omega_j + \frac{1+i}{2}\omega_j^{-1}\right) f\right) = \left(\frac{1-i}{2} i^{N_{\mathbf{m}}(j)} + \frac{1+i}{2}\, i^{-N_{\mathbf{m}}(j)}\right) w_{\mathbf{m}}(f).$$

It can be checked by cases that

$$\frac{1-i}{2}(i^N) + \frac{1+i}{2}(i^{-N}) = (-1)^{\binom{N}{2}} = \begin{cases} 1 & \text{if } N \cong 0 \text{ or } 1 \mod 4 \\ -1 & \text{if } N \cong 2 \text{ or } 3 \mod 4. \end{cases} \tag{3.9}$$

Therefore, (a) holds by (3.2).

(b) If S is not j,k-alternating, then $C_{\mathbf{m}}(j,k) = 0$ and $L_{jk} = 1$, so (b) holds. If S is j,k-alternating, then

$$w_{\mathbf{m}}(L_{jk}f) = \frac{1}{2}\left(1 + (-1)^{N_{\mathbf{m}}(j)} + (-1)^{N_{\mathbf{m}}(k)} - (-1)^{N_{\mathbf{m}}(j)+N_{\mathbf{m}}(k)}\right) w_{\mathbf{m}}(f). \tag{3.10}$$

It can be checked by cases that

$$\begin{aligned} W &:= \frac{1}{2}\left(1 + (-1)^{N_{\mathbf{m}}(j)} + (-1)^{N_{\mathbf{m}}(k)} - (-1)^{N_{\mathbf{m}}(j)+N_{\mathbf{m}}(k)}\right) \\ &= (-1)^{N_{\mathbf{m}}(j)\cdot N_{\mathbf{m}}(k)} \\ &= \begin{cases} -1 & \text{if } N_{\mathbf{m}}(j) \text{ and } N_{\mathbf{m}}(k) \text{ are both odd} \\ 1 & \text{if } N_{\mathbf{m}}(j) \text{ or } N_{\mathbf{m}}(k) \text{ is even,} \end{cases} \end{aligned}$$

and so (b) holds by (3.3) and Lemma 1 holds in all cases.

□

By Lemma 1,

$$w_{\mathbf{m}}\left(\prod_{1\le j\le k\le n} L_{jk}\right) = \left(\prod_{1\le j\le k\le n} (-1)^{C_{\mathbf{m}}(j,k)}\right) w_{\mathbf{m}}(1) = (-1)^{C_{\mathbf{m}}} w_{\mathbf{m}}(1).$$

By (3.6), $w_{\mathbf{m}}(1) = w_{\mathbf{m}}$. By (3.5), $w_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{C_{\mathbf{m}}} W_{\mathbf{m}}$. Therefore,

$$\begin{aligned} w_{\mathbf{m}}\left(\prod_{1\le j\le k\le n} L_{jk}\right) &= (-1)^{C_{\mathbf{m}}} \epsilon_0 \cdot (-1)^{C_{\mathbf{m}}} W_{\mathbf{m}} \\ &= \epsilon_0 W_{\mathbf{m}}. \end{aligned}$$

Since

$$W_G(f) = \sum_{\mathbf{m}} w_{\mathbf{m}}(f),$$

we conclude that

$$W_G\left(\prod_{1\le j\le k\le n} L_{jk}\right)$$

is a linear combination of Pfaffians that counts each perfect matching with weight $\epsilon_0 W_{\mathbf{m}}$, as desired.

□

We have just described Tesler's rocedure for identifying a linear combination of Pfaffians to enumerate perfect matchings on a crossing-oriented graph. To follow through on the Klein bottle, recall that the surface is $1,2$-alternating and 1-nonoriented. Accordingly, we evaluate

$$\begin{aligned}
f(\omega_1,\omega_2) &= \frac{1-i}{2}(\omega_1 + i\omega_1^{-1}) \cdot \frac{1}{2}(1+\omega_1^2+\omega_2^2-\omega_1^2\omega_2^2) \\
&= \frac{1-i}{4}\left(\omega_1+\omega_1^3+\omega_1\omega_2^2-\omega_1^3\omega_2^2+i\omega_1^{-1}+i\omega_1+i\omega_1^{-1}\omega_2^2-i\omega_1\omega_2^2\right).
\end{aligned}$$

Reducing exponents modulo 4,

$$\begin{aligned}
f(\omega_1,\omega_2) &= \frac{1-i}{4}\left(\omega_1+\omega_1^3+\omega_1\omega_2^2-\omega_1^3\omega_2^2+i\omega_1^3+i\omega_1+i\omega_1^3\omega_2^2-i\omega_1\omega_2^2\right) \\
&= \frac{1-i}{4}\left(\omega_1+\omega_1^3+\omega_1\omega_2^2-\omega_1^3\omega_2^2+i\omega_1^3+i\omega_1+i\omega_1^3\omega_2^2-i\omega_1\omega_2^2\right) \\
&= \frac{1-i}{4}\left((1+i)(\omega_1+\omega_1^3)+(1-i)(\omega_1\omega_2^2-\omega_1^3\omega_2^2)\right) \\
&= \frac{1}{2}(\omega_1+\omega_1^3-i\omega_1\omega_2^2+i\omega_1^3\omega_2^2).
\end{aligned}$$

Computing the f-weight of G,

$$\begin{aligned}
W_G(f(\omega_1,\omega_2)) &= \frac{1}{2}(\mathrm{Pf}\,B(i^1,i^0)+\mathrm{Pf}\,B(i^3,i^0)-i\mathrm{Pf}\,B(i^1,i^2)+i\mathrm{Pf}\,B(i^3,i^2)) \\
&= \frac{1}{2}(\mathrm{Pf}\,B(i,1)+\mathrm{Pf}\,B(-i,1)-i\mathrm{Pf}\,B(i,-1)+i\mathrm{Pf}\,B(-i,-1)).
\end{aligned}$$

The last expression will compute the number of perfect matchings of a graph embedded in the Klein bottle.

With the derivation of these linear combinations established, let G be a projective plane graph of dimension $M \times N$, with M even. Since the projective plane is symmetric, and an odd-by-odd projective plane, having an odd number of vertices, cannot be tiled, we adopt the practice of always taking M to be even. We will use the terms *even* or *odd projective plane* in reference to the parity of N. The graph G will be 4×3 in the illustrations that follow, but as with the Möbius strips, the crossing orientations follow a regular pattern in higher dimensional graphs. Figure 3.5 shows the graph G with a superimposed crossing orientation.

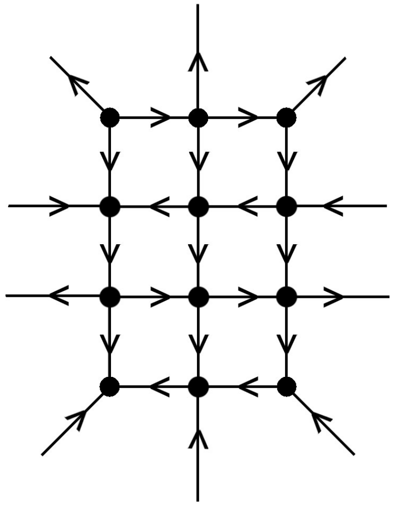


Figure 3.5: Crossing orientation for the 4×3 projective grid. The general crossing orientation extends this pattern.

Let $B(x_1, x_2)$ be the corresponding signed x-adjacency matrix:

$$B(x_1,x_2) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x_1 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2x_1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -x_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & x_2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2x_1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ -2x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

We associate the projective plane with the pasting word $a_1a_2a_1a_2$, and observe that it is 1-nonoriented, 2-nonoriented, and $1,2$-alternating. Accordingly, we compute the product

$$\begin{aligned} f(\omega,\omega_2) &= \frac{1-i}{2}(\omega_1 + i\omega_1^{-1}) \cdot \frac{1-i}{2}(\omega_2 + i\omega_2^{-1}) \cdot \frac{1}{2}(1 + \omega_1^2 + \omega_2^2 - \omega_1^2\omega_2^2) \\ &= \frac{-i}{4}(2\omega_1\omega_2 + 2i\omega_1\omega_2 + 2i\omega_1^3\omega_2^3 - 2\omega_1^3\omega_2^3) \end{aligned}$$

substituting computing the f-weight,

$$\begin{aligned}
W_G(f(\omega_1,\omega_2)) &= \frac{-i}{4}(2\operatorname{Pf} B(i,i) + 2i\operatorname{Pf} B(i,i) + 2i\operatorname{Pf} B(-i,-i) - 2\operatorname{Pf} B(-i,-i)) \\
&= \frac{-i}{2}(i+1)\mathrm{Pf}\, B(i,i) - \frac{i}{2}(i-1)\mathrm{Pf}\, B(-i,-i) \\
&= \frac{1-i}{2}\mathrm{Pf}\, B(i,i) + \frac{1+i}{2}\mathrm{Pf}\, B(-i,-i),
\end{aligned}$$

and since these matrices are complex conjugates,

$$W_G(f(\omega_1,\omega_2)) = \mathrm{Re}(1-i)\mathrm{Pf}\, B(i,i). \tag{3.11}$$

In the case of the projective plane, $x_1 = x_2$ when $B(x_1,x_2)$ is evaluated, so hereafter we will simply write $B(x)$.

$$B(i) = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2i & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -i & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & -i & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & i & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & i & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -2i & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\
-2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0
\end{pmatrix}$$

Up to sign, the number of tilings of the $M \times N$ projective plane for even M is equal to $\mathrm{Re}\,(1-i)\mathrm{Pf}\, B(i)$.

Chapter 4

Application of Classical Methods to the Projective Plane

We now attempt to apply the methods from the previous two chapters to the projective grid graph. Using Tesler's methods, we found in the last chapter that the expression

$$\mathrm{Re}\,(1-i)\mathrm{Pf}\,B(i)$$

would produce the number of perfect matchings of an even or odd projective plane. In the even case, this is equal to $\mathrm{Pf}\,B(1)$, which is more similar to Lu and Wu's Möbius adjacency matrix. We begin with this matrix so as to borrow their techniques to the greatest possible extent.

Let G be a $2M \times 2N$ projective plane grid graph, drawn in the plane. By exhaustive search by a computer, I identified the crossing orientation of the projective plane shown in Figure 4.1.

A general $M \times N$ projective grid, when drawn as in Figure 4.1, can be crossing-oriented as follows:

1. Direct the top row of horizontal edges in the underlying grid to the right, then direct subsequent rows left and right in alternation until the bottom is reached.

2. Direct all vertical edges of the underlying grid downward.

3. Direct crossing edges on the top side of the graph outward. Direct crossing edges on the left side of the graph inward and outward in alternation, beginning with an edge directed into the $(N+1)$st vertex.

4. Direct the bottom and right crossing edges to be consistent with the left and top.

The 4×5 projective plane adjacency matrix is shown in Figure 4.1:

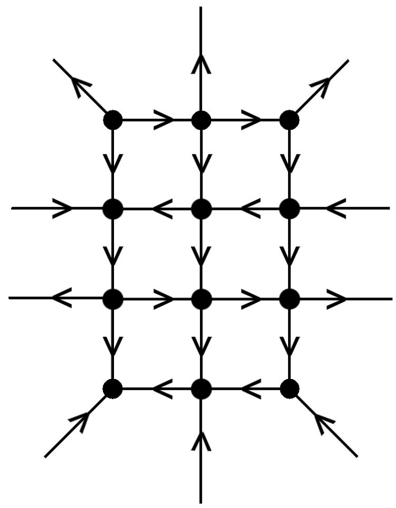


Figure 4.1: Crossing orientation of the 4×3 projective grid.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Figure 4.2: The 4×5 projective grid adjacency matrix.

As in the previous section, define the 2×2 matrices

$$
a(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a(0,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
a(1,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a(-1,0) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}
$$

$$
a(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

the $N \times N$ analogs of the following matrices,

$$F_N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the $2M \times 2M$ analog of this matrix:

$$J_{2M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$A = I_{2M} \otimes A_{2N} + \left[F_{2M} - F_{2M}^t\right] \otimes I_N \otimes a(0,1) + J_{2M} \otimes B_{2N} + (K_{2M} - K_{2M}^t) \otimes L_N \otimes a(1,1)$$

where

$$A_{2N} = I_N \otimes a(0,0) + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0),$$

$$B_{2N} = -\, K_N \otimes a(1,0) + K_N^t \otimes a(-1,0).$$

The decomposition of A given above is the same as that given for the Möbius grid in Chapter (2), with the addition of a single term representing the additional set of crossing edges.

As before, we simultaneously conjugate $J_{2M}, F_{2M} = F_{2M}^t$, and I_{2M} by U_{2M}:

$$\begin{aligned} U_{m,m'} &= i^m \sqrt{\frac{2}{2M+1}} \sin\left(\frac{mm'\pi}{2M+1}\right) \\ U^{-1}_{m,m'} &= (-i)^{m'} \sqrt{\frac{2}{2M+1}} \sin\left(\frac{mm'\pi}{2M+1}\right) \end{aligned}$$

to obtain diagonalizations

$$\begin{aligned} (U^{-1} I_{2M} U)_{m,m'} &= \delta_{m,m'} \\ (U^{-1} J_{2M} U)_{m,m'} &= i\,(-1)^{M+m} \delta_{m,m'} \\ (U(F_{2M} - F_{2M}^t) U^{-1})_{m,m'} &= (2i \cos\phi_m) \delta_{m,m'} \end{aligned}$$

for $m, m' = 1, 2, \ldots 2M$, where

$$\phi_m = \frac{m\pi}{2M+1}.$$

We now examine the effect of conjugation on $K_{2M} - K^t_{2M}$. To start,

$$U^{-1}K_{2M} = U^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{2M+1}}\sin\left(\frac{\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ \sqrt{\frac{2}{2M+1}}\sin\left(\frac{2\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ -\sqrt{\frac{2}{2M+1}}\sin\left(\frac{3\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{\frac{2}{2M+1}}\sin\left(\frac{2M\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -\sqrt{\frac{2}{2M+1}}\sin\left(\frac{\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ \sqrt{\frac{2}{2M+1}}\sin\left(\frac{2\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ -\sqrt{\frac{2}{2M+1}}\sin\left(\frac{3\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{\frac{2}{2M+1}}\sin\left(\frac{2M\pi}{2M+1}\right) & 0 & \dots & 0 & 0 \end{pmatrix} U_{2M} = \left[(-1)^m \frac{2i}{2M+1}\sin\left(\frac{m\pi}{2M+1}\right)\sin\left(\frac{m'\pi}{2M+1}\right)\right]$$

for $m, m' = 1 \dots 2M$.

So,

$$(U^{-1}_{2M}K_{2M}U_{2M})_{m,m'} = \left[(-1)^m \frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'}\right]$$

and as it turns out,

$$(U^{-1}_{2M}K^t_{2M}U_{2M})_{m,m'} = \left[(-1)^{m'+1} \frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'}\right].$$

This leads to

$$
\begin{aligned}
Z_{2M} &:= (U_{2M}^{-1}(K_{2M} - K_{2M}^t)U_{2M})_{m,m'} \\
&= \left[\frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'} - (-1)^{m'+1}\frac{2i}{2M+1}\sin\phi_m\sin\phi_{m'}\right] \\
&= \left[(-1)^m\frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'} + (-1)^{m'}\frac{2i}{2M+1}\sin\phi_m\sin\phi_{m'}\right] \\
&= \left[(-1)^m\frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'} + (-1)^{m'}\frac{2i}{2M+1}\sin\phi_m\sin\phi_{m'}\right] \\
&= \left[(-1)^m\frac{2i}{2M+1}\sin\phi_m \sin\phi_{m'} + (-1)^{m'}\frac{2i}{2M+1}\sin\phi_m\sin\phi_{m'}\right] \\
&= \left[\frac{2i\sin\phi_m\sin\phi_{m'}}{2M+1}\left((-1)^m + (-1)^{m'}\right)\right].
\end{aligned}
$$

The term $\left((-1)^m + (-1)^{m'}\right)$ makes Z_{2M} a checkerboard matrix. We refer to its entries by

$$
z_{m,m'} = \left[\frac{2i\sin\phi_m\sin\phi_{m'}}{2M+1}\left((-1)^m + (-1)^{m'}\right)\right]. \tag{4.1}
$$

Having traced the effect of conjugation by U on $K_{2M} - K_{2M}^t$, we now conjugate all $2M$-dimensional matrices in the decomposition of A by U and write

$$
\begin{aligned}
\det A = &\Big|\left[\delta_{m,m'}\right]\otimes A_{2N} + \left[2i\cos\phi_m\delta_{m,m'}\right]\otimes I_N\otimes a(0,1) + \left[i\,(-1)^{M+m}\delta_{m,m'}\right]\otimes B_{2N} \\
&+ Z_{2M}\otimes L_N\otimes a(1,1)\Big| \\
= &\left|\bigoplus_{m=1}^{2M}\left[A_{2N} + 2i\cos\phi_m I_N\otimes a(0,1) + i\,(-1)^{M+m}B_{2N}\right] + Z_{2M}\otimes L_N\otimes a(1,1)\right|.
\end{aligned} \tag{4.2}
$$

Let the $N\times N$ matrix T_N be defined

$$
T_N = F_N + i\,(-1)^{M+m+1}K_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ i(-1)^{M+m+1} & 0 & 0 & \dots & 0 \end{pmatrix}.
$$

Substituting the definition of A_{2N} into (4.2),

$$\begin{aligned}
\det A = \Bigg| \bigoplus_{m=1}^{2M} \Bigg[\Bigg[\Big[I_N \otimes a(0,0) + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \Big] + 2i\cos\phi_m I_N \otimes a(0,1) \\
+ i\,(-1)^{M+m} B_{2N} \Bigg] + Z_{2M} \otimes L_{2N} \Bigg| \\
= \Bigg| \bigoplus_{m=1}^{2M} \Bigg[I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \\
+ i\,(-1)^{M+m} B_{2N} \Bigg] + Z_{2M} \otimes L_N \otimes a(1,1) \Bigg|,
\end{aligned}$$

expanding B_{2N},

$$\begin{aligned}
= \Bigg| \bigoplus_{m=1}^{2M} \Bigg[I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + F_N \otimes a(1,0) + F_N^t \otimes a(-1,0) \otimes F_N^t \\
+ i\,(-1)^{M+m} \Big[- K_N \otimes a(1,0) + K_N^t \otimes a(-1,0) \Big] \Bigg] + Z_{2M} \otimes L_N \otimes a(1,1) \Bigg| \\
= \Bigg| \bigoplus_{m=1}^{2M} \Bigg(I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + \Big[F_N + (-1)^{M+m+1} i K_N \Big] \otimes a(1,0) \\
+ \Big[F_N^t - (-1)^{M+m+1} i K_n^t \Big] \otimes a(-1,0) \Bigg) + Z_{2M} \otimes L_{2N} \Bigg| \\
= \Bigg| \bigoplus_{m=1}^{2M} \Bigg[I_N \otimes \Big[a(0,0) + 2i\cos\phi_m a(0,1) \Big] + T_N \otimes a(1,0) + T_N^\dagger \otimes a(-1,0) \Bigg] \\
+ Z_{2M} \otimes L_N \otimes a(1,1) \Bigg|.
\end{aligned}$$

We found in Chapter (2) that T_N and $T_N^\dagger$ are simultaneously diagonalized by the matrix X_N, with eigenvalues $e^{i\theta_n}$ and $e^{-i\theta_n}$, where

$$\begin{aligned}
X_{n,n'} &= \Big[e^{-i(n-1)\theta_{n'}} \Big], \\
X_{n,n'}^{-1} &= \Big[\frac{1}{N} e^{i(n'-1)\theta_n} \Big],
\end{aligned}$$

and

$$\theta_n = (-1)^{M+m+1} \left[\frac{(4n-1)\pi}{2N} \right]$$

for $n = 1, 2 \ldots, N$. As before, note that θ_n depends implicitly on m. When m is unclear from context, we will refer to $\theta_{n,m}$. The matrix X_N inherits this dependence on m, so we use X_m to indicate the matrix above evaluated at $\theta_{n,m}$, which will be understood to be $N \times N$ in dimension.

We must now examine the effect of conjugation by X_N on L_N. To do so, we adopt the notation

$$\lambda_n = e^{-i\theta_n}$$

and

$$c = \frac{1}{N}.$$

Again, we will sometimes need to specify

$$\lambda_{n,m} = e^{-i\theta_{n,m}}.$$

Then,

$$X_N^{-1} L_N = \begin{pmatrix} c & c\lambda_1^{-1} & c\lambda_1^{-2} & \dots & \dots & c\lambda_1^{1-N} \\ c & c\lambda_2^{-1} & c\lambda_2^{-2} & \dots & \dots & c\lambda_2^{1-N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c & c\lambda_N^{-1} & c\lambda_N^{-2} & \dots & \dots & c\lambda_N^{1-N} \end{pmatrix} \begin{pmatrix} 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & \iddots & & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 \\ 1 & 0 & & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c\lambda_1^{1-N} & c\lambda_1^{2-N} & \dots & \dots & c\lambda_1^{-1} & c \\ c\lambda_2^{1-N} & c\lambda_2^{2-N} & \dots & \dots & c\lambda_2^{-1} & c \\ c\lambda_3^{1-N} & c\lambda_3^{2-N} & \dots & \dots & c\lambda_3^{-1} & c \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c\lambda_N^{1-N} & c\lambda_N^{2-N} & \dots & \dots & c\lambda_N^{-1} & c \end{pmatrix}$$

and so,

$$X_N^{-1} L_N X_N = \begin{pmatrix} c\lambda_1^{1-N} & c\lambda_1^{2-N} & \dots & \dots & c\lambda_1^{-1} & c \\ c\lambda_2^{1-N} & c\lambda_2^{2-N} & \dots & \dots & c\lambda_2^{-1} & c \\ c\lambda_3^{1-N} & c\lambda_3^{2-N} & \dots & \dots & c\lambda_3^{-1} & c \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c\lambda_N^{1-N} & c\lambda_N^{2-N} & \dots & \dots & c\lambda_N^{-1} & c \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \dots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \dots & \lambda_n^{n-1} \end{pmatrix}$$

$$= \Omega_N$$

where Ω_N is an $N \times N$ matrix with

$$\begin{aligned}\Omega_{n,n'} &= c\left(\lambda_n^{1-N}, \lambda_n^{2-N}, \ldots, 1\right) \cdot \left(1, \lambda_{n'}, \lambda_{n'}^2, \ldots \lambda_{n'}^{N-1}\right) \\ &= \sum_{k=1}^{N} \lambda_n^{-(N-k)} \lambda_{n'}^{k-1} \\ &= \sum_{k=1}^{N} e^{i(N-k)\theta_n} e^{-i(k-1)\theta_{n'}} \\ &= \sum_{k=1}^{N} \left(e^{i\theta_n}\right)^{N-k} \left(e^{i\theta_{n'}}\right)^{-(k-1)}.\end{aligned}$$

When necessary, we will refer to Ω^m, with entries defined

$$\Omega^m_{n,n'} = \sum_{k=1}^{N} (e^{i\theta_{n,m}})^{N-k} (e^{i\theta_{n',m}})^{-(k-1)}.$$

The next objective is to conjugate the matrix by

$$\bigoplus_{m=1}^{2M} [X_m \otimes I_2],$$

but things will be easier to consider if we leave out the 2×2 component and focus on the conjugation of $Z_{2M} L_N$ by

$$\bigoplus_{m=1}^{2M} X_m.$$

The matrix Z_{2M} is a checkerboard matrix that is symmetric about the main diagonal and conjugate-symmetric about the main anti-diagonal:

$$\begin{pmatrix}
z_{1,1} & 0 & z_{1,3} & 0 & \cdots & \cdots & z_{1,2M-3} & 0 & z_{1,2M-1} & 0 \\
0 & z_{2,2} & 0 & z_{2,4} & \cdots & \cdots & 0 & z_{2,2M-2} & 0 & z^*_{1,2M-1} \\
z_{1,3} & 0 & z_{3,3} & 0 & \cdots & \cdots & z_{3,2M-3} & 0 & z^*_{2,2M-2} & 0 \\
0 & z_{2,4} & 0 & z_{4,4} & \cdots & \cdots & 0 & z^*_{3,2M-3} & 0 & z^*_{1,2M-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \iddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \iddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
z_{1,2M-3} & 0 & z_{3,2M-3} & 0 & \cdots & \cdots & z^*_{4,4} & 0 & z^*_{2,4} & 0 \\
0 & z_{2,2M-2} & 0 & z^*_{3,2M-3} & \cdots & \cdots & 0 & z^*_{3,3} & 0 & z^*_{1,3} \\
z_{1,2M-1} & 0 & z^*_{2,2M-2} & 0 & \cdots & \cdots & z^*_{2,4} & 0 & z^*_{2,2} & 0 \\
0 & z^*_{1,2M-1} & 0 & z^*_{1,2M-3} & \cdots & \cdots & 0 & z^*_{1,3} & 0 & z^*_{1,1}
\end{pmatrix}.$$

To avoid problems with notation, we proceed with the case when $M = 4$:

$$\begin{pmatrix}
z_{1,1} & 0 & z_{1,3} & 0 & z_{1,5} & 0 & z_{1,7} & 0 \\
0 & z_{2,2} & 0 & z_{2,4} & 0 & z_{2,6} & 0 & z^*_{1,7} \\
z_{1,3} & 0 & z_{3,3} & 0 & z_{3,5} & 0 & z^*_{2,6} & 0 \\
0 & z_{2,4} & 0 & z_{4,4} & 0 & z^*_{3,5} & 0 & z^*_{1,5} \\
z_{1,5} & 0 & z_{3,5} & 0 & z^*_{4,4} & 0 & z^*_{2,4} & 0 \\
0 & z_{2,6} & 0 & z^*_{3,5} & 0 & z^*_{3,3} & 0 & z^*_{1,3} \\
z_{1,7} & 0 & z^*_{2,6} & 0 & z^*_{2,4} & 0 & z^*_{2,2} & 0 \\
0 & z^*_{1,7} & 0 & z^*_{1,5} & 0 & z^*_{1,3} & 0 & z^*_{1,1}
\end{pmatrix}.$$

The block matrix $Z_{2M} \otimes L_N$ takes form

$$\begin{pmatrix} z_{1,1}L & 0 & z_{1,3}L & 0 & z_{1,5}L & 0 & z_{1,7}L & 0 \\ 0 & z_{2,2}L & 0 & z_{2,4}L & 0 & z_{2,6}L & 0 & z_{1,7}^*L \\ z_{1,3}L & 0 & z_{3,3}L & 0 & z_{3,5}L & 0 & z_{2,6}^*L & 0 \\ 0 & z_{2,4}L & 0 & z_{4,4}L & 0 & z_{3,5}^*L & 0 & z_{1,5}^*L \\ z_{1,5}L & 0 & z_{3,5}L & 0 & z_{4,4}^*L & 0 & z_{2,4}^*L & 0 \\ 0 & z_{2,6}L & 0 & z_{3,5}^*L & 0 & z_{3,3}^*L & 0 & z_{1,3}^*L \\ z_{1,7}L & 0 & z_{2,6}^*L & 0 & z_{2,4}^*L & 0 & z_{2,2}^*L & 0 \\ 0 & z_{1,7}^*L & 0 & z_{1,5}^*L & 0 & z_{1,3}^*L & 0 & z_{1,1}^*L \end{pmatrix},$$

and

$$\bigoplus_{m=1}^{2M} X_m^{-1}(Z_{2M} \otimes L_N) \bigoplus_{m=1}^{2M} X_m$$

takes form

$$\begin{pmatrix} z_{1,1}\Omega^1 & 0 & z_{1,3}\Omega^1 & 0 & z_{1,5}\Omega^1 & 0 & z_{1,7}\Omega^1 & 0 \\ 0 & z_{2,2}\Omega^2 & 0 & z_{2,4}\Omega^2 & 0 & z_{2,6}\Omega^2 & 0 & z_{1,7}^*\Omega^2 \\ z_{1,3}\Omega^3 & 0 & z_{3,3}\Omega^3 & 0 & z_{3,5}\Omega^3 & 0 & z_{2,6}^*\Omega^3 & 0 \\ 0 & z_{2,4}\Omega^4 & 0 & z_{4,4}\Omega^4 & 0 & z_{3,5}^*\Omega^4 & 0 & z_{1,5}^*\Omega^4 \\ z_{1,5}\Omega^5 & 0 & z_{3,5}\Omega^5 & 0 & z_{4,4}^*\Omega^5 & 0 & z_{2,4}^*\Omega^5 & 0 \\ 0 & z_{2,6}\Omega^6 & 0 & z_{3,5}^*\Omega^6 & 0 & z_{3,3}^*\Omega^6 & 0 & z_{1,3}^*\Omega^6 \\ z_{1,7}\Omega^7 & 0 & z_{2,6}^*\Omega^7 & 0 & z_{2,4}^*\Omega^7 & 0 & z_{2,2}^*\Omega^7 & 0 \\ 0 & z_{1,7}^*\Omega^8 & 0 & z_{1,5}^*\Omega^8 & 0 & z_{1,3}^*\Omega^8 & 0 & z_{1,1}^*\Omega^8 \end{pmatrix}.$$

Let

$$\Sigma_{2MN} = \bigoplus_{m=1}^{2M} X_m^{-1}(Z_{2M} \otimes L_N) \bigoplus_{m=1}^{2M} X_m,$$

as above. Then Σ_{2MN} is a $2M$-block by $2M$-block block-checkerboard matrix with block dimensions $N \times N$ that retains the symmetries of Z_{2M} at the block level. Revisiting the determinant equation,

$$\begin{aligned} \det A &= \left| \bigoplus_{m=1}^{2M} \left[I_N \otimes \left[a(0,0) + 2i\cos\phi_m a(0,1) \right] + T_N \otimes a(1,0) + T_N^\dagger \otimes a(-1,0) \right] \right. \\ &\qquad \left. + Z_{2M} \otimes L_N \otimes a(1,1) \right| \\ &= \left| \bigoplus_{m=1}^{2M} \bigoplus_{n=1}^{N} \begin{pmatrix} -2i\cos\phi_m & 1 - e^{-i\theta_n} \\ e^{i\theta_n} - 1 & 2i\cos\phi_m \end{pmatrix} + \Sigma_{2MN} \otimes a(1,1) \right|. \end{aligned}$$

We now have the determinant of A as the determinant of the sum of a block 2×2 diagonal matrix and a block $2N \times 2N$ checkerboard matrix.
Let

$$\omega_{i,j}^k = z_{i,j}\Omega^k.$$

Conjugation by the permutation matrix $P \otimes I_N$, where P is the $2M$-dimensional permutation matrix generated by

$$[(1, 2M)(3, 2(M-1)), \ldots (M, M+1)],$$

transforms the matrix Σ as follows:

$$\Sigma = \begin{pmatrix}
\omega^1_{1,1} & 0 & \omega^1_{1,3} & 0 & \omega^1_{1,5} & 0 & \omega^1_{1,7} & 0 \\
0 & \omega^2_{2,2} & 0 & \omega^2_{2,4} & 0 & \omega^2_{2,6} & 0 & \omega^{2*}_{1,7} \\
\omega^3_{1,3} & 0 & \omega^3_{3,3} & 0 & \omega^3_{3,5} & 0 & \omega^{3*}_{2,6} & 0 \\
0 & \omega^4_{2,4} & 0 & \omega^4_{4,4} & 0 & \omega^{4*}_{3,5} & 0 & \omega^{4*}_{1,5} \\
\omega^5_{1,5} & 0 & \omega^5_{3,5} & 0 & \omega^{5*}_{4,4} & 0 & \omega^{5*}_{2,4} & 0 \\
0 & \omega^6_{2,6} & 0 & \omega^{6*}_{3,5} & 0 & \omega^{6*}_{3,3} & 0 & \omega^{6*}_{1,3} \\
\omega^7_{1,7} & 0 & \omega^{7*}_{2,6} & 0 & \omega^{7*}_{2,4} & 0 & \omega^{7*}_{2,2} & 0 \\
0 & \omega^{8*}_{1,7} & 0 & \omega^{8*}_{1,5} & 0 & \omega^{8*}_{1,3} & 0 & \omega^{8*}_{1,1}
\end{pmatrix}$$

$$(P \otimes I_N)\Sigma_{2MN}(P \otimes I_N) = \begin{pmatrix}
\omega^{8*}_{1,1} & \omega^{8*}_{1,7} & \omega^{8*}_{1,3} & \omega^{8*}_{1,5} & 0 & 0 & 0 & 0 \\
\omega^{2*}_{1,7} & \omega^2_{2,2} & \omega^2_{2,6} & \omega^2_{2,4} & 0 & 0 & 0 & 0 \\
\omega^{6*}_{1,3} & \omega^6_{2,6} & \omega^{6*}_{3,3} & \omega^{6*}_{3,5} & 0 & 0 & 0 & 0 \\
\omega^{4*}_{1,5} & \omega^4_{2,4} & \omega^{4*}_{3,5} & \omega^4_{4,4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^{5*}_{4,4} & \omega^5_{3,5} & \omega^{5*}_{2,4} & \omega^5_{1,5} \\
0 & 0 & 0 & 0 & \omega^3_{3,5} & \omega^3_{3,3} & \omega^{3*}_{2,6} & \omega^3_{1,3} \\
0 & 0 & 0 & 0 & \omega^{7*}_{2,4} & \omega^{7*}_{2,6} & \omega^{7*}_{2,2} & \omega^7_{1,7} \\
0 & 0 & 0 & 0 & \omega^1_{1,5} & \omega^1_{1,3} & \omega^1_{1,7} & \omega^1_{1,1}
\end{pmatrix}.$$

The matrix $\Sigma_{2MN} \otimes a(1,1)$ has $2N \times 2N$ blocks running down the diagonal. The Möbius component of A at this stage is block 2×2 diagonal, and so may as well be considered block $2N \times 2N$ diagonal, and so the entire matrix may be considered block-checkerboard. Letting S equal the Möbius component and following through with the transformation above on the whole matrix, we find that

$$\begin{aligned}
\det A &= \left| S + \Sigma_{2MN} \otimes a(1,1) \right| \\
&= \det \begin{pmatrix}
\omega^1_{11} + S_1 & 0 & \omega^1_{13} & 0 & \omega^1_{15} & 0 & \omega^1_{17} & 0 \\
0 & \omega^2_{22} + S_2 & 0 & \omega^2_{24} & 0 & \omega^2_{26} & 0 & \omega^{2*}_{17} \\
\omega^3_{13} & 0 & \omega^3_{33} + S_3 & 0 & \omega^3_{35} & 0 & \omega^{3*}_{26} & 0 \\
0 & \omega^4_{24} & 0 & \omega^4_{44} + S_4 & 0 & \omega^{4*}_{35} & 0 & \omega^{4*}_{15} \\
\omega^5_{15} & 0 & \omega^5_{35} & 0 & \omega^{5*}_{44} + S_5 & 0 & \omega^{5*}_{24} & 0 \\
0 & \omega^6_{26} & 0 & \omega^{6*}_{35} & 0 & \omega^{6*}_{33} + S_6 & 0 & \omega^{6*}_{13} \\
\omega^7_{17} & 0 & \omega^{7*}_{26} & 0 & \omega^{7*}_{24} & 0 & \omega^{7*}_{22} + S_7 & 0 \\
0 & \omega^{8*}_{17} & 0 & \omega^8_{15} & 0 & \omega^{8*}_{13} & 0 & \omega^{8*}_{11} + S_8
\end{pmatrix} \\
&= \det \begin{pmatrix}
\omega^{8*}_{11} + S_8 & \omega^{8*}_{17} & \omega^{8*}_{13} & \omega^{8*}_{15} & 0 & 0 & 0 & 0 \\
\omega^{2*}_{17} & \omega^2_{22} + S_2 & \omega^2_{26} & \omega^2_{24} & 0 & 0 & 0 & 0 \\
\omega^{6*}_{13} & \omega^6_{26} & \omega^{6*}_{33} + S_6 & \omega^{6*}_{35} & 0 & 0 & 0 & 0 \\
\omega^{4*}_{15} & \omega^4_{24} & \omega^{4*}_{35} & \omega^4_{44} + S_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^{5*}_{44} + S_5 & \omega^5_{35} & \omega^{5*}_{24} & \omega^5_{15} \\
0 & 0 & 0 & 0 & \omega^3_{35} & \omega^3_{33} + S_3 & \omega^{3*}_{26} & \omega^3_{13} \\
0 & 0 & 0 & 0 & \omega^{7*}_{24} & \omega^{7*}_{26} & \omega^{7*}_{22} + S_7 & \omega^7_{17} \\
0 & 0 & 0 & 0 & \omega^1_{15} & \omega^1_{13} & \omega^1_{17} & \omega^1_{11} + S_1
\end{pmatrix},
\end{aligned}$$

where each entry represents a $2N \times 2N$ block, and the entries from $\Sigma \otimes a(1,1)$ are $N \times N$ matrices tensored with $a(1,1)$. The larger $2MN \times 2MN$ blocks each have determinant equal to the Pfaffian of the original matrix, so we narrow our consideration to the upper block:

$$\begin{pmatrix}
\omega^{8*}_{11} + S_8 & \omega^{8*}_{17} & \omega^{8*}_{13} & \omega^{8*}_{15} \\
\omega^{2*}_{17} & \omega^2_{22} + S_2 & \omega^2_{26} & \omega^2_{24} \\
\omega^{6*}_{13} & \omega^6_{26} & \omega^6_{33} + S_6 & \omega^{6*}_{35} \\
\omega^{4*}_{15} & \omega^4_{24} & \omega^{4*}_{35} & \omega^4_{44} + S_4
\end{pmatrix}.$$

After swapping rows and columns to get the elements of S in the original order, this becomes
$$\begin{pmatrix} \omega_{22}^{2}+S_2 & \omega_{24}^{2} & \omega_{26}^{2} & \omega_{17}^{2*} \\ \omega_{24}^{4} & \omega_{44}^{4}+S_4 & \omega_{35}^{4*} & \omega_{15}^{4*} \\ \omega_{26}^{6} & \omega_{35}^{6*} & \omega_{33}^{6*}+S_6 & \omega_{13}^{6*} \\ \omega_{17}^{8*} & \omega_{15}^{8*} & \omega_{13}^{8*} & \omega_{11}^{8*}+S_8 \end{pmatrix}.$$
Recall that
$$S = \bigoplus_{m=1}^{2M} \bigoplus_{n=1}^{N} \begin{pmatrix} -2i\cos\phi_m & 1-e^{-i\theta_n} \\ e^{i\theta_n}-1 & 2i\cos\phi_m \end{pmatrix}.$$
Considering the entries from $\Sigma_{2MN} \otimes a(1,1)$ as tensor products of $N \times N$ blocks, and S as an $N \times N$ block-diagonal matrix, we expand this to

$$\det \begin{pmatrix} s_{21} & \omega_{22}^{2}+s_{22} & 0 & \omega_{24}^{2} & 0 & \omega_{26}^{2} & 0 & \omega_{17}^{2*} \\ \omega_{22}^{2}+s_{23} & s_{24} & \omega_{24}^{2} & 0 & \omega_{26}^{2} & 0 & \omega_{17}^{2*} & 0 \\ 0 & \omega_{24}^{4} & s_{41} & \omega_{44}^{4}+s_{42} & 0 & \omega_{35}^{4*} & 0 & \omega_{15}^{4*} \\ \omega_{24}^{4} & 0 & \omega_{44}^{4}+s_{43} & s_{44} & \omega_{35}^{4*} & 0 & \omega_{15}^{4*} & 0 \\ 0 & \omega_{26}^{6} & 0 & \omega_{35}^{6*} & s_{61} & \omega_{33}^{6*}+s_{62} & 0 & \omega_{13}^{6*} \\ \omega_{26}^{6} & 0 & \omega_{35}^{6*} & 0 & \omega_{33}^{6*}+s_{63} & s_{64} & \omega_{13}^{6*} & 0 \\ 0 & \omega_{17}^{8*} & 0 & \omega_{15}^{8*} & 0 & \omega_{13}^{8*} & s_{81} & \omega_{11}^{8*}+s_{82} \\ \omega_{17}^{8*} & 0 & \omega_{15}^{8*} & 0 & \omega_{13}^{8*} & 0 & \omega_{11}^{8*}+s_{83} & s_{84} \end{pmatrix}.$$

Conjugating again by the appropriate-dimensional version of P, this matrix transforms to

$$\begin{pmatrix} s_{84} & 0 & 0 & 0 & \omega_{13}^{8*} & \omega_{15}^{8*} & \omega_{11}^{8*}+s_{83} & \omega_{17}^{8*} \\ 0 & s_{24} & 0 & 0 & \omega_{26}^{2} & \omega_{24}^{2} & \omega_{17}^{2*} & \omega_{22}^{2}+s_{23} \\ 0 & 0 & s_{64} & 0 & \omega_{33}^{6*}+s_{63} & \omega_{35}^{6*} & \omega_{13}^{6*} & \omega_{26}^{6} \\ 0 & 0 & 0 & s_{44} & \omega_{35}^{4*} & \omega_{44}^{4}+s_{43} & \omega_{15}^{4*} & \omega_{24}^{4} \\ \omega_{13}^{6*} & \omega_{26}^{6} & \omega_{33}^{6*}+s_{62} & \omega_{35}^{6*} & s_{61} & 0 & 0 & 0 \\ \omega_{15}^{4*} & \omega_{24}^{4} & \omega_{35}^{4*} & \omega_{44}^{4}+s_{42} & 0 & s_{41} & 0 & 0 \\ \omega_{11}^{8*}+s_{82} & \omega_{17}^{8*} & \omega_{13}^{8*} & \omega_{15}^{8*} & 0 & 0 & s_{81} & 0 \\ \omega_{17}^{2*} & \omega_{22}^{2}+s_{22} & \omega_{26}^{2} & \omega_{24}^{2} & 0 & 0 & 0 & s_{21} \end{pmatrix}.$$

With another series of row and column operations to get the elements of S on the diagonals of the dense blocks, this becomes

$$\begin{pmatrix} s_{84} & 0 & 0 & 0 & \omega_{11}^{8*}+s_{83} & \omega_{17}^{8*} & \omega_{13}^{8*} & \omega_{15}^{8*} \\ 0 & s_{24} & 0 & 0 & \omega_{17}^{2*} & \omega_{22}^{2}+s_{23} & \omega_{26}^{2} & \omega_{24}^{2} \\ 0 & 0 & s_{64} & 0 & \omega_{13}^{6*} & \omega_{26}^{6} & \omega_{33}^{6*}+s_{63} & \omega_{35}^{6*} \\ 0 & 0 & 0 & s_{44} & \omega_{15}^{4*} & \omega_{24}^{4} & \omega_{35}^{4*} & \omega_{44}^{4}+s_{43} \\ \omega_{11}^{8*}+s_{82} & \omega_{17}^{8*} & \omega_{13}^{8*} & \omega_{15}^{8*} & s_{81} & 0 & 0 & 0 \\ \omega_{17}^{2*} & \omega_{22}^{2}+s_{22} & \omega_{26}^{2} & \omega_{24}^{2} & 0 & s_{21} & 0 & 0 \\ \omega_{13}^{6*} & \omega_{26}^{6} & \omega_{33}^{6*}+s_{62} & \omega_{35}^{6*} & 0 & 0 & s_{61} & 0 \\ \omega_{15}^{4*} & \omega_{24}^{4} & \omega_{35}^{4*} & \omega_{44}^{4}+s_{42} & 0 & 0 & 0 & s_{41} \end{pmatrix}. \tag{4.3}$$

This is equal to

$$a(0,1) \otimes \Big[\bigoplus_{m=1}^{M} \bigoplus_{n=1}^{N} (2i\cos\phi_m) \Big] + a(1,1) \otimes \begin{pmatrix} \omega_{13}^{6*} & \omega_{26}^{6} & \omega_{33}^{6*} & \omega_{35}^{6*} \\ \omega_{15}^{4*} & \omega_{24}^{4} & \omega_{35}^{4*} & \omega_{44}^{4} \\ \omega_{11}^{8*} & \omega_{17}^{8*} & \omega_{13}^{8*} & \omega_{15}^{8*} \\ \omega_{17}^{2*} & \omega_{22}^{2} & \omega_{26}^{2} & \omega_{24}^{2} \end{pmatrix}$$
$$+ a(1,1) \otimes \Big[\bigoplus_{m=1}^{M} \bigoplus_{n=1}^{N} (e^{i\theta_n}-1) \Big] - a(-1,0) \otimes \Big[\bigoplus_{m=1}^{M} \bigoplus_{n=1}^{N} (e^{i\theta_n}-1) \Big].$$

For a 2×2 block matrix of form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

it is true in general [6] that if D is invertible,

$$\det M = \det(A - BD^{-1}C) \det D.$$

Applying this to our present situation, we find the determinant of the block in the lower right of the matrix above is 1, so the problem can be reduced to finding the determinant of the difference between the upper left block of our matrix and the conjugation of the lower right block's inverse by the off-diagonal blocks. We observe that this matrix is skew-Hermitian, and so the original problem can be reduced to the computation of the determinant of a skew-Hermitian matrix of dimension $MN \times MN$, but we have not made further progress with this approach.

Chapter 5

Experimental Approach

We further explore the structure of the signed adjacency matrices.

5.0.1 Summary of Results

Conjecture 1. *For even M and N, every $M \times N$ projective plane grid graph, $M \times N$ Möbius grid graph, and $M \times N$ rectangular grid graph can be associated with a finite sequence ("Aitken list") of rational numbers of form*

$$1,\ a_2,\ \frac{a_3}{a_2}, \ldots \frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \frac{a_{n+3}}{a_{n+2}}, \ldots, \frac{a_{MN}}{a_{MN-1}}$$

in which a_{MN} is the number of perfect matchings. The product of the list is also the number of perfect matchings. Alternatively, such a graph can be associated with the corresponding integer sequence ("Aitken sequence")

$$1, a_{\frac{N}{2}-1} \ldots a_n, a_{n+1}, a_{n+2}, a_{n+3} \ldots, a_{MN}$$

which terminates in the number of perfect matchings.

Conjecture 2. *The following statements are believed to hold for every $M \times N$ rectangle, Möbius strip, and projective plane grid graph for even M and N:*

1. *The first $\frac{N}{2}$ elements of the Aitken sequence are equal, and equal either ± 1, depending on the crossing orientation used for the adjacency matrix.*

2. *The Aitken sequence is strictly increasing in absolute value.*

3. *The elements of the Aitken lists cluster into M distinct "chunks" (see Figures 5.3-5.8)*

4. *The final element of the Aitken list is the largest in absolute value.*

5. *The $M \times N$ rectangle, Möbius strip, and projective plane agree on the first $\frac{MN}{2}$ elements of their respective Aitken lists.*

6. *The Möbius strip and projective plane differ only in the final N elements of their Aitken lists.*

Conjecture 3. *In the case of the $2 \times N$ projective plane, with N even,*

1. *If the Aitken sequence is taken to start at the final element of the first chunk, the Nth element of the sequence is $U_{n-1}(2)$.*

2. *With the exception of the final element, the second chunk of the Aitken list converges to $2 + \sqrt{3}$.*

3. *As N increases, the final element of the $2 \times 2N$ Aitken list converges to $2+2\sqrt{3}$.*

4. *As N increases, the final element of the $2n \times N$ Aitken list converges to $2+2\sqrt{3}$.*

5. *The kth element of the $2n \times 2$ Aitken sequence is the kth Fibonacci number.*

6. *The final element of the Aitken sequence is equal to $2\,U_{n-1}(2) - 2\,U_{n-2}(2)$.*

Conjecture 4. *In the case of the $4 \times N$ projective plane, when N is even,*

1. *The $\frac{N}{2}$ through Nth elements of the Aitken sequence are given by*

$$U_k\left(\frac{3}{2}\right) - U_{k-1}\left(\frac{3}{2}\right)$$

   for $k \geq 1$.

2. *The ith element of the first chunk of a $4 \times N$ Aitken sequence is the number of domino tilings of the $2 \times 2i$ rectangular grid.*

Conjecture 5. *In the $M \times (2N - 1)$ projective plane,*

1. *The elements of the Aitken list are typically complex.*

2. *The real parts of the elements of the Aitken list are very close to the elements of the corresponding $M \times 2N$ Aitken list.*

3. *The complex parts of the Aitken list are typically very small, with all the variation happening at a small number of localized points.*

5.0.2 Method

Recall from previous sections that the signed adjacency matrix of any directed graph is skew symmetric. If A is a skew-symmetric matrix, then by the LDL^t decomposition, there exists a lower triangular matrix L and block-diagonal matrix D such that

$$A = LDL^t. \tag{5.1}$$

In particular, the matrix D has form

$$\begin{pmatrix}
0 & \lambda_0 & & & & & & & & & & & & & & \\
-\lambda_0 & 0 & & & & & & & & & & & & & & \\
& & 0 & \lambda_1 & & & & & & & & & & & & \\
& & -\lambda_1 & 0 & & & & & & & & & & & & \\
& & & & 0 & \lambda_2 & & & & & & & & & & \\
& & & & -\lambda_2 & 0 & & & & & & & & & & \\
& & & & & & \ddots & \ddots & & & & & & & & \\
& & & & & & \ddots & \ddots & & & & & & & & \\
& & & & & & & & \ddots & \ddots & & & & & & \\
& & & & & & & & \ddots & \ddots & & & & & & \\
& & & & & & & & & & 0 & \lambda_{k-2} & & & & \\
& & & & & & & & & & -\lambda_{k-3} & 0 & & & & \\
& & & & & & & & & & & & 0 & \lambda_{k-1} & & \\
& & & & & & & & & & & & -\lambda_{k-1} & 0 & & \\
& & & & & & & & & & & & & & 0 & \lambda_k \\
& & & & & & & & & & & & & & -\lambda_k & 0
\end{pmatrix}$$

where zeros have been omitted. The λs will be real if A is real, and complex if A is complex. Since column and row operations are determinant-preserving, and $(\operatorname{Pf} A)^2 = \det A$, D and A have the same Pfaffian up to sign. The determinant of a block diagonal matrix is the product of the determinants of the diagonal blocks, and each diagonal block of D clearly has determinant λ_i^2. Therefore,

$$\det A = \prod_{i=1}^{k} \lambda_i^2$$

and

$$\operatorname{Pf} A = \prod_{i=1}^{k} \lambda_i.$$

When the LDL^t decomposition is applied to the signed adjacency matrix of a projective plane, the off-diagonal elements take interesting values. The 4×4 case is shown below:

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ & & 0 & 1 \\ & & -1 & 0 \\ & & & & 0 & -2 \\ & & & & 2 & 0 \\ & & & & & & 0 & -\frac{5}{2} \\ & & & & & & \frac{5}{2} & 0 \\ & & & & & & & & 0 & \frac{7}{5} \\ & & & & & & & & -\frac{7}{5} & 0 \\ & & & & & & & & & & 0 & \frac{13}{7} \\ & & & & & & & & & & -\frac{13}{7} & 0 \\ & & & & & & & & & & & & 0 & -\frac{46}{13} \\ & & & & & & & & & & & & \frac{46}{13} & 0 \\ & & & & & & & & & & & & & & 0 & -\frac{228}{46} \\ & & & & & & & & & & & & & & \frac{228}{46} & 0 \end{pmatrix}.$$

In general, the off-diagonals are rational for even-by-even matrices (the *even case*) and complex for even-by-odd (the *odd case*). The patterns are easier to recognize in the even case, so we turn our attention there, for now.

In reference to the 4×4 matrix above, notice that the nonzero entries of the 2×2 blocks are inverses, and so are described by the list

$$1, 1, -2, -\frac{5}{2}, \frac{7}{5}, \frac{13}{7}, -\frac{46}{13}, -\frac{228}{46}$$

and the negatives thereof. We refer to this list as the *Aitken list*, and make some observations:

1. The denominator of each element of an even Aitken list is the numerator of the previous element. We refer to this as the *numerator-denominator pattern.*

2. The numerator of the final element is the Pfaffian of the associated matrix.

3. Aitken lists can be associated with *Aitken sequences*, which are integer sequences describing the list, given knowledge of (1). In the 4×4 case, the Aitken sequence is

$$1, 2, 5, 7, 13, 46, 228$$

4. As a consequence of (2), an Aitken sequence terminates in the number of perfect matchings of the associated surface.

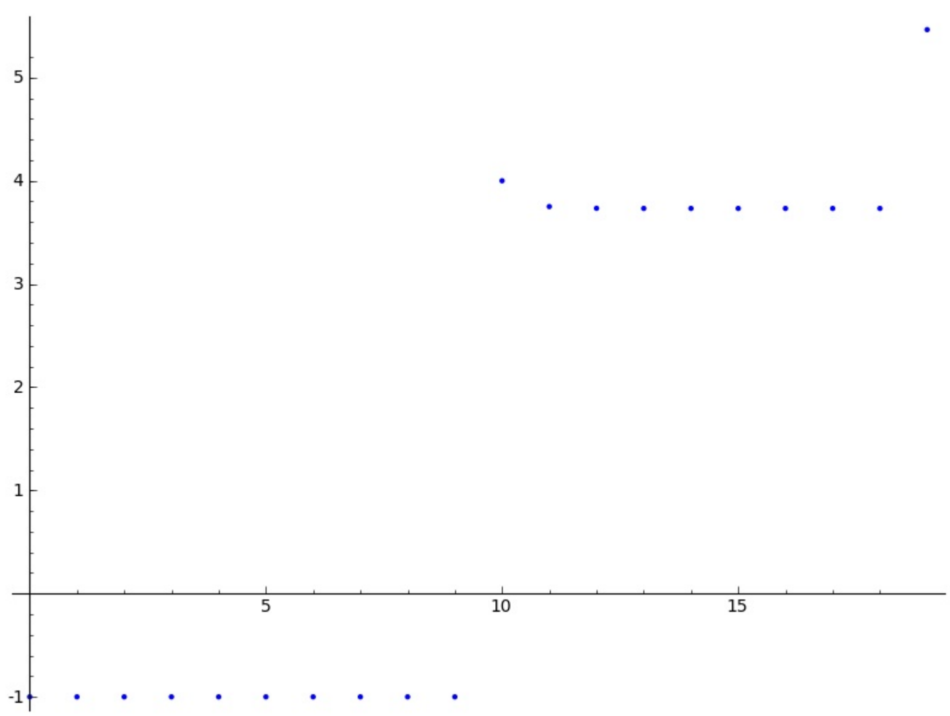
The Aitken block decomposition algorithm, after which these lists are named, is in the appendix.

Continuing the investigation, we examine the Aitken list for the 2×20 projective plane:

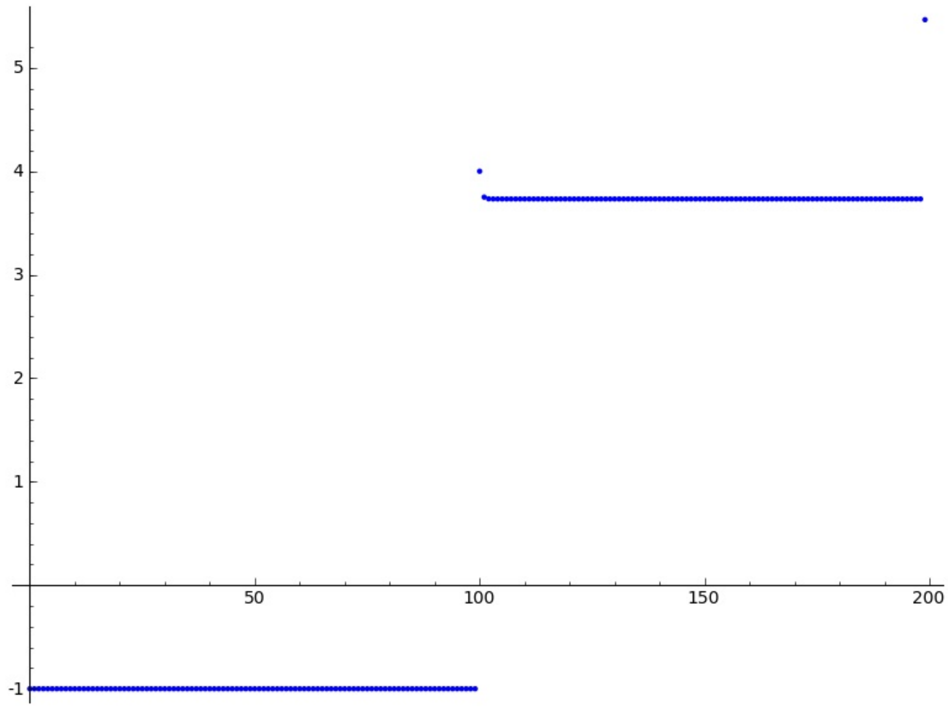
$$-1, \ldots, -1, 4, \frac{15}{4}, \frac{56}{15}, \frac{209}{56}, \frac{780}{209}, \frac{2911}{780}, \frac{10864}{2911}, \frac{40545}{10864}, \frac{151316}{40545}, \frac{826806}{151316}$$

where I have replaced eight copies of -1 with dots.

Here is a plot of the Aitken values:



For comparison, here is the Aitken plot of the 2×200 projective plane:



The final Aitken values of these two lists are, respectively,

$$\frac{826806}{151316} \text{ and } \frac{1234960030599837928682339736709998512373739432964939784153}{226013371928974192395830842015030781678432595000982162108}$$

with numerical approximations 5.46410161516297 and 5.46410161513775. The numerators of these fractions are the numbers of perfect matchings of the corresponding projective grids. A WolframAlpha search for the second value indicates that it is approximately equal to $2 + 2\sqrt{3}$. The second to last Aitken value has numerical approximation $3.73205080756888 \approx 2+\sqrt{3}$. This leads us to believe that as N increases, the final element of a $2 \times 2N$ Aitken list converges to $2 + 2\sqrt{3}$ and the sequence up to that element converges to $2 + \sqrt{3}$.

Turning back to the 2×20 example, observe that the list is divided into two sign-separated chunks of length 10. In general, the list for an $M \times N$ projective plane will contain M chunks of length $\frac{N}{2}$, but the potential difference in signs depends on the crossing orientation used. For example, Tesler and Lu and Wu use different crossing orientations of the Möbius strip. Here are the Aitken plots corresponding to their respective orientations of the 4×10 Möbius grids:

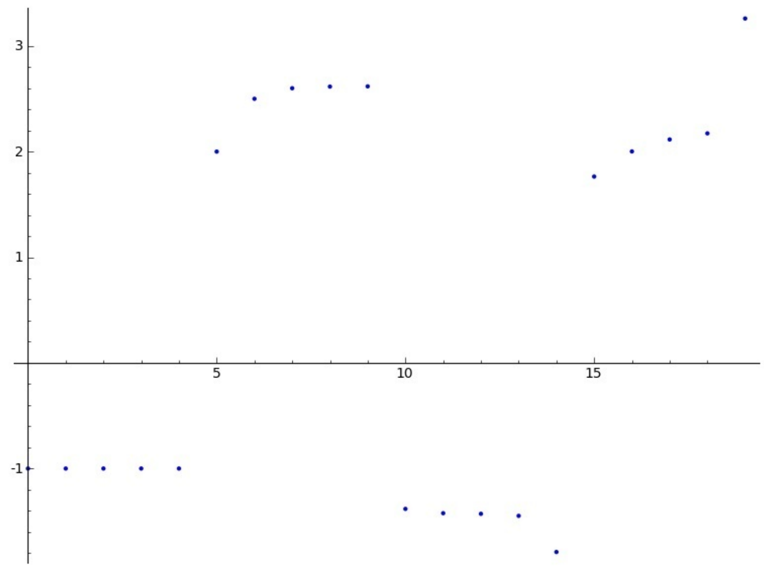


Figure 5.1: Aitken plot resulting from Tesler's crossing orientation of the Möbius strip

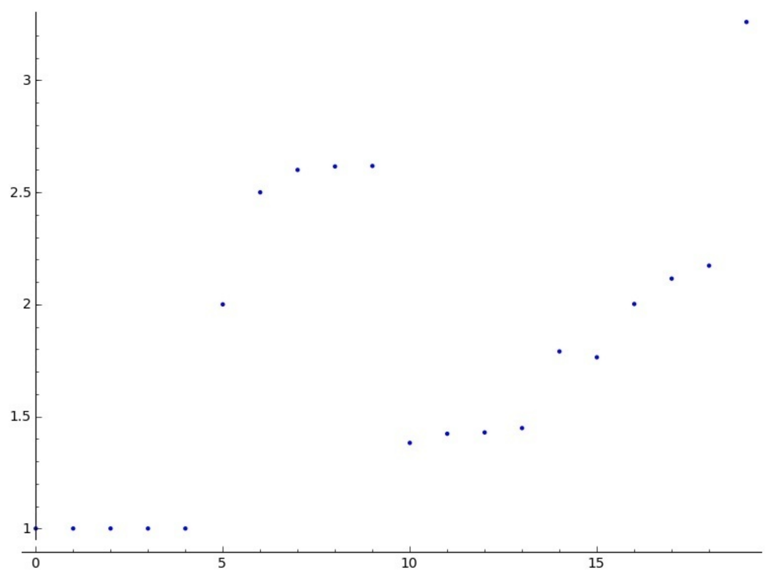


Figure 5.2: Aitken plot resulting from Lu and Wu's crossing orientation of the Möbius strip

The corresponding Aitken lists are

$$-1, \ldots, -1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, -\frac{123}{89}, -\frac{175}{123}, -\frac{10}{7}, -\frac{181}{125}, -\frac{324}{181}, \frac{127}{72}, \frac{2288}{1143}, \frac{4839}{2288}, \frac{10514}{4839}, \frac{17138}{5257}$$

and

$$1, \ldots, 1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, \frac{123}{89}, \frac{175}{123}, \frac{10}{7}, \frac{181}{125}, \frac{324}{181}, \frac{127}{72}, \frac{2288}{1143}, \frac{4839}{2288}, \frac{10514}{4839}, \frac{17138}{5257}.$$

where the ellipses represent three of the appropriate number. The Aitken lists differ in sign, but the corresponding integer sequences are identical.

The Aitken sequence for the 2×20 plane is

$$1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 826806.$$

All but the final value match to the sequence $A001353$ in OEIS. The sequence is defined by the recurrence

$$a_n = 4\, a_{n-1} - a_{n-2},$$

with $a_0 = 0$ and $a_1 = 1$. Incidentally, the same recurrence was found in the combinatorial approach to describe the number of tilings of the $2 \times 2N$ projective plane, only with $a_0 = 4, a_1 = 6$. One formula listed on OEIS is

$$a(n) = U_{n-1}(2).$$

The final element a_n in an Aitken sequence seems to equal $a_n = 6\, a_{n-1} - 2\, a_{n-2}$. In the context of the formula above, this would give us

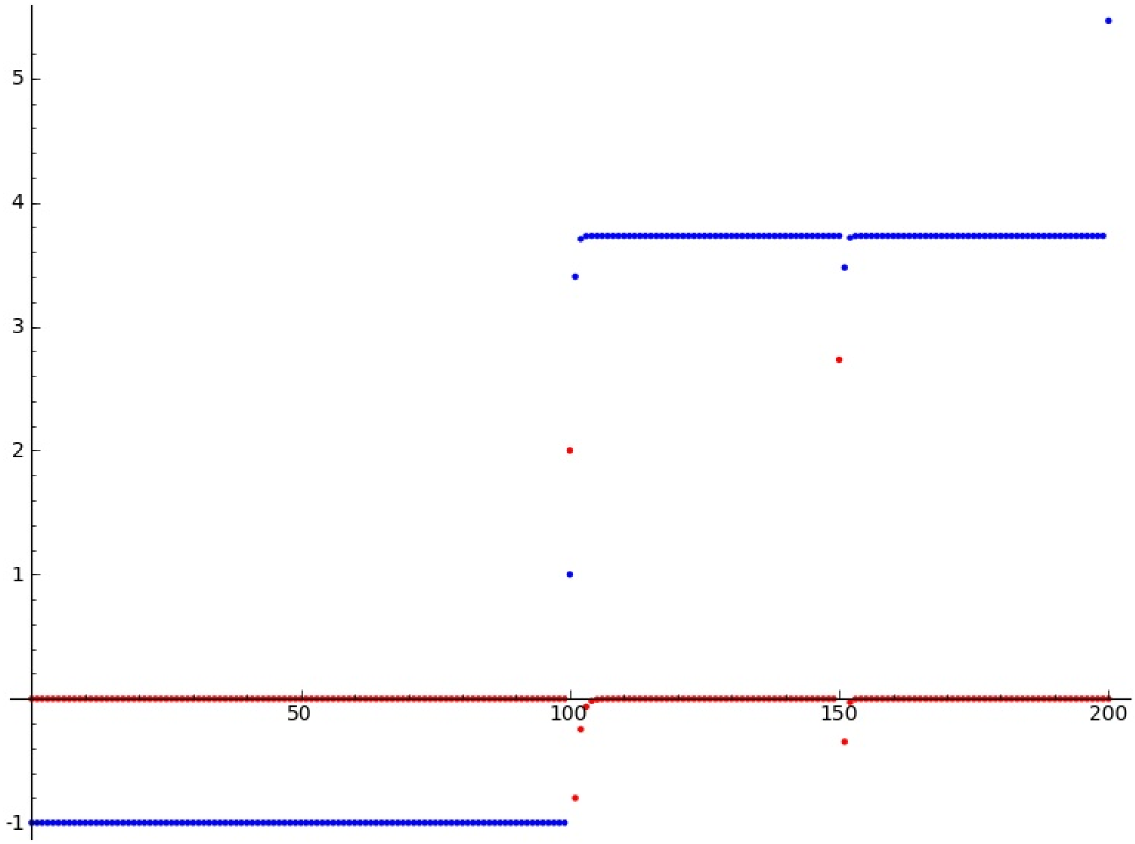
$$T_n = 6\, U_{n-2}(2) - 2\, U_{n-3}(2)$$

for the number of tilings of the $2 \times 2N$ projective plane. By some Chebyshev identities, this formula can be shown to equal

$$2\, U_{n-1}(2) - 2\, U_{n-2}(2)$$

which was the formula derived in the conditioning argument.

The $2 \times (2n+1)$ case is also interesting:



In the plot above, the real parts of the 2×201 Aitken values are plotted in blue and the imaginary parts are plotted in red. The real parts of the 2×201 Aitken values are close to the 2×200 Aitken values, but they do not follow the numerator-denominator pattern. In the plot above, the only completely real-valued Aitken values are the -1s, but the imaginary parts are almost always very close to zero, with deviations occurring in two distinct "swerves" at indices 100 and 150.

The even and odd Aitken plots in higher dimensions bear a familial resemblance to the $2 \times n$ case:

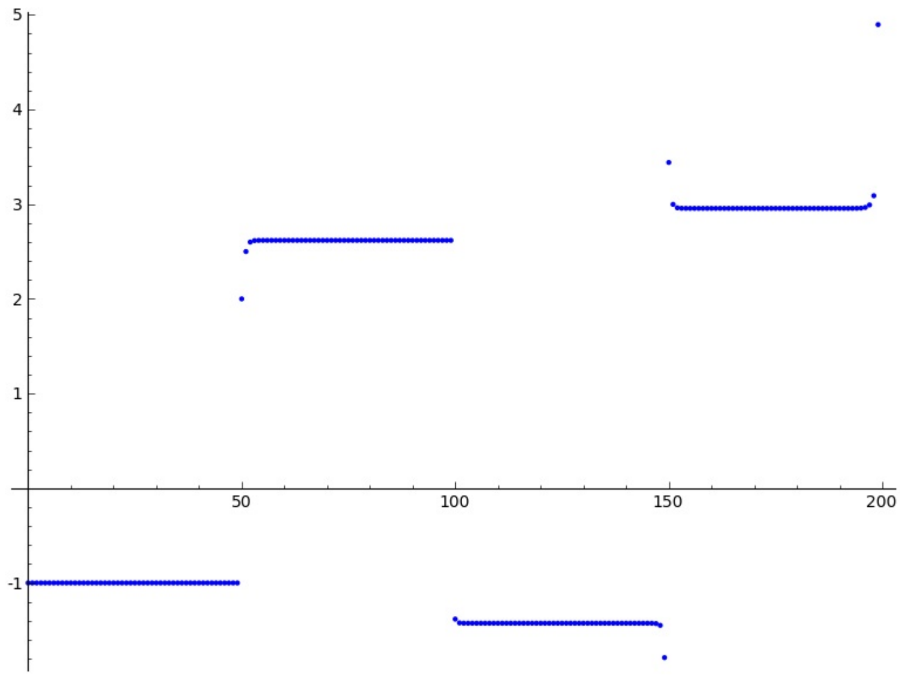


Figure 5.3: The 4×100 Aitken plot.

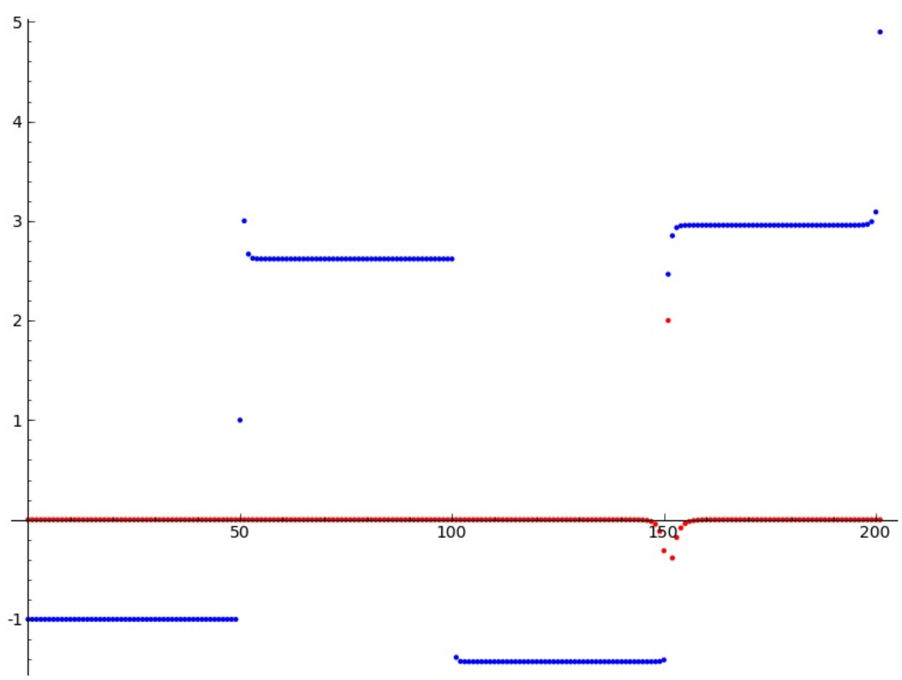


Figure 5.4: 4×101; imaginary parts in red

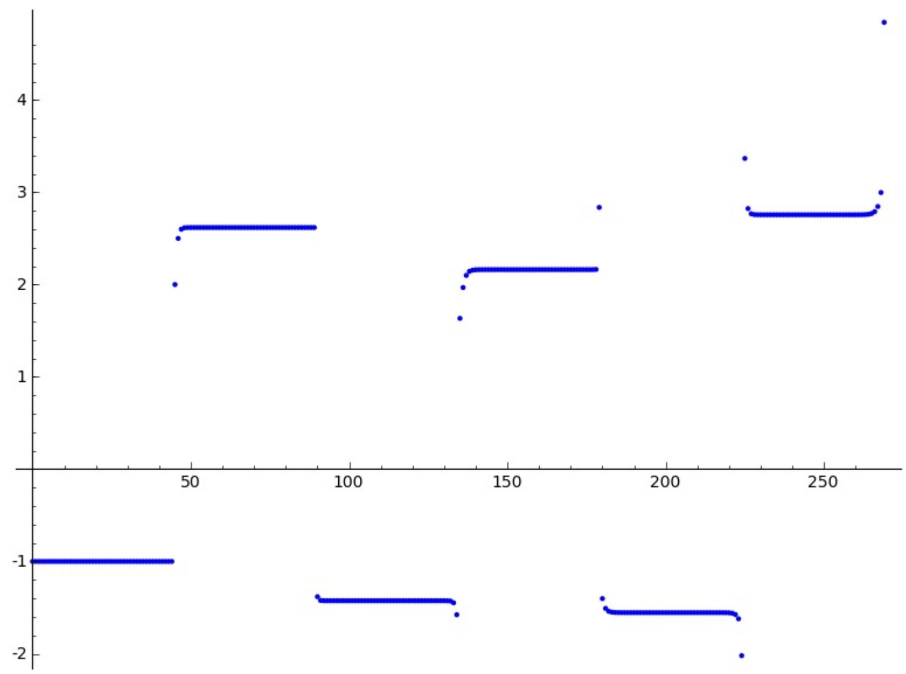


Figure 5.5: 6×90

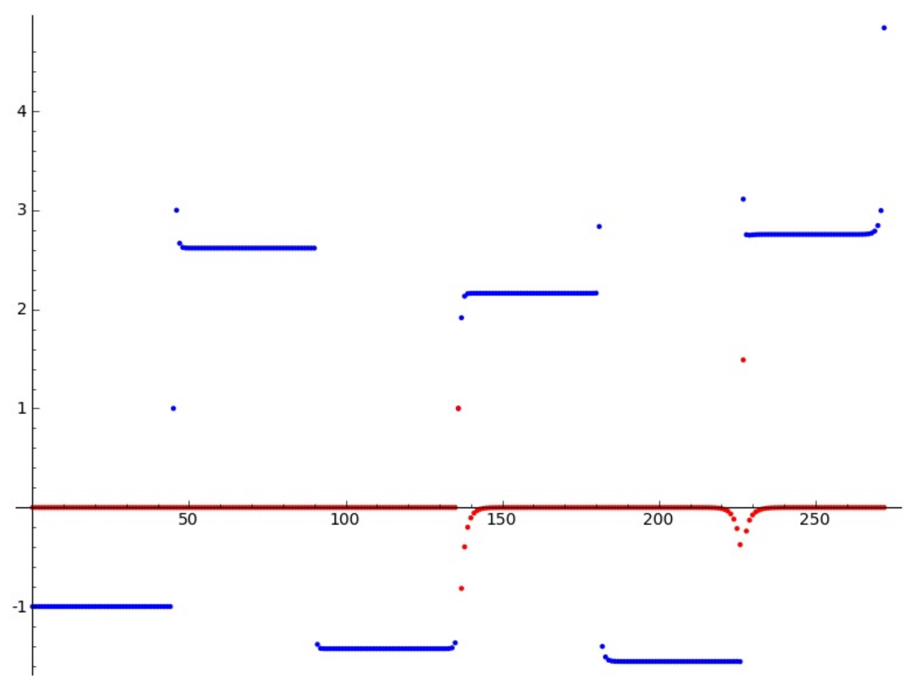


Figure 5.6: 6×91

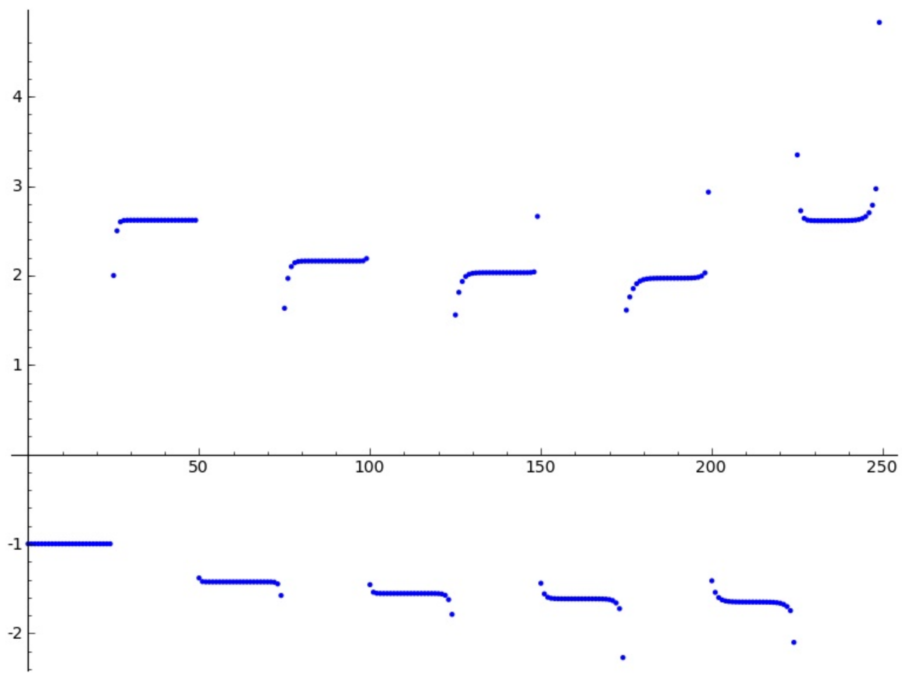


Figure 5.7: 10×50

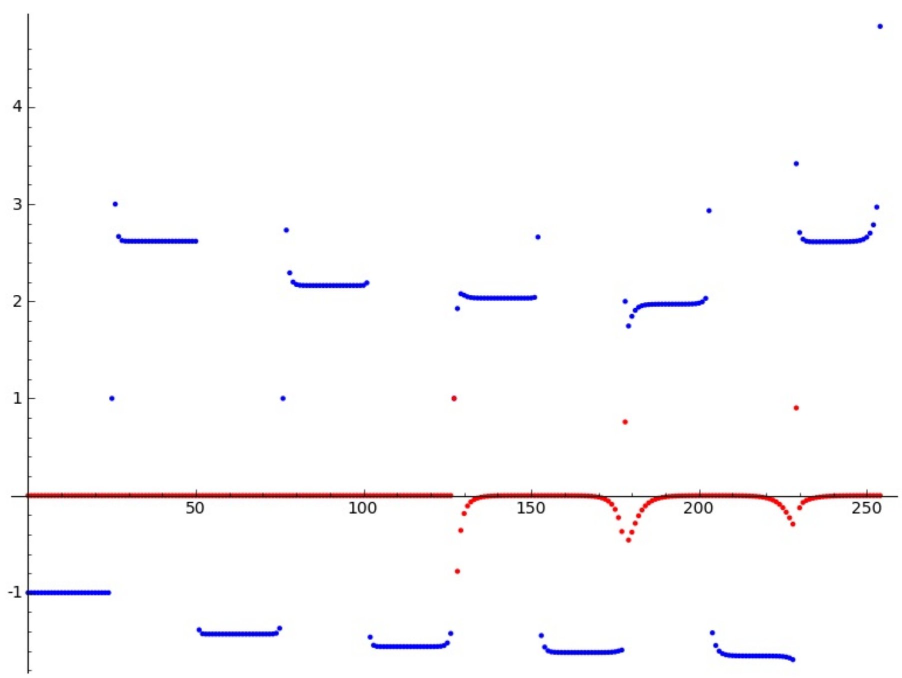


Figure 5.8: 10×51

Turning to the $4 \times 2N$ case, the Aitken list for the 4×12 projective plane is

$$\ldots, -1, 2, \tfrac{5}{2}, \tfrac{13}{5}, \tfrac{34}{13}, \tfrac{89}{34}, \tfrac{233}{89}, -\tfrac{322}{233}, -\tfrac{229}{161}, -\tfrac{653}{458}, -\tfrac{933}{653}, -\tfrac{1351}{933}, -\tfrac{2417}{1351}, \tfrac{8318}{2417}, \tfrac{12480}{4159}, \tfrac{18539}{6240}, \tfrac{221933}{74156}, \tfrac{685678}{221933}, \tfrac{1678802}{342839}$$

where I have omitted all but the final -1, with the corresponding Aitken sequence

$$1, 2, 5, 13, 34, 89, 233, 322, 458, 653, 933, 1351, 2417, 8318, 24960, 74156, 221933, 685678, 3357604.$$

Note from the sign changes in the Aitken list that the first "chunk" in the Aitken plot corresponds to the subsequence

$$2, 5, 13, 34, 89, 233.$$

This sequence matches $A001519$ on OEIS, satisfies the recurrence

$$a_n = 3\,a_{n-1} - a_{n-2}$$

and is generated by

$$a_n = U_n(\frac{3}{2}) - U_{n-1}(\frac{3}{2}).$$

The same sequence seems to define the first chunk of every $2M \times 2N$ projective grid Aitken list for $M > 1$. I have also observed that the first element of the second chunk is the sum of the two elements preceding it. Finally, an overlay of the 4×50 Aitken values for the rectangular grid, Möbius strip, and projective plane is shown in Figure 5.9.

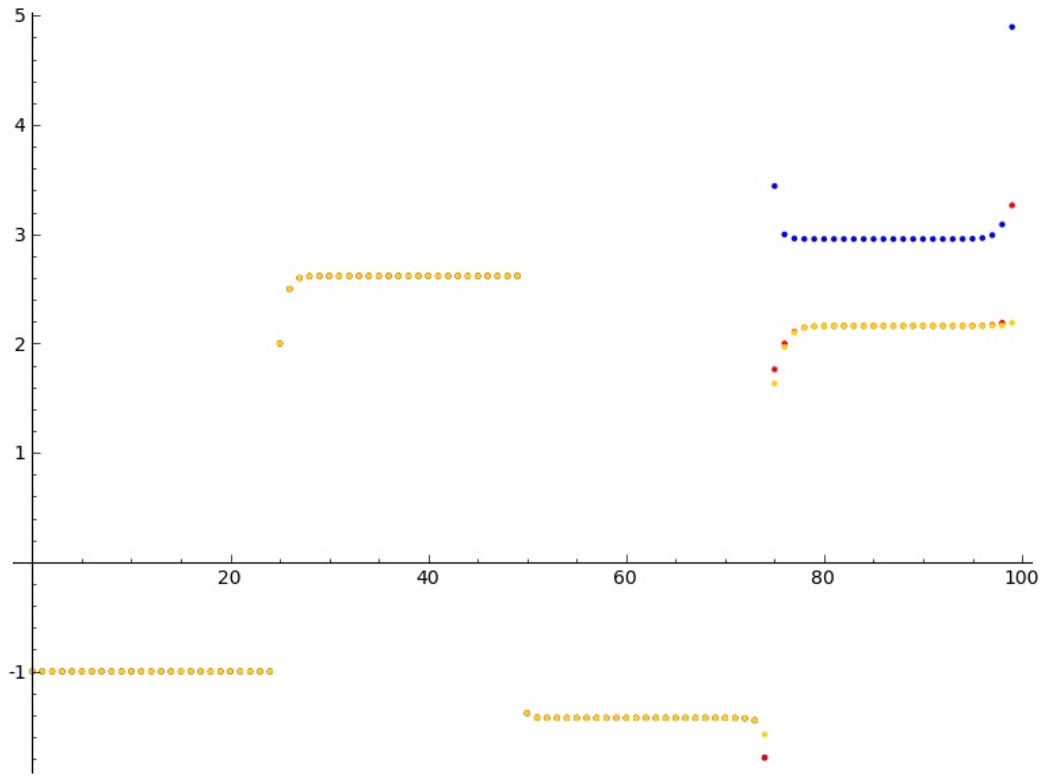


Figure 5.9: Aitken values for 4×50 rectangle, Möbius strip, projective plane in gold, red, and blue, respectively.

The Aitken values of the three surfaces agree up to the final Aitken value of the third chunk, at which only the rectangle differs, and the projective plane and Möbius strip differ only in the final chunk. This seems to be the case for general $M \times 2N$ Aitken lists.

Conclusion

Although we failed to find a general formula to enumerate perfect matchings on the $M \times N$ projective grid, we have uncovered a few possible approaches. A general formula will quickly follow from

1. A general formula for the lower triangular matrix that induces the LDL^t decomposition, as described in 5.0.2.

2. A solution, in the context of the projective grid, to the recursion relation that defines the Aitken block decomposition described in 5.0.2 and in the Appendix.

3. A simultaneous diagonalization of the upper and lower blocks in the matrix in (4.3), from which would follow a tridiagonalization of the entire matrix.

4. Diagonalization, or tridiagonalization, of the skew-Hermitian matrix described at the end of Chapter 4.

Appendix A

Aitken block diagonalization

Let A be a skew-symmetric matrix. Then,

$$A = \begin{pmatrix} R & Q \\ -Q^t & S \end{pmatrix}$$

where R and S are square, skew-symmetric and Q is a rectangular matrix filling the space left by R and S. In practice, we will always take S to be 2×2. Aitken's algorithm harnesses the identity

$$\mathrm{Pf}\,(A) = \mathrm{Pf}\,(R + QS^{-1}Q^t)\,\mathrm{Pf}\,(S).$$

For a skew-symmetric matrix A,

Aitken_list (A, Slist = []):
    if A is 2×2:
        Slist += $\mathrm{Pf}\,(A)$
        return Slist
    else:
        $R = A[: -2, : -2]$
        $S = A[-2 :, -2 :]$
        $Q = A[: -2, -2 :]$
        Slist += $\mathrm{Pf}\,(S)$
        return *Aitken_list*$(R + QS^{-1}Q^t$, Slist)

For example, we'll look at the 4×2 case. We begin by setting A_1 to the signed adjacency matrix:

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2i \\ -1 & 0 & 0 & 1 & 0 & 0 & 2i & 0 \\ -1 & 0 & 0 & -1 & 1 & -i & 0 & 0 \\ 0 & -1 & 1 & 0 & -i & 1 & 0 & 0 \\ 0 & 0 & -1 & i & 0 & 1 & 1 & 0 \\ 0 & 0 & i & -1 & -1 & 0 & 0 & 1 \\ 0 & -2i & 0 & 0 & -1 & 0 & 0 & -1 \\ -2i & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \qquad R1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & -i \\ 0 & -1 & 1 & 0 & -i & 1 \\ 0 & 0 & -1 & i & 0 & 1 \\ 0 & 0 & i & -1 & -1 & 0 \end{pmatrix}$$

$$Q1 = \begin{pmatrix} 0 & 2i \\ 2i & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$A_2 = R_1 + Q_1 S_1^{-1} Q_1^t = \begin{pmatrix} 0 & 5 & 1 & 0 & -2i & 0 \\ -5 & 0 & 0 & 1 & 0 & 2i \\ -1 & 0 & 0 & -1 & 1 & -i \\ 0 & -1 & 1 & 0 & -i & 1 \\ 2i & 0 & -1 & i & 0 & 2 \\ 0 & -2i & i & -1 & -2 & 0 \end{pmatrix}.$$

Moving forward,

$$R_2 = \begin{pmatrix} 0 & 5 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} -2i & 0 \\ 0 & 2i \\ 1 & -i \\ -i & 1 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Accordingly,

$$A_3 = R_2 + Q_2 S_2^{-1} Q_2^t = \begin{pmatrix} 0 & 3 & 2 & i \\ -3 & 0 & i & 2 \\ -2 & -i & 0 & -2 \\ -i & -2 & 2 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Finally,

$$A_4 = R_3 + Q_3 S_3^{-1} Q_3^t = \begin{pmatrix} 0 & \frac{11}{2} \\ -\frac{11}{2} & 0 \end{pmatrix} = S_4.$$

The tridiagonalized form of the original matrix is thus

$$\bigoplus_{k=1}^{2MN} S_k = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \end{pmatrix}.$$

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