Compression of Graphical Structures: Fundamental Limits, Algorithms, and Experiments *

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Abstract

Information theory traditionally deals with "conventional data," be it textual data, image, or video data. However, databases of various sorts have come into existence in recent years for storing "unconventional data" including biological data, social data, web data, topographical maps, and medical data. In compressing such data, one must consider two types of information: the information conveyed by the *structure itself*, and the information conveyed by the data labels implanted in the structure. In this paper, we attempt to address the former problem by studying information of graphical structures (i.e., unlabeled graphs). As the first step, we consider the Erdős-Rényi graphs $\mathcal{G}(n,p)$ over n vertices in which edges are added randomly with probability p. We prove that the *structural entropy* of $\mathcal{G}(n,p)$ is

$$\binom{n}{2}h(p) - \log n! + o(1) = \binom{n}{2}h(p) - n\log n + O(n),$$

where $h(p) = -p\log p - (1-p)\log(1-p)$ is the entropy rate of a conventional memoryless binary source. Then, we propose a two-stage compression algorithm that asymptotically achieves the structural entropy up to the $n\log n$ term (the first two leading terms) of the structural entropy. Our algorithm runs either in time $O(n^2)$ in the worst case for any graph or in time $O(n+e)$ on average for graphs generated by $\mathcal{G}(n,p)$, where e is the average number of edges. To the best of our knowledge, this is the first provable (asymptotically) optimal graph compressor for Erdős-Rényi graph models. We use combinatorial and analytic techniques such as generating functions, Mellin transform, and poissonization to establish these findings. Our experiments confirm the theoretical results and show the usefulness of our algorithm for some real-world graphs such as the Internet, biological networks, and social networks.

Index Terms: Unlabeled graphs, structural entropy, Erdős-Rényi graphs, graph automorphism, arithmetic encoder, digital trees, poissonization, Mellin transform, analytic information theory.

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1 Introduction

Shannon introduced in 1948 a metric for information launching the field of information theory. However, as observed by Brooks [5] and others [25, 34], there is no theory that gives us a useful metric for information embodied in structure. Shannon himself in his 1953 less known paper [30] argued for an extension of information theory to "non-conventional data" (i.e., lattices). Indeed, data is increasingly available in various forms (e.g., sequences, expressions, interactions, structures) and in exponentially increasing amounts. For example, in biology large amounts of data are now in public domain on gene regulation, protein interactions, and metabolic pathways. Most of such data is multidimensional and context dependent. Therefore, it necessitates novel theory and efficient algorithms for extracting meaningful information from non-conventional data structures. Typically, a data file of this new type (e.g., biological data, topographical maps, medical data, volumetric data) is a "data structure" conveying a "shape" and consisting of labels implanted in the structure. In understanding such data structures, one must take into account two types of information: the information conveyed by the structure itself and the data labels implanted in the structure.[1] In this paper, we address the former problem in order to understand how much structural information these data structures possess (measured in terms of the entropy induced by a probabilistic graph model discussed below). A larger goal is to quantify the amount of information in networks such as the Internet, social networks, biological networks, and economic networks.

Unconventional data often contains more sophisticated structural relations. For example, a graph can be represented by a binary matrix that further can be viewed as a binary sequence. However, such a string does not exhibit internal symmetries that are conveyed by the so-called *graph automorphism* (making certain sequences/matrices "indistinguishable"). The main challenge in dealing with such structural data is to identify and describe these structural relations. In fact, these "regular properties" constitute "useful (extractable) information" understood in the spirit of Rissanen "learnable information" [26] (cf. also [21, 22]).

As the first step in understanding structural information, we restrict our attention to structures on graphs. More specifically, we study *unlabeled graphs* (or structures) generated by a memoryless source known as the Erdős-Rényi model [3] in which edges are added randomly with probability p. This model induces a probability distribution on structures so that one can compute the Shannon entropy giving us a fundamental limit on lossless unlabeled graph compression. We prove that this *structural entropy* $H_{\mathcal{S}}$ is

$$\binom{n}{2}h(p) - \log n! + o(1) = \binom{n}{2}h(p) - n\log n + O(n),$$

where n is the number of vertices and $h(p) = -p\log p - (1-p)\log(1-p)$ is the entropy rate of a conventional memoryless binary source.[2] In addition, we prove that, for almost every structure S from this model, the probability of S is close to 2^{-H_S} for large n, which is a manifestation of AEP (asymptotic equipartition property) for the Erdős-Rényi graphs.

In the next step, we design and analyze a graphical (structure) compression algorithm,

[1] Given the well known expression for the entropy: $H(S,X) = H(S) + H(X|S)$, where S is the structure and X labels, our approach is first to describe (compress) structure and then, if needed, use more conventional methods to describe labels.

[2] All logarithms are to the base 2 throughout this paper.

called SZIP, that asymptotically achieves the compression rate on Erdős-Rényi graph models,

$$\binom{n}{2} h(p) - n \log n + O(n),$$

matching the lower bound up to the first two leading terms of the structural entropy with high probability. Our algorithm consists of two stages. It first encodes a structure into two binary strings that are then compressed using an arithmetic encoder. We provide two different implementation of our algorithm. One runs in time $O(n^2)$ in the worst case for any graph. The other runs in time $O(n + e)$ on average when the graph is generated by $\mathcal{G}(n, p)$, where e is the average number of edges. This is faster than $O(n^2)$-time algorithm, also discussed in [23], theoretically as well as in practice since most real-world graphs are very sparse. Experimental results on both real-world networks and the Erdős-Rényi graphs confirm the efficiency and utility of our algorithm.

There are other possible metrics of information content of a graph. For example, "topological entropy" discussed in [25, 34] attempts to characterize the distinctiveness of vertex degrees by partitioning all vertices into subsets of the same long term connectivity (i.e., neighborhoods). As a by-product of our analysis, we prove that such topological entropy is equal to $\log n + o(1)$ for the Erdős-Rényi random graph model. Furthermore, the most popular "graph entropy" due to Körner generalizes standard Shannon entropy to "undistinguished symbols" [31]. Körner graph entropy is a function of the graph and a probability distribution on the vertices. Roughly speaking, graph entropy reflects the number of bits you need to transmit to describe the vertex when one distinguishes only between vertices that are connected (connected vertices represent "distinguishable symbols"). For example, if the graph is complete, then one must distinguish between any two vertices. In this case, the Körner entropy achieves the highest value that coincides with the Shannon entropy. But a complete graph has the simplest structure to describe, thus it should be clear that our structural entropy is quite different than the Körner graph entropy.

Literature on graphical structure compression is scarce. In 1984, Turan [35] raised the question of finding efficient coding method for general unlabeled graphs on n vertices, suggesting a lower bound of $\binom{n}{2} - n \log n + O(n)$ bits. In 1990, Naor [23] proposed such a representation that is optimal up to the first two leading terms when all unlabeled graphs are equally likely. Naor's result is asymptotically a special case of ours when $p = 1/2$. Finally, in a recent paper Kieffer et al. [19] presented a structural complexity of a binary tree, in a spirit similar to ours. There also have been some heuristic methods for real-world graphs compression including Adler and Mitzenmacher [1] (see also [6]), who proposed an encoding technique for web graphs, and a similar idea has been used in [32] for compressing sparse graphs. Recently, attention has been paid to grammar compression for some data structures: Peshkin [24] proposed an algorithm for a graphical extension of the one-dimensional SEQUITUR compression method. However, SEQUITUR is known not to be asymptotically optimal [28]. Therefore, the Peshkin method already lacks asymptotic optimality in the 1D case. To the best of our knowledge our algorithm is the first provable asymptotically optimal compression scheme for graphical structures generated according to Erdős-Rényi model.

The paper is organized as follows. The structural entropy of a graph is defined in Section 2 and compared to the conventional graph entropy. Our algorithm is described in Section 3, where we derive the structural entropy for $\mathcal{G}(n, p)$. We also present there our experimental results. Our main results are proved in Sections 4 and 5, where we introduce random bi-

nary trees that resemble tries and digital search trees. We use analytic techniques such as generating functions, Mellin transform, and poissonization to establish our results.

2 Structural Entropy

In this section, we formally define the structural entropy of a random (unlabeled) graph model. Given n distinguishable vertices, a random graph is generated by adding edges randomly. This random graph model $\mathcal{G}$ produces a probability distribution on graphs, and the graph entropy $H_{\mathcal{G}}$ is defined naturally as

$$H_{\mathcal{G}} = \mathbf{E}[-\log P(G)] = -\sum_{G \in \mathcal{G}} P(G) \log P(G),$$

where $P(G)$ is the probability of a graph G. We now introduce a *random structure model* $\mathcal{S}$ for the unlabeled version of a random graph model $\mathcal{G}$. In such a model, graphs are generated in the same manner as in $\mathcal{G}$, but they are thought of as unlabeled graphs. That is, the vertices are indistinguishable, and the graphs having "the same structure" are considered to be the same even if their labeled versions are different. Thus, we shall use the terms *unlabeled graphs* and *structures* interchangeably. For a given structure $S \in \mathcal{S}$, the probability of S can be computed as

$$P(S) = \sum_{G \cong S, G \in \mathcal{G}} P(G).$$

Here $G \cong S$ means that G and S have the same structure, that is, S is *isomorphic* to G. If all isomorphic labeled graphs have the same probability, then for any labeled graph $G \cong S$,

$$P(S) = N(S) \cdot P(G), \tag{1}$$

where $N(S)$ is the number of different labeled graphs that have the same structure as S. The *structural entropy* $H_{\mathcal{S}}$ of a random graph $\mathcal{G}$ can be defined as the entropy of a random structure $\mathcal{S}$, that is,

$$H_{\mathcal{S}} = \mathbf{E}[-\log P(S)] = -\sum_{S \in \mathcal{S}} P(S) \log P(S),$$

where the summation is over all distinct structures.

Example: In Figure 1(a), we draw different graphs built on three vertices. Let us assume that they are equally probable, that is, $P(G_i) = 1/8$ for $1 \le i \le 8$. Then the entropy of this random graph $\mathcal{G}$ is $H_{\mathcal{G}} = -8 \cdot \frac{1}{8} \log \frac{1}{8} = 3$ bits. Let $\mathcal{S}$ be the random structure that corresponds to $\mathcal{G}$. In Figure 1(b), we present all different structures that can be generated by $\mathcal{S}$. Since $N(S_1) = N(S_4) = 1$ and $N(S_2) = N(S_3) = 3$, thus $P(S_1) = P(S_4) = 1/8$ and $P(S_2) = P(S_3) = 3/8$. The entropy of the random structure $\mathcal{S}$ is $H_{\mathcal{S}} = -2 \cdot \frac{1}{8} \log \frac{1}{8} - 2 \cdot \frac{3}{8} \log \frac{3}{8} \approx 1.811$ bits. ∎

In order to compute the probability of a given structure S, one needs to estimate the number of ways, $N(S)$, to construct a given structure S. For this, we need to consider the automorphisms of a graph. An *automorphism* of a graph G is an adjacency preserving permutation of vertices of G. The collection $\mathrm{Aut}(G)$ of all automorphisms of G is called *the*

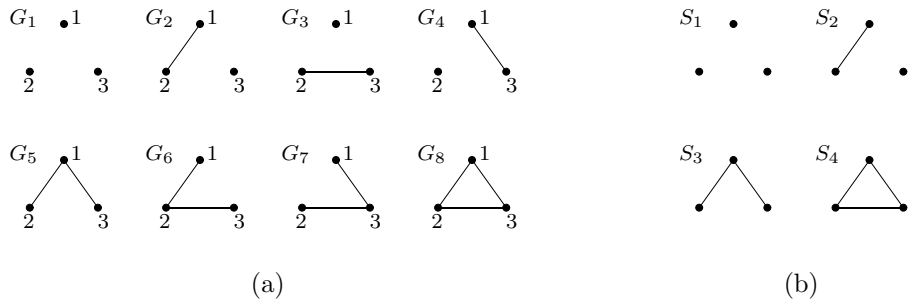


Figure 1: All different graphs and structures built on three vertices.

automorphism group of G. In the sequel, $\mathrm{Aut}(S)$ of a structure S denotes $\mathrm{Aut}(G)$ for some labeled graph G such that $G \cong S$. In group theory, it is well known that [14, 15]

$$N(S) = \frac{n!}{|\mathrm{Aut}(S)|}. \tag{2}$$

Trivially, $1 \le |\mathrm{Aut}(S)| \le n!$.

Example: In Figure 2(a), the graph G has exactly four automorphisms, that is, in the usual cyclic permutation representation: $(v_1)(v_2)(v_3)(v_4)$, $(v_1)(v_4)(v_2v_3)$, $(v_1v_4)(v_2)(v_3)$, and $(v_1v_4)(v_2v_3)$. For example, $(v_1)(v_4)(v_2v_3)$ stands for a permutation π such that $\pi(v_1) = v_1$, $\pi(v_4) = v_4$, $\pi(v_2) = v_3$, and $\pi(v_3) = v_2$. Thus, by (2), G has $4!/4 = 6$ different labeling as shown in Figure 2(b). ∎

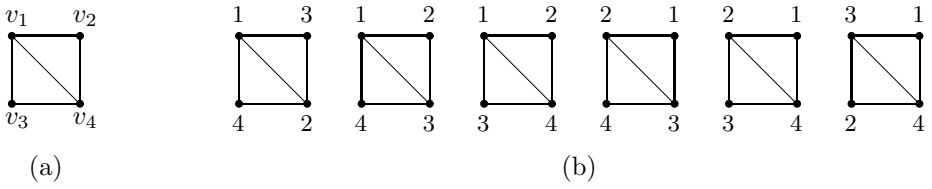


Figure 2: The six different labeling of a graph.

With these preliminary definitions, we are now in the position to present a relationship between $H_{\mathcal{G}}$ and $H_{\mathcal{S}}$.

Lemma 1 *If all isomorphic graphs have the same probability, then*

$$H_{\mathcal{S}} = H_{\mathcal{G}} - \log n! + \sum_{S \in \mathcal{S}} P(S) \log |\mathrm{Aut}(S)|$$

for any random graph $\mathcal{G}$ and its corresponding random structure $\mathcal{S}$, where $\mathrm{Aut}(S)$ *is the automorphism group of S.*

Proof: Observe that for any $\mathcal{G}$ and $\mathcal{S}$

$$\begin{aligned} H_{\mathcal{G}} &= -\sum_{G\in\mathcal{G}} P(G)\log P(G) \\ &= -\sum_{S\in\mathcal{S}} \sum_{G\cong S, G\in\mathcal{G}} P(G)\log P(G) \\ &= -\sum_{S\in\mathcal{S}} \sum_{G\cong S, G\in\mathcal{G}} \frac{P(S)}{N(S)}\log\frac{P(S)}{N(S)} \qquad \text{(by (1))} \\ &= -\sum_{S\in\mathcal{S}} N(S)\cdot\frac{P(S)}{N(S)}\log\frac{P(S)}{N(S)} \\ &= H_{\mathcal{S}} + \sum_{S\in\mathcal{S}} P(S)\log\frac{n!}{|\mathrm{Aut}(S)|} \qquad \text{(by (2))} \\ &= H_{\mathcal{S}} + \log n! - \sum_{S\in\mathcal{S}} P(S)\log|\mathrm{Aut}(S)|. \end{aligned}$$

This proves the lemma. ∎

The last term of the structural entropy $\sum_{S\in\mathcal{S}} P(S)\log|\mathrm{Aut}(S)|$ can vary from 0 to $n\log n$ since $1 \le |\mathrm{Aut}(S)| \le n!$. However, as we shall see in most *random* graph models there is not much symmetry, and hence $\sum_{S\in\mathcal{S}} P(S)\log|\mathrm{Aut}(S)| = o(1)$. In order to develop further the idea of information in a random structure, hereafter we will focus on the Erdős-Rényi random graph [3].

3 Main Results

In this section, we first compute the structural entropy for the Erdős-Rényi random graph. As it is well known, such entropy constitutes a lower bound for lossless compression. Then we describe our optimal compression algorithm that asymptotically achieves this lower bound up to the second leading term with high probability. Finally, we present experimental results.

3.1 Structural Entropy of the Erdős-Rényi Model

In the Erdős-Rényi random graph model $\mathcal{G}(n,p)$, graphs are generated randomly on n vertices with edges chosen independently with probability $0 < p < 1$. If a graph G in $\mathcal{G}(n,p)$ has k edges, then

$$P(G) = p^k q^{\binom{n}{2}-k},$$

where $q = 1-p$. Let $\mathcal{S}(n,p)$ be the random structure model (unlabeled graphs) corresponding to $\mathcal{G}(n,p)$. Then, by (1) if $S \in \mathcal{S}(n,p)$ has k edges,

$$P(S) = N(S)\cdot p^k q^{\binom{n}{2}-k}.$$

To compute the entropy of $\mathcal{S}(n,p)$ we need to estimate $N(S)$. For this, we must study an important property of $\mathcal{S}(n,p)$ (or equivalently, $\mathcal{G}(n,p)$), namely *asymmetry*. A graph is said to be *asymmetric* if its automorphism group does not contain any permutation other than the identity (i.e., $(v_1)(v_2)\cdots(v_n)$) so that $|\mathrm{Aut}(G)| = 1$; otherwise it is called *symmetric*. It

is known that almost every graph from $\mathcal{G}(n,p)$ is asymmetric [11, 20]. In the sequel, we write $a_n \ll b_n$ to mean $a_n = o(b_n)$ when $n \to \infty$. For completeness, we present in Appendix A a proof of Kim et al.'s result [20].

Lemma 2 (Kim, Sudakov, and Vu, 2002) *For all p satisfying $\frac{\ln n}{n} \ll p$ and $1-p \gg \frac{\ln n}{n}$, a random graph $G \in \mathcal{G}(n,p)$ is symmetric with probability $O(n^{-w})$ for any positive constant $w > 1$.*

Remark 1. While Lemma 2 proves asymmetry for Erdős-Rényi graphs, we conjecture the property holds for almost all known random generation of graphs (e.g., power law graphs and preferential attachment graphs).

Using this property, we next present the structural entropy of $\mathcal{G}(n,p)$ and establish the asymptotic equipartition property (AEP), that is, the typical probability of a structure S.

Theorem 1 *For large n and all p satisfying $\frac{\ln n}{n} \ll p$ and $1-p \gg \frac{\ln n}{n}$, the following holds:*
(i) *The structural entropy $H_{\mathcal{S}}$ of $\mathcal{G}(n,p)$ is*

$$H_{\mathcal{S}} = \binom{n}{2} h(p) - \log n! + O\left(\frac{\log n}{n^{\alpha}}\right), \quad \text{for some } \alpha > 0,$$

(ii) *(AEP) For a structure $S \in \mathcal{S}(n,p)$ and $\epsilon > 0$,*

$$P\left(\left|-\frac{1}{\binom{n}{2}} \log P(S) - h(p) + \frac{\log n!}{\binom{n}{2}}\right| < \epsilon\right) > 1 - 2\epsilon, \tag{3}$$

where $h(p) = -p\log p - (1-p)\log(1-p)$ is the entropy rate of a binary memoryless source.

Proof: Let us first compute the entropy $H_{\mathcal{G}}$ of $\mathcal{G}(n,p)$. In $\mathcal{G}(n,p)$, $m = \binom{n}{2}$ distinct edges are independently selected with probability p, and thus there are 2^m different labeled graphs. That is, each graph instance can be considered as a binary sequence X of length m. Thus,

$$H_{\mathcal{G}} = -\mathbf{E}[\log P(X_1^m)] = -m\mathbf{E}[\log P(X_1)] = \binom{n}{2} h(p).$$

By Lemma 1,

$$H_{\mathcal{S}} = \binom{n}{2} h(p) - \log n! + A$$

where

$$A = \sum_{S \in \mathcal{S}} P(S) \log |\mathrm{Aut}(S)|.$$

Now we show that $A = o(1)$ to prove part (i).

$$\begin{aligned}
A &= \sum_{S \in \mathcal{S}(n,p) \text{ is symmetric}} P(S) \log |\mathrm{Aut}(S)| + \sum_{S \in \mathcal{S}(n,p) \text{ is asymmetric}} P(S) \log |\mathrm{Aut}(S)| \\
&= \sum_{S \in \mathcal{S}(n,p) \text{ is symmetric}} P(S) \log |\mathrm{Aut}(S)| \quad (\because |\mathrm{Aut}(S)| = 1 \text{ for all asymmetric } S) \\
&\le \sum_{S \in \mathcal{S}(n,p) \text{ is symmetric}} P(S) \cdot n \log n \quad (\because |\mathrm{Aut}(S)| \le n! \le n^n) \\
&= O\left(\frac{\log n}{n^{w-1}}\right) \quad \text{for any positive constant } w > 1 \quad \text{(by Lemma 2)}.
\end{aligned}$$

To prove part (ii), we define the *typical set* T_ϵ^n as the set of structures S on n vertices having the following two properties: (a) S is asymmetric; (b) for $G \cong S$,

$$2^{-\binom{n}{2}(h(p)+\epsilon)} \leq P(G) \leq 2^{-\binom{n}{2}(h(p)-\epsilon)}.$$

Let T_1^n and T_2^n be the sets of structures satisfying the properties (a) and (b), respectively. Then, $T_\epsilon^n = T_1^n \cap T_2^n$. By the asymmetry of $\mathcal{G}(n,p)$, we know that $P(T_1^n) > 1-\epsilon$ for large n. As explained above, a labeled graph G can be viewed as a binary sequence of length $\binom{n}{2}$. Thus, by the property (b) and the AEP for binary sequences, we also know that $P(T_2^n) > 1-\epsilon$ for large n. Thus, $P(T_\epsilon^n) = 1 - P(\overline{T_1^n} \cup \overline{T_2^n}) > 1 - 2\epsilon$. Now let us compute $P(S)$ for S in T_ϵ^n. By the property (a), $P(S) = n!P(G)$ for any $G \cong S$. By this and the property (b), we can see that any structure S in T_ϵ^n satisfies the condition in (3). This completes the proof. ∎

Remark 2. The structural entropy can be equivalently written as

$$H_{\mathcal{S}} = \binom{n}{2} h(p) - n\log n + n\log e - \frac{1}{2}\log n - \frac{1}{2}\log(2\pi) + o(1) \tag{4}$$

by Stirling's approximation, $n! \approx \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n$.

Remark 3. Roughly speaking, Theorem 1(ii) means that the probability of a typical graph structure is $P(S) \sim 2^{-\binom{n}{2}h(p)+\log n!}$.

By Shannon's source coding theorem, the structural entropy computed in Theorem 1 is a fundamental lower bound on the lossless compression of structures from $\mathcal{S}(n,p)$. In the next section, we design an asymptotically optimal compression algorithm matching the first two leading terms as in (4) of the structural entropy with high probability.

As already observed in the introduction, there are other measures of information content of a graph. For example, consider partitioning vertices of a graph G into subsets, $O_i(G)$, with vertices belonging to the same subset having neighbors of the same node degree. For example, in Figure 2(a) we find that $O_1 = \{v_1, v_4\}$ and $O_2 = \{v_2, v_3\}$. These subsets turn out to be the so-called *orbits* of the underlying graph automorphism [14]. Clearly, all graphs G of the same structure $S \in \mathcal{S}$ have the same orbits. Assigning some probability measure on the set of orbits, one can define another information metric that can be called the *topological entropy*, H_T [25, 34]. For a given structure S we define the probability of an orbit $O_i(S)$ to be $|O_i(S)|/n$. Then the topological entropy is defined as

$$H_T = -\sum_{S\in\mathcal{S}} P(S) \sum_i \frac{|O_i(S)|}{n} \log \frac{|O_i(S)|}{n},$$

where the sum is over all structures $S \in \mathcal{S}$ and over all enumeration of orbits.

Let us again consider the Erdős-Rényi model for graph generation. By Lemma 2 we conclude that all orbits are singletons with high probability. This leads to the following corollary.

Corollary 1 *Assume graphs are generated according to the Erdős-Rényi process $\mathcal{G}(n,p)$. For all p satisfying $\frac{\ln n}{n} \ll p$ and $1-p \gg \frac{\ln n}{n}$, the topological entropy is*

$$H_T = \log n - O\left(\frac{\log n}{n^\alpha}\right)$$

for some $\alpha > 0$.

3.2 Compression Algorithm

Our algorithm, called Szip (Structural zip), is a compression scheme for unlabeled graphs. In other words, given a labeled graph G, it compresses G into a codeword, from which one can construct a graph S that is isomorphic to G. The algorithm consists of two stages. First it encodes G into two binary sequences and then compresses them using an arithmetic encoder.

The main idea behind our algorithm is quite simple: We select a vertex, say v_1, and store the number of neighbors of v_1 in binary. Then we partition the remaining $n-1$ vertices into two sets: the neighbors of v_1 and non-neighbors of v_1. We continue by selecting a vertex, say v_2, from the neighbors of v_1 and store two numbers: the number of neighbors of v_2 among each of these two sets. Then we partition the remaining $n-2$ vertices into four sets: the neighbors of both v_1 and v_2, the neighbors of v_1 that are non-neighbors of v_2, the non-neighbors of v_1 that are neighbors of v_2, and the non-neighbors of both v_1 and v_2. This procedure continues until all vertices are processed. During the construction, the number of neighbors for each set in the partition is appended to either B_1 or B_2, where B_2 contains those numbers for singleton sets (i.e., we store either "0" when there is no neighbor or "1" otherwise). We shall conclude that the length of B_2 (in compressed form) dominates the compression rate (we also observe that by the construction B_2 can be viewed as generated by a memoryless source). This allows us to prove that the algorithm achieves the structural entropy up to the first two leading terms shown in (4). We provide two different implementation of our algorithm, which generate the same codeword but only differ in their running time.

In Section 4, we prove our main findings that we summarize below.

Theorem 2 *Let $L(S)$ be the length of the codeword generated by our algorithm for Erdős-Rényi graphs $G \in \mathcal{G}(n,p)$ isomorphic to a structure S. The following holds:*
(i) *For large n,*

$$\mathbf{E}[L(S)] \leq \binom{n}{2} h(p) - n\log n + (c + \Phi(\log n))\, n + o(n),$$

where c is an explicitly computable constant, and $\Phi(\log n)$ is a fluctuating function with a small amplitude independent of n.
(ii) *Furthermore, for any $\epsilon > 0$,*

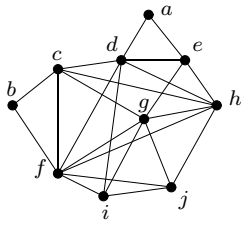
$$P\left(L(S) - \mathbf{E}[L(S)] \leq \epsilon n \log n\right) \geq 1 - o(1).$$

(iii) *Finally, our algorithm runs either in time $O(n^2)$ in the worst case for any graph or in time $O(n+e)$ on average for graphs generated by $\mathcal{G}(n,p)$, where e is the average number of edges.*

We next describe the general framework of the algorithm that runs in time $O(n^2)$ in the worst case. Then we propose some data structures that allow us to reduce the time complexity to $O(n+e)$ on average.

3.2.1 Worst-Case $O(n^2)$-Time Algorithm

First we need some definitions and notations. An *ordered partition* of a set X is a sequence of nonempty subsets of X such that every element in X is in exactly one of these subsets. For example, one ordered partition of $\{a,b,c,d,e\}$ is $\{a,b\},\{e\},\{c,d\}$ that is denoted by



k	v	$\mathcal{P}_{k-1} - v$	encoding	$\mathcal{P}_k$
0				$abcdefghij$
1	a	$bcdefghij$	*0010*	$de/bcfghij$
2	d	$e/bcfghij$	1, *100*	$e/cfhi/bgj$
3	e	$cfhi/bgj$	*001*, *01*	$h/cfi/g/bj$
4	h	$cfi/g/bj$	*10*, 1, *01*	$cf/i/g/j/b$
5	c	$f/i/g/j/b$	1, 0, 1, 0, 1	$f/i/g/j/b$
6	f	$i/g/j/b$	1, 1, 1, 1	$i/g/j/b$
7	i	$g/j/b$	1, 1, 0	$g/j/b$
8	g	j/b	1, 0	j/b
9	j	b	0	b
10	b			

Figure 3: An example for our encoding algorithm, given the graph on the left.

$ab/e/cd$. It is equivalent to $ba/e/dc$, but distinct from $e/ab/cd$. Given an ordered partition $\mathcal{P}$ of a set X, we also define a partial order of the elements of X as follows: $a < b$ in $\mathcal{P}$ if the subset containing a precedes the subset containing b in $\mathcal{P}$. For example, $a < c$ and $e < c$ in $\mathcal{P} = ab/e/cd$, but $e \not< a$. An ordered partition $\mathcal{P}_1$ of a set X is called *finer* than an ordered partition $\mathcal{P}_2$ of X if the following two conditions hold: (1) every element (i.e., subset of X) of $\mathcal{P}_1$ is a subset of some element of $\mathcal{P}_2$, and (2) for all $a, b \in X$, $a < b$ in $\mathcal{P}_1$ if $a < b$ in $\mathcal{P}_2$. For example, both $a/b/e/cd$ and $ab/e/d/c$ are finer than $ab/e/cd$. Finally, a subtraction of an element from an ordered partition gives us another ordered partition (e.g., for $\mathcal{P} = ab/e/cd$ we find that $\mathcal{P} - c$ and $\mathcal{P} - e$ are $ab/e/d$ and ab/cd, respectively).

The first stage of our encoding algorithm consists of n steps, updating in each step an ordered partition $\mathcal{P}$ of a subset of $V(G)$. Let $\mathcal{P}_i$ be the partition after the i-th step. At the beginning, $\mathcal{P}_0 = V(G)$. In the i-th step, a vertex v is selected to be removed from the first subset in $\mathcal{P}_{i-1}$. Then, for each subset U in $\mathcal{P}_{i-1} - v$ (in its order), we encode the *number of neighbors* of v in U using $\lceil \log(|U| + 1) \rceil$ bits. After that, $\mathcal{P}_{i-1} - v$ becomes a finer partition $\mathcal{P}_i$ such that for each subset U in $\mathcal{P}_{i-1} - v$, U is divided into two smaller subsets U_1 and U_2, and U_1 precedes U_2 in $\mathcal{P}_i$ where U_1 is the set of all neighbors of v in U and U_2 is the set of all non-neighbors of v in U. These steps are repeated until $\mathcal{P}$ becomes empty.

While the algorithm is running, the binary encodings of the number of neighbors are concatenated in the order they are generated. During the course of the algorithm, we separately maintain two types of encodings – those of length more than one bit (i.e., for subsets $|U| > 1$) and those of length exactly one bit (i.e., for subsets $|U| = 1$). The former type of encodings are appended to a binary sequence B_1, while the latter encodings form a binary sequence B_2.

Example: Figure 3 shows the progress of our algorithm step by step. Here k denotes the step number, and v denotes the chosen vertex in each step. All encodings whose length is larger than one (denoted by *italic* font) are appended to B_1. The other encodings (those of length one) form B_2. After ten steps, B_1 and B_2 are *0010100001011001* and 11101011111110100, respectively. ■

In the second stage, B_1 and B_2 are compressed to $\hat{B}_1$ and $\hat{B}_2$ by a binary arithmetic encoder [8]. Finally, the encoding of G consists of n, $\hat{B}_1$, and $\hat{B}_2$.

We next describe our decoding algorithm constructing from n, $\hat{B}_1$, and $\hat{B}_2$ a graph isomorphic to the original graph. First we restore B_1 and B_2 by decompressing $\hat{B}_1$ and $\hat{B}_2$. Then, we create a graph G having n vertices and no edges. The general framework of our decoding algorithm is very similar to that of our encoding algorithm. Again, one ordered partition $\mathcal{P}$ of a subset of $V(G)$ is maintained. Let $\mathcal{P}_i$ be the ordered partition after the

i-th step. At the beginning, $\mathcal{P}_0 = V(G)$. In the i-th step, we remove a vertex v from the first subset in $\mathcal{P}_{i-1}$. Then, for each subset U in $\mathcal{P}_{i-1} - v$ (in its order), we extract the first $\ell = \lceil \log(|U|+1) \rceil$ bits from either B_1 (if $|U| > 1$) or B_2 (if $|U| = 1$), and we select any ℓ vertices in U and make an edge between v and each of those ℓ vertices. After that, $\mathcal{P}_{i-1} - v$ becomes a finer partition $\mathcal{P}_i$ in the same way as in our encoding algorithm. These steps are repeated until $\mathcal{P}$ becomes empty.

Example: Let us reconstruct a graph from the encoding in the previous example. After decompressing we have n=10, B_1=*0010100001011001*, and B_2=11101011111110100. We start with a graph of 10 isolated vertices, and proceed as described above. Figure 4 shows the details. Again, k denotes the step number, and v denotes the chosen vertex (here we always select the first vertex.) The last column shows the edges created in the k-th step. The extracted bits from B_1 are denoted by *italic* font. On the right is shown the reconstructed graph, which is isomorphic to the original graph. ■

k	v	$\mathcal{P}_{k-1} - v$	Extracted bits	$\mathcal{P}_k$	Created edges
0				$abcdefghij$	
1	a	$bcdefghij$	*0010*	$bc/defghij$	$\{a,b\}, \{a,c\}$
2	b	$c/defghij$	1, *100*	$c/defg/hij$	$\{b,c\}, \{b,d\}, \{b,e\}, \{b,f\}, \{b,g\}$
3	c	$defg/hij$	*001*, *01*	$d/efg/h/ij$	$\{c,d\}, \{c,h\}$
4	d	$efg/h/ij$	*10*, 1, *01*	$ef/g/h/i/j$	$\{d,h\}, \{d,i\}, \{d,e\}, \{d,f\}$
5	e	$f/g/h/i/j$	1, 0, 1, 0, 1	$f/g/h/i/j$	$\{e,f\}, \{e,h\}, \{e,j\}$
6	f	$g/h/i/j$	1, 1, 1, 1	$g/h/i/j$	$\{f,g\}, \{f,h\}, \{f,i\}, \{f,j\}$
7	g	$h/i/j$	1, 1, 0	$h/i/j$	$\{g,h\}, \{g,i\}$
8	h	i/j	1, 0	i/j	$\{h,i\}$
9	i	j	0	j	
10	j				

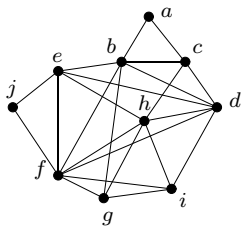


Figure 4: An example for our decoding algorithm, given n=10, B_1=*0010100001011001*, and B_2=11101011111110100 (the reconstructed graph is shown on the right.)

In a naive implementation of the general framework of our encoding algorithm, the time complexity is $O(n^2)$ as follows. In each step of the first stage, we need to count the number of neighbors in each disjoint subset in $\mathcal{P}$ and split it into two smaller subsets. This can be done in $O(n)$ time by scanning all remaining vertices in $\mathcal{P}$. Thus the first stage takes $O(n^2)$ time in total. In the second stage, a linear-time arithmetic encoder takes $O(n^2)$ time since the lengths of B_1 and B_2 are $O(n^2)$.

3.2.2 Average-Case $O(n+e)$-Time Algorithm

We describe another implementation of our algorithm that runs in time $O(n+e)$ on average when the input graph is generated from $\mathcal{G}(n,p)$. Note that we still use the same general framework and thus have the same compression performance as the previous implementation. To reduce the time complexity, we shall use the following three novel techniques. First, we use efficient data structures for maintaining the partition $\mathcal{P}$ and encoding the number of neighbors in each subset. Second, in the arithmetic encoding, we process the intermediate sequence B_2 not in bitwise manner, but instead we process a run of consecutive zeroes in one step. Third, when outputting the code in the arithmetic encoder, we use the greedy outputting method proposed in [18].

Data Structures To describe our data structures, we define the *position* of a vertex v in the partition $\mathcal{P}$ as the number of vertices on the right side of v in $\mathcal{P}$. Note that here $\mathcal{P}$ refers to a specific representation of a partition (e.g., $ba/e/cd$). Similarly, we define the *rank* of a subset U and all vertices $v \in U$ as the number of vertices on the right side of U in $\mathcal{P}$ (i.e., as the position of the rightmost vertex in U).

The partition $\mathcal{P}$ of a subset of $V(G)$ is maintained by the following five arrays, each of which is of size n. Arrays $pos[v]$ and $rank[v]$ store the position and the rank of a vertex v in $\mathcal{P}$, respectively. An array $vertex[i]$ stores the vertex at position i (i.e., $pos[vertex[i]] = i$). An array $size[r]$ stores the size of the subset whose rank is r. Lastly, for r such that $size[r] > 1$, an array $next[r]$ stores the largest rank r' such that $r' < r$ and $size[r'] > 1$. We also have a variable *head* containing the largest rank r such that $size[r] > 1$. These arrays are updated while $\mathcal{P}$ becomes smaller and finer in each step. Figure 5 shows some examples of such representation.

We observe the following properties: (1) The vertices with the same rank are in the same subset in $\mathcal{P}$; (2) The division of a subset U does not affect the ranks of vertices outside U (in fact, it affects only the rank of vertices in U that are graph neighbors of the chosen vertex); (3) Once the size of a subset becomes one, its rank is the same as its position and does not change until the end; (4) Using *head* and *next*, one can traverse only the subsets whose size is larger than one.

Algorithm Now we describe our algorithm in some detail. The first stage consists of n steps. Let $\mathcal{P}_i$ be the ordered partition after the i-th step, which is maintained implicitly by the arrays described above. Here we assume that the input graph is given as an adjacency list and $N(v)$ denotes the list of neighbors of vertex v. We also have a temporary array $\mathcal{B}$ of size n that stores sets of integers as discussed below. This array allows us to reconstruct B_2 by storing the step number i (for the positions of neighbors) or $-i$ (for the positions of subsets). An array $count[r]$ is used for counting the number of neighbors whose rank is r, which is set to zero initially. In the i-th step, the algorithm works as follows:

1. *[Selecting a vertex]*
   Remove any vertex v from the leftmost subset in $\mathcal{P}_{i-1}$ and update the arrays accordingly.
2. *[Counting the number of neighbors]*
   For each neighbor $u \in N(v)$ that is still in $\mathcal{P}_{i-1} - v$,
   - 2.1. Let r be $rank[u]$.
   - 2.2. If $size[r] > 1$, increase $count[r]$ by one.
   - 2.3. If $size[r] = 1$, mark its position by inserting the step number i in $\mathcal{B}[r]$.
3. *[Encoding the numbers for B_1 and dividing the partition]*
   While traversing subsets U such that $|U| > 1$ using *head* and *next*,
   - 3.1. Let r be the rank of U.
   - 3.2. Encode the number of neighbors in U (stored in $count[r]$) using $\lceil \log(size[r] + 1) \rceil$ bits and append it to B_1.
   - 3.3. Mark the position of U (i.e., the positions of both ends of U) by inserting $-i$ in both $\mathcal{B}[r]$ and $\mathcal{B}[r + size[r] - 1]$.
   - 3.4. If the division of U occurs (i.e, some vertices are neighbors, and some are non-neighbors), update arrays *size*, *next*, and *head* accordingly. If not, reset $count[r]$ to 0.

4. *[Moving vertices in the partition]*
   For each neighbor $u \in N(v)$ such that $count[rank[u]] > 0$ and u is still in $\mathcal{P}_{i-1} - v$,
   4.1. Let r be $rank[u]$.
   4.2. Decrease $count[r]$ by one.
   4.2. Move u to its correct position by updating *pos* and *vertex* (i.e., swap u and the vertex at position $r + size[r] + count[r]$).
   4.3. Update the rank of u (i.e., increase it by $size[r]$).

Example: Figure 5 shows the changes of our data structure step by step. For simplicity, here we select the leftmost vertex (i.e., vertex with the highest position) in each step. The updated values from the previous step are denoted by bold font. After ten steps, B_1 becomes 0010100001011001, and the information about B_2 is stored in array $\mathcal{B}$. ■

After repeating the above steps until $\mathcal{P}$ becomes empty, we extract B_2 from $\mathcal{B}$ in the form of a run length code. That is, B_2 is encoded as a sequence of lengths of the runs of zeroes between any two consecutive '1's (including both ends). For example, $B_2 = 11101011111110100$ is encoded as $0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 2$. First, we show how to extract B_2 from $\mathcal{B}$. Recall that, for each step i, B_2 is generated from the subsets of size one (i.e., singleton sets) in $\mathcal{P}_{i-1} - v$. In Figure 6, each column $\mathcal{B}[k]$ (for $k = 9, 8, \cdots, 0$) shows all the elements stored in $\mathcal{B}[k]$. Here we put the elements stored in step i at row i so that it better illustrates the process. From the information stored in $\mathcal{B}$, one can infer bits of B_2 generated in the i-th step as follows. In the i-th step, there are $n - i$ vertices in $\mathcal{P}_{i-1} - v$, and their positions are from 0 to $n - i - 1$. In substep 3.3, the position of each subset of size larger than one in $\mathcal{P}_{i-1} - v$ is marked by a pair of $-i$'s. Thus, one can infer the positions of singleton sets (e.g., shaded cells in Figure 6). In substep 2.3, the position of each singleton set containing a neighbor is marked by i. Each of these marked positions contributes a '1' while others contribute '0's. Thus the concatenation c_i of the bits in decreasing order of position is the contribution to B_2 in the i-th step. Therefore, B_2 is nothing but $c_1 c_2 \cdots c_{n-1}$. To generate B_2 as a run length code, we directly generate the run length code r_i of c_i without explicitly generating c_i. When merging r_i's, we need to treat the first and the last numbers in r_i's in a special way (i.e., by adding the last number in r_i and the first number in r_{i+1}). The last two columns of Figure 6 show c_i's and r_i's, respectively. The time complexity of the construction of B_2 is analyzed in the following lemma, which will be used later to analyze the overall time complexity of our algorithm.

Lemma 3 *The sequence B_2 can be constructed from $\mathcal{B}$ in $O(n + \ell)$ time, where ℓ is the total number of elements stored in $\mathcal{B}$.*

Proof: For each r_i, the number of zeroes between each two consecutive '1's can be inferred from the positions of i's and $-i$'s in $\mathcal{B}$. This process can be performed for all i's simultaneously by scanning $\mathcal{B}$ once from $\mathcal{B}[n-1]$ to $\mathcal{B}[0]$. This takes $O(n + \ell)$ time, while the concatenation of r_i's can be performed in $O(n)$ time. ■

In the second stage, both B_1 and B_2 are compressed by a binary arithmetic encoder, but B_2 is compressed by a modified arithmetic encoder, which uses the greedy outputting method as described in [18]. We first briefly describe a general (non-adaptive) binary arithmetic encoder and then describe our modified arithmetic encoder. Given a probability p for a bit

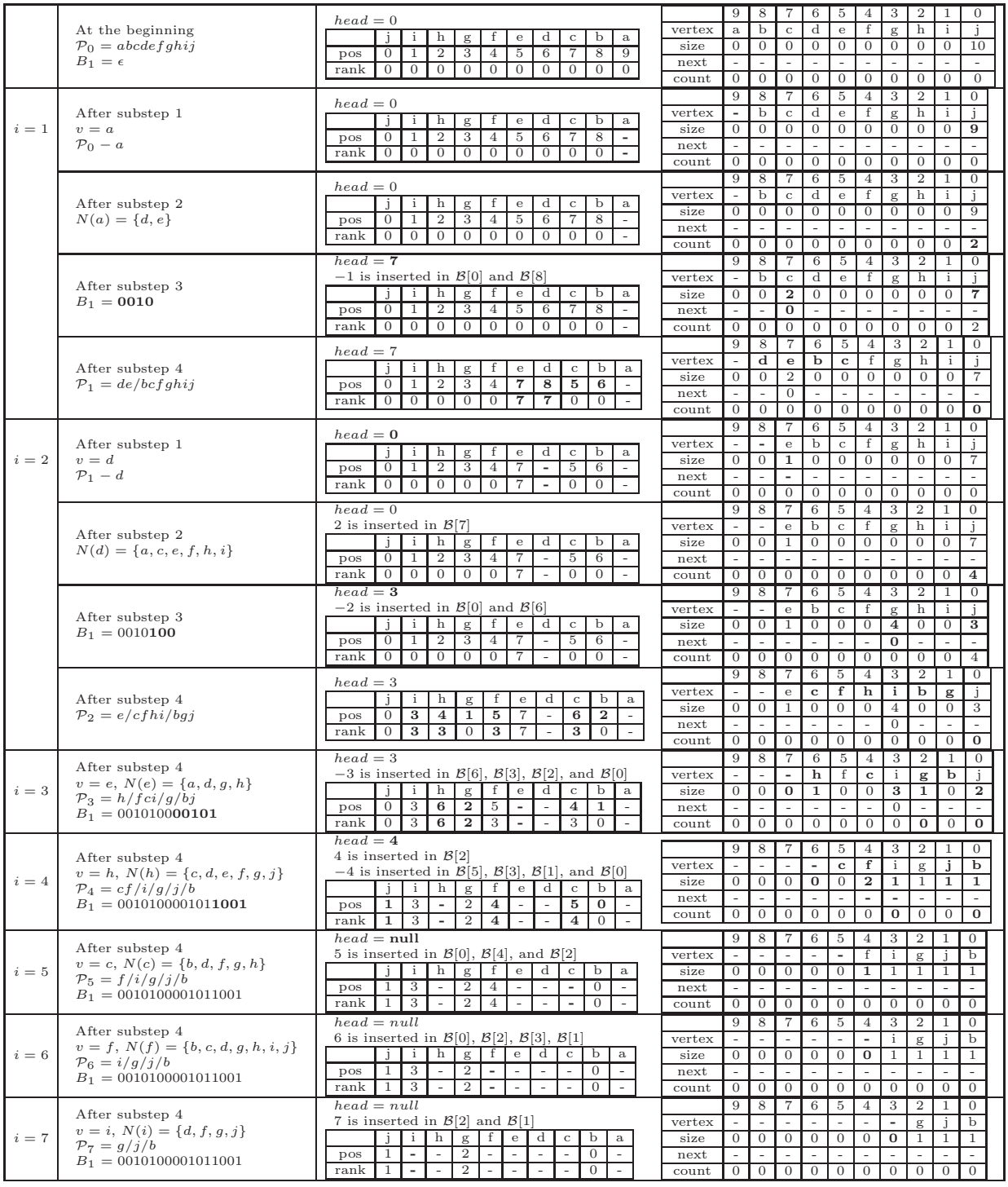


Figure 5: An example for our average $O(n+e)$-time encoding algorithm, given the graph in Figure 3.

step i	$\mathcal{P}_{i-1} - v$	$\mathcal{B}[9]$	$\mathcal{B}[8]$	$\mathcal{B}[7]$	$\mathcal{B}[6]$	$\mathcal{B}[5]$	$\mathcal{B}[4]$	$\mathcal{B}[3]$	$\mathcal{B}[2]$	$\mathcal{B}[1]$	$\mathcal{B}[0]$	c_i	r_i
1	$bcdefghij$		-1								-1		0
2	$e/bcfghij$			2	-2						-2	1	0,0
3	$cfhi/bgj$				-3			-3	-3		-3		0
4	$cfi/g/bj$					-4		-4	4	-4	-4	1	0,0
5	$f/i/g/j/b$						5		5		5	10101	0,1,1,0
6	$i/g/j/b$							6	6	6	6	1111	0,0,0,0,0
7	$g/j/b$								7	7		110	0,0,1
8	j/b									8		10	0,1
9	b											0	1

Figure 6: An example of extraction of B_2 from $\mathcal{B}$

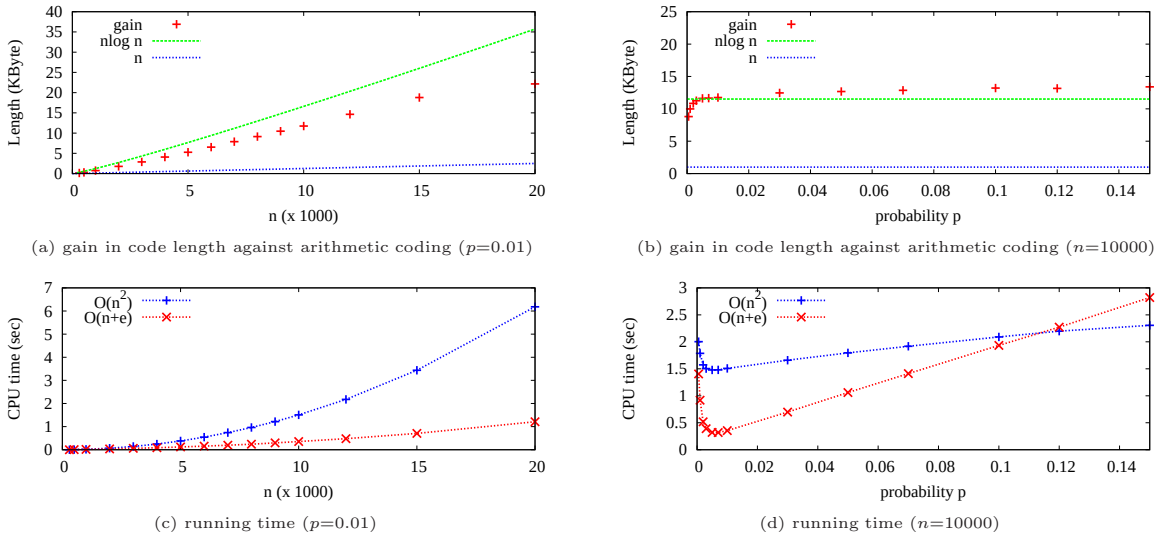
Table 1: The average code length and running time for some real-world networks.

			Code length (bits)				CPU time (secs)	
Networks	# of nodes	# of edges	Szip	adj. mat. $\binom{n}{2}$	adj. list $e\lceil \log n \rceil$	arithmetic coding	$O(n+e)$	$O(n^2)$
US Airports	332	2,126	8,108	54,946	19,134	12,947	<0.01	<0.01
Protein interaction (Yeast)	2,329	6,646	46,853	2,785,980	79,752	67,063	0.11	0.12
Collaboration (Geometry)	6,167	21,535	113,684	19,012,861	279,955	241,549	0.47	0.68
Collaboration (Erdős)	6,934	11,857	60,263	24,043,645	154,141	147,121	1.02	1.08
Genetic interaction (Human)	8,595	26,066	221,226	37,018,710	364,924	310,459	1.22	1.54
Internet (AS level)	25,881	52,407	301,463	334,900,140	786,105	737,851	12.97	13.81

'1', the encoder starts with an initial interval $[0, N)$ where N is a large positive integer. For each bit, it first calculates the new interval from the current interval. Then, from the newly calculated interval, it outputs code bits and normalize the interval so that its length is greater than a predefined threshold. If we use this encoder, the complexity would be $O(n^2)$ since the length of B_2 is $\Theta(n^2)$ in bits. Thus, in our modified encoder, we process a run of zeroes in one step. When we extract B_2 from $\mathcal{B}$, we compute the probability p of having '1' in B_2 and also precompute in a table the probability of a run of k zeroes, which is $(1-p)^k$ for $k = 1, 2, \cdots$. We recall that B_2 stores lengths of run of zeroes. When the encoder receives a number from B_2, it calculates the new interval for a run of zeroes in constant time by looking up the precomputed table. After this, it outputs code bits in a constant time using the greedy outputting method in [18], and then it processes a bit '1' in the usual way. Here we need one restriction on k since for very large k the probability $(1-p)^k$ becomes too small to represent the new interval precisely. Thus, we set $k_{max} = \lceil 1/p \rceil$, and if $k > k_{max}$, then we process only the first k_{max} zeroes in every step until all k zeroes are exhausted.

3.3 Experimental Results

To test our algorithm, we applied it to Erdős-Rényi random graphs and real-world networks including biological, social, and technological networks. Table 1 summarizes the results for the real-world networks. For comparison, we list the lengths of three other encodings of graphs, namely, the usual implementations of adjacency matrix of $\binom{n}{2}$ bits, and adjacency list of *at least* $e\lceil \log n \rceil$ bits (normally, $2e\lceil \log n \rceil$ bits) where e is the number of edges. Finally, we applied an arithmetic encoder to the adjacency matrix, which can achieve $\binom{n}{2}h(p)$ bits. For many real-world networks, our algorithm achieves twice better compression than the standard arithmetic encoder. For comparison of running time, we implemented the two versions of our algorithm. For all of our real-world test data, our $O(n+e)$-time implementation is faster than $O(n^2)$-time implementation. We measured CPU time on a machine equipped with Pentium D 3.0GHz processor and 2GB of RAM, running Linux. All the numbers are averages over



(a) gain in code length against arithmetic coding (p=0.01)

(b) gain in code length against arithmetic coding (n=10000)

(c) running time (p=0.01)

(d) running time (n=10000)

Figure 7: The average gain in code length and running time for the Erdős-Rényi random graphs.

100 measurements.

Figure 7 shows the results for $\mathcal{G}(n,p)$ graphs. In (a) and (b), we plot the gain of our encoding against arithmetic encoding of the adjacency matrix, that is, the difference between the two encodings. We plot it for a fixed p in (a) and for a fixed n in (b). The plots confirm our analysis that the gain is asymptotically close to $n \log n$. In (c) and (d), we plot the CPU time consumed by $O(n+e)$-time and $O(n^2)$-time implementations, and it shows that our $O(n+e)$-time implementation is faster unless the graph is too dense.

One of the referees asked us to compare our compression algorithm to the one discussed in [1] (applied to web data WT2g from TREC). A direct comparison is not very revealing and informative since the graphs tested in [1] are directed graphs and the type of data compressed is different. Nevertheless, we manage to compare compression rates per edge. The graph in [1] is a directed graph with 247428 nodes and 1166702 edges. The best result shown in Table 2 of [1] is 8.35 bits/edge. To compress this graph using our algorithm, we transformed it into an undirected graph with 989109 edges. Our algorithm Szip compresses the graph structure to 3454719 bits in total, that is, 3.49 bits/edge. This is better compression rate per edge than the one reported in [1]. Again, we should be cautious to draw too fast conclusion since both algorithms are not completely compatible.

Let us make some final observations. Our results predict that for structures generated by Erdős-Rényi model, $\mathcal{S}(n,p)$, one can achieve compression up to

$$\binom{n}{2} h(p) - n \log n + O(n)$$

bits which should be compared to $\binom{n}{2} h(p)$ bits, if conventional algorithms are used (i.e., arithmetic encoder to the adjacency matrix). The redundancy, $n \log n$ of our compression scheme is confirmed for randomly generated graphs from $\mathcal{G}(n,p)$. For many real-world graphs presented in Table 1, our algorithm achieves more than twice better compression when compared to standard arithmetic encoder. While these graphs are not randomly generated according to $\mathcal{G}(n,p)$ (rather by a *power-law* distribution), we believe their good compression rate is a

consequence of small p. Indeed, consider for our $\mathcal{G}(n,p)$ model the behavior of the structural entropy $H_{\mathcal{S}}$ when $p \to 0$ satisfying the conditions of Theorem 1. Let then $p \sim \omega(n)(\log n/n)$ for slowly growing $\omega(n) \to \infty$ as $n \to \infty$. In this case

$$h(p) \sim \omega(n)\frac{\log^2 n}{n},$$

and therefore the structural entropy becomes

$$H_{\mathcal{S}} \sim \frac{1}{2}(n-1)\omega(n)\log^2 n - n\log n + O(n).$$

Clearly, the second leading term $n \log n$ plays a significant role in the compression of such graphs. This may explain why our encoding is much better than arithmetic coding for real-world networks that are usually sparse graphs. However, to establish this fact rigorously we need to extend our analysis to other graph generation models such as the power law graphs.

4 Analysis

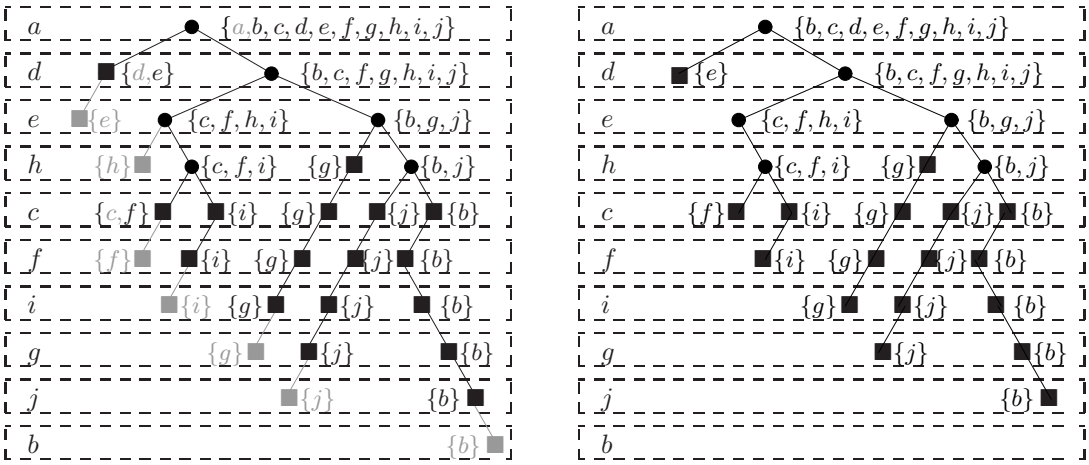
In this section, we analyze the compression performance and time complexity of our algorithm, proving Theorem 2. To accomplish it we apply a variety of combinatorial and analytic techniques such as generating functions, Mellin transform, poissonization, and combinatorics.

We start with a description of two binary trees that better capture the progress of our algorithm. Given a graph G on n vertices, the binary tree T_n is built as follows. At the beginning, the root node contains all n graph vertices, $V(G)$, that one can also visualize as n balls. In the first step, a graph vertex (ball) v is removed from the root node, and the other $n-1$ graph vertices move down to the left or to the right depending whether they are adjacent vertices in G to v or not; adjacent vertices go to the left child node and the others go to the right child node. We create a new child node in T_n if there is at least one graph vertex in that node. After the first step, the tree is of height 1 with $n-1$ graph vertices in the nodes at level 1. Similarly, in the i-th step, we remove one graph vertex (ball) v from the (level-wise) leftmost node at level $i-1$. If this removal makes the node empty, we remove the node. The other graph vertices at level $i-1$ move down to the left or to the right depending whether they are adjacent to v or not. We repeat these steps until all graph vertices are removed (i.e., after n steps).

For our example from Figure 3, the construction of the tree T_n and the progress of the algorithm are presented in Figure 8. The removed graph vertices are shown on the left. At each level, the subsets of graph vertices (before removing a vertex from the leftmost node in (a) and after in (b)) are shown next to the nodes. We observe that the subsets at each level (from left to right) in T_n are the same as the subsets in $P_{k-1} - v$ at each step of our algorithm in Figure 3.

Let N_x denote the number of graph vertices that pass through node x in T_n (excluding the graph vertex removed at x, if any). In Figure 8(b), for example, N_x is the number of graph vertices shown next to the node x. Our algorithm needs to encode, for each node x in T_n, the number of neighbors (of the removed graph vertex) among N_x vertices. This requires $\lceil \log(N_x+1) \rceil$ bits. Let $L(B_1)$ and $L(B_2)$ be the lengths of sequences B_1 and B_2, respectively. By construction, these lengths are defined as

$$L(B_1) = \sum_{x \in T_n \text{ and } N_x > 1} \lceil \log(N_x+1) \rceil,$$



(a) T_n with removed nodes and vertices in gray (b) T_n

Figure 8: A binary tree T_n, with square-shaped nodes containing exactly one ball and circle-shaped nodes containing more than one ball.

$$\text{and} \quad L(B_2) = \sum_{x \in T_n \text{ and } N_x = 1} \lceil \log(N_x + 1) \rceil = \sum_{x \in T_n \text{ and } N_x = 1} 1.$$

In Figure 8(b), $L(B_1)$ and $L(B_2)$ are sums over all circle-shaped nodes and over all square-shaped nodes, respectively. Here we can observe an important property of B_2 presented next.

Lemma 4 *Given a graph from $\mathcal{G}(n,p)$, the sequence B_2 constructed by our algorithm is probabilistically equivalent to a binary sequence generated by a memoryless source(p) with p being the probability of generating a '1'.*

Proof: Consider any bit $b \in B_2$. It represents the number of neighbors of a vertex u in a subset, which contains only one vertex, say v. Then the probability that $b =$'1' is the same as the probability that u and v are connected, which is p in the Erdős-Rényi model. Let us consider any two bits b_1 and b_2. Assume that b_i corresponds to vertices u_i and v_i (i.e., b_i corresponds to the potential edge between u_i and v_i.) These two potential edges are chosen independently according to the Erdős-Rényi model. This shows the memoryless property. ■

To set up precise recurrence relations for our analysis, we need to define a random binary tree $T_{n,d}$ for integers $n \geq 0$ and $d \geq 0$, which is generated similarly to T_n as follows. If $n = 0$, then it is just an empty tree. For $n > 0$, we create a root node, in which we put n balls. In each step, all balls independently move down to the left (with probability p) or right (with probability $1 - p$). We create a new node if there is at least one ball in that node. Thus, after the i-th step, the balls will be at level i. If the balls are at level d or greater, then we remove one ball from the leftmost node before the balls move down to the next level. These steps are repeated until all balls are removed (i.e., after $n + d$ steps). We observe that, if T_n is generated by a graph from $\mathcal{G}(n,p)$, T_n is nothing but the random binary tree $T_{n,0}$. Thus, by analyzing $T_{n,0}$, we can compute both $L(B_1)$ and $L(B_2)$.

4.1 Proof of Theorem 2(i): Average Performance

In this section, we prove part (i) of our main result, that is, we derive the average length of the compressed string representing graphical structure.

Let us first estimate $L(B_1)$. Recall, N_x denotes the number of balls that pass through node x (excluding the ball removed at x, if any). Let

$$A_{n,d} = \sum_{x \in T_{n,d} \text{ and } N_x > 1} \lceil \log(N_x + 1) \rceil,$$

and $a_{n,d} = \mathbf{E}[A_{n,d}]$. Then $\mathbf{E}[L(B_1)] = a_{n,0}$. Clearly, $a_{0,d} = a_{1,d} = 0$ and $a_{2,0} = 0$. For $n \geq 2$ and $d = 0$, we observe that

$$a_{n+1,0} = \lceil \log(n+1) \rceil + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a_{k,0} + a_{n-k,k}). \tag{5}$$

This follows from the fact that starting with $n+1$ balls in the root node, and removing one ball we are left with n balls passing through the root node. The root contributes $\lceil \log(n+1) \rceil$. Then, those n balls move down to the left or right subtrees. Let us assume k balls move down to the left subtree (the other $n-k$ balls must move down to the right subtree, and this happens with probability $\binom{n}{k} p^k q^{n-k}$.) At level one, one ball is removed from those k balls in the root of the left subtree. This contributes $a_{k,0}$. There will be no removal among $n-k$ balls in the right subtree until all k balls in the left subtree are removed. This contributes $a_{n-k,k}$. Similarly, for $d > 0$, we can see that

$$a_{n,d} = \lceil \log(n+1) \rceil + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a_{k,d-1} + a_{n-k,k+d-1}). \tag{6}$$

This recurrence is quite complex, but we only need a good upper bound that is presented in the next lemma.

Lemma 5 *For all integers $n \geq 0$ and $d \geq 0$, we have*

$$a_{n,d} \leq x_n$$

where x_n satisfies $x_0 = x_1 = 0$ and for $n \geq 2$

$$x_n = \lceil \log(n+1) \rceil + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}). \tag{7}$$

Proof: We use induction on both n and d. Clearly, $a_{n,d} \leq x_n$ for $n = 0$ or 1 ($d \geq 0$). For $n = 2$ and $d = 0$, $a_{2,0} \leq x_2$ since $a_{2,0} = 0$ and $x_2 \geq 2$. For other cases ($n = 2$ and $d > 0$, or $n > 2$), we assume that $a_{i,j} \leq x_i$ holds for $i < n$, and for $i = n$ and $j < d$. Now we want to show that $a_{n,d} \leq x_n$. We divide it into two cases.
(i) Case $d = 0$. We observe that

$$a_{n,0} \leq a_{n+1,0} = \lceil \log(n+1) \rceil + \sum_{k=1}^{n-1} \binom{n}{k} p^k q^{n-k} (a_{k,0} + a_{n-k,k}) + q^n a_{n,0} + p^n a_{n,0}.$$

Thus,

$$(1 - p^n - q^n) a_{n,0} \leq \lceil \log(n+1) \rceil + \sum_{k=1}^{n-1} \binom{n}{k} p^k q^{n-k} (a_{k,0} + a_{n-k,k}). \tag{8}$$

Similarly, from (7), we get

$$(1-p^n-q^n)x_n = \lceil \log(n+1) \rceil + \sum_{k=1}^{n-1} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}). \tag{9}$$

Therefore,

$$\begin{aligned}
(1-p^n-q^n)a_{n,0} &\leq \lceil \log(n+1) \rceil + \sum_{k=1}^{n-1} \binom{n}{k} p^k q^{n-k} (a_{k,0} + a_{n-k,k}) \qquad \text{(by (8))} \\
&\leq \lceil \log(n+1) \rceil + \sum_{k=1}^{n-1} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}) \quad \text{(by induction hypothesis)} \\
&= (1-p^n-q^n)x_n. \qquad \text{(by (9))}
\end{aligned}$$

(ii) Case $d > 0$. By (6) and the induction hypothesis,

$$a_{n,d} \leq \lceil \log(n+1) \rceil + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}) = x_n.$$

This completes the proof. ∎

The next step involves solving asymptotically recurrence (7). We do it in Section 5 proving the following lemma.

Lemma 6 *Consider the following recurrence for x_n with $x_0 = x_1 = 0$ and for $n \geq 2$*

$$x_n = a_n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}),$$

where $a_n = \lceil \log(n+1) \rceil$ for $n \geq 2$ and $a_0 = a_1 = 0$. Then:
(i) *If $\log p / \log q$ is irrational, then*

$$x_n = \frac{n}{h(p)} A^*(-1) \log e + o(n), \tag{10}$$

where

$$A^*(-1) = \sum_{b \geq 2} \frac{\lceil \log(b+1) \rceil}{b(b-1)}. \tag{11}$$

(ii) *If $\log p / \log q = r/d$ (rational) with $\gcd(r,d) = 1$, then*

$$x_n = \frac{n}{h(p)} \left(A^*(-1) + \Phi(\log_p n) \right) \log e + O(n^{1-\eta}) \tag{12}$$

for some $\eta > 0$, where

$$\Phi(x) = \sum_{k \neq 0} A^*(-1 + 2k\pi r i / \log p) \exp(2k\pi r x i) \tag{13}$$

is a fluctuating function with a small amplitude.

Finally, the average length of $L(B_1)$ can be derived. We present it in the next theorem.

Theorem 3 *For large n,*

$$\mathbf{E}[L(B_1)] \leq \frac{n}{h(p)}\left(\beta + \Phi_1(\log n)\right) + o(n),$$

where

$$\beta = \log e \cdot \sum_{b\geq 2} \frac{\lceil \log(b+1) \rceil}{b(b-1)} = 3.760\cdots,$$

and $\Phi_1(\log n)$ is a fluctuating function for $\log p/\log q$ rational with small amplitude and asymptotically zero otherwise.

Proof. It follows directly from Lemmas 5 and 6. ∎

The next step is to estimate the average length of B_2. Let $S_{n,d}$ be the total number of nodes x in $T_{n,d}$ such that $N_x = 1$, that is,

$$S_{n,d} = \sum_{x\in T_{n,d} \text{ and } N_x=1} 1 \;=\; \sum_{x\in T_{n,d} \text{ and } N_x=1} N_x \;=\; \sum_{x\in T_{n,d}} N_x - \sum_{x\in T_{n,d} \text{ and } N_x>1} N_x.$$

Let $B_{n,d} = \sum_{x\in T_{n,d}, N_x>1} N_x$. We observe that

$$L(B_2) = S_{n,0} = \sum_{x\in T_{n,0}} N_x - B_{n,0} = \frac{n(n-1)}{2} - B_{n,0}. \tag{14}$$

The last equality follows from the fact that the sum of N_x's for all x at level ℓ in $T_{n,0}$ is equal to $n-1-\ell$.

Let $b_{n,d} = \mathbf{E}[B_{n,d}]$. For our analysis we only need $b_{n,0}$. Clearly, $b_{0,d} = b_{1,d} = 0$ and $b_{2,0} = 0$. For $n \geq 2$, we can find the following recurrence (similarly to $a_{n,d}$):

$$b_{n+1,0} = n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (b_{k,0} + b_{n-k,k}), \tag{15}$$

$$\text{and} \quad b_{n,d} = n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (b_{k,d-1} + b_{n-k,k+d-1}) \quad \text{for } d > 0. \tag{16}$$

To prove our main result, we only need a lower bound that is established in the next lemma.

Lemma 7 *For all $n \geq 0$ and $d \geq 0$,*

$$b_{n,d} \geq y_n - \frac{n}{2}$$

such that y_n satisfies $y_0 = 0$ and for $n \geq 0$

$$y_{n+1} = n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (y_k + y_{n-k}). \tag{17}$$

Proof: We prove it by induction on both n and d. Clearly, $b_{n,d} \geq y_n - n/2$ for $n = 0$ or 1 ($d > 0$). For $n = 2$ and $d = 0$, $b_{2,0} \geq y_2 - 2$ since $b_{2,0} = 0$ and $y_2 = 1$. For other cases ($n = 2$ and $d > 0$, or $n > 2$), we assume that $b_{i,j} \geq y_i - \frac{i}{2}$ holds for $i < n$, and for $i = n$ and $j < d$. Now we want to show that $b_{n,d} \geq y_n - \frac{n}{2}$. We divide it into two cases.
(i) Case $d = 0$. By (15) and the induction hypothesis, we have

$$\begin{aligned} b_{n,0} &\geq (n-1) + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-1-k} (y_k - \frac{k}{2} + y_{n-1-k} - \frac{n-1-k}{2}) \\ &= y_n - \frac{n-1}{2} > y_n - \frac{n}{2}. \quad \text{(by (17))} \end{aligned}$$

(ii) Case $d > 0$. By (16) and the induction hypothesis, we have

$$\begin{aligned} b_{n,d} &\geq n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (y_k - \frac{k}{2} + y_{n-k} - \frac{n-k}{2}) \\ &= y_{n+1} - \frac{n}{2} \geq y_n - \frac{n}{2}. \end{aligned}$$

This completes the proof. ∎

It is easy to see that y_n represents the expected path length in a *digital search tree* over n strings as discussed in [16, 33]. A digital search tree stores strings (keys, items, balls) directly in the nodes. At level k, the branching is based on the kth symbol of an inserted string (see [33] for details). The authors of [16] proved, among others, that

$$\begin{aligned} y_n &= \frac{n}{h(p)} \left(\log n + \frac{h_2}{2h(p)} + \gamma - 1 - \alpha + \Phi_2(\log n) \right) \\ &\quad + \frac{1}{h(p)} \left(\log n + \frac{h_2}{2h(p)} - \gamma - \log p - \log q + \alpha \right) + O(1), \end{aligned} \tag{18}$$

where $h_2 = p \log^2 p + q \log^2 q$, $\gamma = 0.577\cdots$ is the Euler constant, and

$$\alpha = -\sum_{k=1}^{\infty} \frac{p^{k+1} \log p + q^{k+1} \log q}{1 - p^{k+1} - q^{k+1}}.$$

In the above, $\Phi_2(\log n)$ is a fluctuating function for $\log p / \log q$ rational with small amplitude and zero otherwise.

In summary, by (14), Lemma 7, and the above, we arrive at our next result.

Theorem 4 *For large n,*

$$\begin{aligned} \mathbf{E}[L(B_2)] &\leq \frac{n(n-1)}{2} - \frac{n}{h(p)} \log n \\ &\quad + \frac{n}{h(p)} \left(\frac{h(p)}{2} - \frac{h_2}{2h(p)} - \gamma + 1 + \alpha - \Phi_2(\log n) \right) - \frac{1}{h(p)} \log n + O(1), \end{aligned}$$

with the notations as below (18).

Finally, we compute $\mathbf{E}[L(S)] = \mathbf{E}[L(\hat{B}_1) + L(\hat{B}_2)] + O(\log n)$, where $\hat{B}_1$ and $\hat{B}_2$ are compressed strings B_1 and B_2, while $O(\log n)$ bits are needed to encode n. This proves the part (i) of Theorem 2. We observe that the arithmetic encoder can compress a binary sequence of length m on average up to $mh + \frac{1}{2}\log m + O(1) = mh + O(logm)$, where h is the entropy rate of the binary source [9, 36]. Thus, by Theorem 3,

$$\mathbf{E}[L(\hat{B}_1)] \leq \frac{h'}{h(p)}(\beta + \Phi_1(\log n))n + o(n),$$

where β and $\Phi_1(\log n)$ are defined in Theorem 3, and h' is the entropy rate of the binary source that B_1 is generated from. Similarly, we can compute $\mathbf{E}[L(\hat{B}_2)]$. In this case, however, we know that the entropy rate for B_2 is $h(p)$. Thus, by Theorem 4,

$$\mathbf{E}[L(\hat{B}_2)] \leq \binom{n}{2} h(p) - n\log n + n\left(\frac{h(p)}{2} - \frac{h_2}{2h(p)} - \gamma + 1 + \alpha - \Phi_2(\log n)\right) + O(\log n),$$

where h_2, γ, α, and $\Phi_2(\log n)$ are defined above. This completes the part (i) of Theorem 2.

4.2 Proof of Theorem 2(ii): Performance with High Probability

Now we prove part (ii) of Theorem 2, that is, we show that $L(S) - \mathbf{E}[L(S)] \leq \epsilon n \log n$ with high probability. Since $L(S) = L(\hat{B}_1) + L(\hat{B}_2)$, we need bounds for $L(\hat{B}_1)$ and $L(\hat{B}_2)$. We start with $L(\hat{B}_1)$. By Markov's inequality,

$$P\left(L(\hat{B}_1) > \epsilon n \log n\right) < \frac{\mathbf{E}[L(\hat{B}_1)]}{\epsilon n \log n} = O\left(\frac{1}{\log n}\right), \quad \epsilon > 0. \tag{19}$$

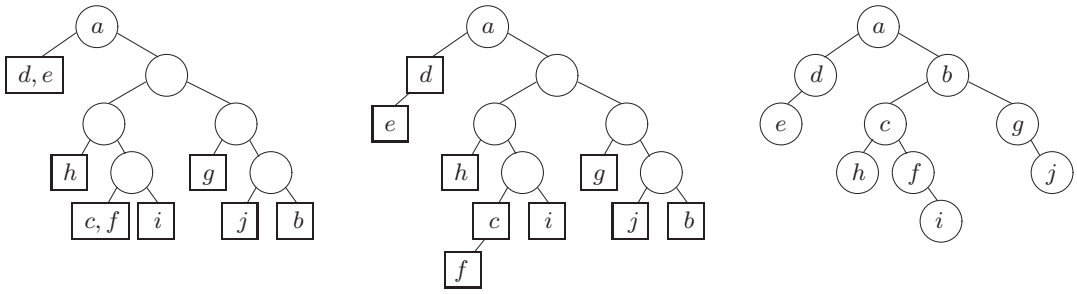
Handling $L(\hat{B}_2)$ is more complicated. In [36] it was proved that for a binary sequence X of length ℓ, the code length generated by an arithmetic encoder is at most $-\log P(X) + \frac{1}{2}\log \ell + 3$. In our case, $B_2 = b_1 b_2 \cdots b_{L(B_2)}$ is memoryless, and then

$$L(\hat{B}_2) < -\log P(B_2) + \frac{1}{2}\log L(B_2) + 3 = L(B_2) \cdot \left[-\frac{1}{L(B_2)} \sum_{i=1}^{L(B_2)} \log P(b_i)\right] + \frac{1}{2}\log L(B_2) + 3. \tag{20}$$

Thus we need good bounds for $L(B_2)$ and the sum of $\log P(b_i)$. With respect to $L(B_2)$, recall that $L(B_2) = \binom{n}{2} - B_{n,0}$ where

$$B_{n,0} = \sum_{x \in T_{n,0}, N_x > 1} N_x,$$

and N_x is the number of balls that pass through node x in tree $T_{n,0}$ (excluding the ball removed at x, if any). We shall show that $B_{n,0}$ is related to the *path lengths* in slightly modified trees that we denote as $\hat{T}_n$ and $\bar{T}_n$. The tree $\hat{T}_n$ is constructed from $T_{n,0}$ first by recreating the nodes removed during the construction of $T_{n,0}$ (e.g., the tree in Figure 8(a)). Then, we put each ball back into the node from where it was removed and keep moving it up until its parent node is a square-shaped node. To construct $\bar{T}_n$ we observe that there might be some nodes with two balls in $\hat{T}_n$. In such a case, we add a child node and move one ball down to the new node to eliminate all nodes with two balls. Figure 9(a,b) illustrates the



(a) $\hat{T}_n$ (b) $\bar{T}_n$ (c) D_n

Figure 9: An example of binary trees $\hat{T}_n$, $\bar{T}_n$, and D_n, given binary choices for 10 balls $\{a,b,c,d,e,f,g,h,i,j\}$.

construction of $\hat{T}_n$ and $\bar{T}_n$ for the tree T_n (equivalently, $T_{n,0}$) in Figure 8(b). Notice that in this figure all circle-shaped nodes and the square-shaped nodes – directly connected to these circle-shaped nodes – are the same in both T_n and $\hat{T}_n$.

Let $\ell(\hat{T}_n)$ and $\ell(\bar{T}_n)$ be the path lengths to all *balls* in $\hat{T}_n$ and $\bar{T}_n$, respectively. From the construction it is clear that

$$B_{n,0} = \ell(\hat{T}_n).$$

Now let us compare $\ell(\hat{T}_n)$ and $\ell(\bar{T}_n)$. Whenever we have two balls in a node of $\hat{T}_n$, we move one ball down in $\bar{T}_n$ to a new node. This results in a path in $\bar{T}_n$ that is longer by one than the corresponding path in $\hat{T}_n$. However, this can happen at most $n/2$ times since there are at most $n/2$ nodes with two balls. Thus we find[3]

$$\ell(\hat{T}_n) + n/2 \geq_{st} \ell(\bar{T}_n).$$

To estimate the path length $\ell(\bar{T}_n)$, we introduce another binary tree D_n that is probabilistically equivalent to the *digital search tree* built over n random binary strings. It is constructed as follows. If $n = 0$, then it is just an empty tree. For $n > 0$, we create a root node in which we put n balls. One ball remains in the root node, and the other balls independently move down to the left or to the right. We create a new child node if there is at least one ball in that node. We recursively repeat it (i.e., we leave one ball in a node while moving others down.) Figure 9(c) illustrates this procedure.

We shall next show that

$$\ell(\bar{T}_n) \geq_{st} \ell(D_n),$$

where $\ell(D_n)$ is the path length to all balls (nodes) in D_n. For this, we consider two actual trees $\bar{t}_n$ and d_n given the same binary choices regarding the action left/right (1/0) for the n balls. We also assume that the input to both trees is the same, that is, balls are inserted in the same order and therefore we always identify the "smallest" ball in input. Whenever a ball remains in a node during the construction of these trees, we assume that the smallest

[3]For two real-valued random variables X and Y, we write $X \geq_{st} Y$ if the value of X is always greater than or equal to that of Y for every event, or equivalently if $P(X > t) \geq P(Y > t)$ for all $t \in (-\infty, \infty)$ [27].

ball is left in the node. Then, in the next lemma we show that the path length in $\bar{t}_n$ is at least the path length in d_n. Thus $\ell(\bar{T}_n) \geq_{st} \ell(D_n)$.

Lemma 8 *Given binary choices for n balls, let $\bar{t}_n$ and d_n be two tree instances of $\bar{T}_n$ and D_n, respectively. Let $u_t \in \bar{t}_n$ and $u_d \in d_n$ be two corresponding nodes in these trees (i.e., nodes that are reached by the same binary choices). We denote by $B(u)$ the set of balls in the subtree rooted at node u. Then, $B(u_t) \supset B(u_d)$ for any $u_t \in \bar{t}_n$ and $u_d \in d_n$.*

Proof: For the root nodes, it is trivial since both sets have the same n balls. Now it is sufficient to show that the statement is true for children if it is true for their parent nodes. Thus let us assume that $B(u_t) \supset B(u_d)$ for $u_t \in \bar{t}_n$ and $u_d \in d_n$. Let s_t and s_d be the smallest ball (in the input ordering) in $B(u_t)$ and $B(u_d)$, respectively. Now we consider two sets of balls S_t and S_d that will move down from u_t and u_d, respectively. Note that $S_d = B(u_d) - s_d$. We shall show that $S_t \supset S_d$ considering two cases: 1) if u_t is not the leftmost node, then $S_t = B(u_t) \supset B(u_d) - s_d = S_d$; 2) if u_t is the leftmost node, then $S_t = B(u_t) - s_t \supset B(u_d) - s_d = S_d$ since either s_t is the same ball as s_d or s_t is not in S_d. Therefore each ball $b \in S_d$ is also in S_t, and b moves down in the same direction for both u_t and u_d. Therefore, the statement is true for both children nodes. ∎

Now we are ready to prove a relation between $B_{n,0}$ and the path length in a digital search tree, discussed in the following lemma.

Lemma 9 *Let $Y_n := \ell(D_n)$ be the path length in a digital search tree. Then,*

$$B_{n,0} + \frac{n}{2} \geq_{st} Y_n.$$

Proof: Given binary choices for n balls, let us consider tree instances $\hat{t}_n$, $\bar{t}_n$, and d_n. As we have observed, $\ell(\hat{t}_n) + \frac{n}{2} \geq \ell(\bar{t}_n) \geq \ell(d_n)$. Therefore, $\ell(\hat{T}_n) + \frac{n}{2} \geq_{st} \ell(D_n)$. We know that $\ell(\hat{T}_n)$ and $\ell(D_n)$ are equivalent to $B_{n,0}$ and Y_n, respectively. This completes the proof. ∎

Finally, we establish the following two lemmas.

Lemma 10 *For any $\epsilon > 0$,*

$$P\left(L(B_2) \leq \binom{n}{2} - y_n + \epsilon y_n\right) \geq 1 - o(1),$$

where y_n is defined in Lemma 7.

Proof: We observe that $y_n = \mathbf{E}[Y_n]$, where Y_n is the path length in a digital search tree. Let us compute the probability $P_n = P\left(L(B_2) > \binom{n}{2} - y_n + \epsilon y_n\right)$ for large n. We shall prove that $P_n \to 0$. We have

$$\begin{aligned}
P_n &= P\left(B_{n,0} < (1-\epsilon)y_n\right) \qquad \text{(by (14), that is, } L(B_2) = \tbinom{n}{2} - B_{n,0}\text{)} \\
&\leq P\left(Y_n - \frac{n}{2} < (1-\epsilon)y_n\right) \qquad \text{(by Lemma 9)} \\
&= P\left(\frac{Y_n - y_n}{\sqrt{\mathbf{Var}\ Y_n}} < \frac{-\epsilon y_n + n/2}{\sqrt{\mathbf{Var}\ Y_n}}\right) \\
&\leq P\left(\left|\frac{Y_n - y_n}{\sqrt{\mathbf{Var}\ Y_n}}\right| > \left|\frac{-\epsilon y_n + n/2}{\sqrt{\mathbf{Var}\ Y_n}}\right|\right) \quad (\because \tfrac{-\epsilon y_n + n/2}{\sqrt{\mathbf{Var}\ Y_n}} < 0 \text{ for large } n) \\
&< A\mu^k \qquad \text{(by Theorem 1A of [16])}
\end{aligned}$$

for positive constants A and $\mu < 1$, where $k = \left|\frac{-\epsilon y_n + n/2}{\sqrt{\mathbf{Var}\ Y_n}}\right| = \Theta(\sqrt{n \log n})$ as proved in Theorem 1A of [16]. Thus, P_n becomes exponentially small as $n \to \infty$. ∎

In view of (20) and Lemma 10, we need to find a bound for $\sum_{i=1}^{L(B_2)} \log P(b_i)$ which we present next.

Lemma 11 *For any $\epsilon > 0$,*

$$P\left(-\frac{1}{L(B_2)} \sum_{i=1}^{L(B_2)} \log P(b_i) \le h(p) + \epsilon \frac{\log n}{n}\right) \ge 1 - o(1).$$

Proof: Let $F_m(X_1, \cdots, X_m) = -\log P(X_1, \cdots, X_m) - mh(p)$, where the X_i's are binary independent random variables with p being the probability of '1' and $q = 1 - p$. Denoting by $\hat{X}_i$ an independent copy of X_i (with the same distribution as X_i), we have

$$|F_m(X_1, \cdots, X_i, \cdots, X_m) - F_m(X_1, \cdots, \hat{X}_i, \cdots, X_m)| \le |\log P(X_i) - \log P(\hat{X}_i)| \le c,$$

where $c = \max\{\log(p/q), \log(q/p)\}$. Thus, by Azuma's inequality [33]

$$P(-\log P(X_1, \cdots, X_m) - mh(p) \ge \epsilon' n \log n) \le \exp\left(-\frac{\epsilon'^2 n^2 \log^2 n}{2mc^2}\right) = o(1)$$

provided that $m = O(n^2)$. Since $L(B_2) = O(n^2)$, this completes the proof. ∎

By (18),(20), and the above two lemmas, after some algebra we conclude that, with probability $1 - o(1)$,

$$L(\hat{B}_2) < \binom{n}{2} h(p) - n \log n + \epsilon n \log n.$$

This and (19) complete the part (ii) of Theorem 2.

4.3 Proof of Theorem 2(iii): Time Complexity

In this section, we prove part (iii) of Theorem 2, that is, we show that the time complexity of our algorithm in Section 3.2.2 is $O(n + e)$ on average. Let us first analyze the first stage, that consists of n steps. Clearly, the substep 1 takes constant time in each step, and thus it takes $O(n)$ time in total. The substeps 2 and 4 take $O(|N(v)|)$ time in each step since each of operations inside the loop takes constant time. Thus, they take $\sum_{v \in V(G)} O(|N(v)|) = O(e)$ time in total. In the i-th step, the substep 3 takes $O(s_i)$ time, where s_i is the number of subsets in $\mathcal{P}_{i-1} - v$ whose size is larger than one. Thus, in total, it takes $O(s)$ time where $s = \sum_{i=1}^{n} s_i$, which is the total number of nodes x in $T_{n,0}$ with $N_x > 1$. In Figure 8, for example, s is the number of circle-shaped nodes in T_n. By the same analysis as in Section 5 (in this case, $a_n = 1$ in (21)), we can prove that the expected value of s is at most $O(n)$.

Finally, by Lemma 3, the construction of B_2 from $\mathcal{B}$ takes $O(n + \ell)$ time where ℓ is the number of elements inserted in $\mathcal{B}$. We can see that $\ell = O(e + s)$ as follows. The number of elements inserted in substep 2.3 is bounded by e since every insertion corresponds to a

distinct edge. Clearly, the number of elements inserted in substep 3.3 is bounded by $O(s)$. Therefore, the first stage takes $O(n+e)$ time on average.

In the second stage, B_1 and B_2 are compressed by an arithmetic encoder. Clearly, B_1 can be compressed in $O(n)$ time since the length of B_1 is $O(n)$. The number of elements in the run length form of B_2 is at most $e+1$. Thus, the time complexity of the compression of B_2 would be $O(e)$ except that there could be long runs of zeroes, which are compressed in multiple steps. Let n_0 and n_1 be the number of '0's and '1's in B_2, respectively. Thus, $p = n_1/(n_0+n_1)$. The number of additional steps to process $k_{max} = \lceil 1/p \rceil$ zeroes is bounded by

$$\frac{n_0}{\lceil 1/p \rceil} \leq \frac{n_0}{1/p} = \frac{n_0 n_1}{n_0+n_1} \leq n_1 \leq e.$$

Therefore, the second stage takes $O(n+e)$ time. This completes the proof.

5 Proof of Lemma 6: Analysis of x_n

In this section, we prove Lemma 6. We shall analyze asymptotically x_n satisfying $x_0 = x_1 = 0$ and for $n \geq 2$

$$x_n = a_n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (x_k + x_{n-k}), \tag{21}$$

where $a_n = \lceil \log(n+1) \rceil$ for $n \geq 2$ and $a_0 = a_1 = 0$. We shall follow the methodology of [33].

Define the exponential generating function (EGF) of x_n as

$$x(z) = \sum_{n=0}^{\infty} x_n \frac{z^n}{n!}$$

for complex z. Then, from (21), for $n \geq 2$ we have

$$\begin{aligned} \frac{x_n}{n!} z^n &= \frac{a_n}{n!} z^n + \sum_{k=0}^{n} \frac{1}{k!} (zp)^k \frac{1}{(n-k)!} (zq)^{n-k} (x_k + x_{n-k}) \\ &= \frac{a_n}{n!} z^n + \sum_{k=0}^{n} \frac{x_k}{k!} (zp)^k \frac{1}{(n-k)!} (zq)^{n-k} + \sum_{k=0}^{n} \frac{1}{k!} (zp)^k \frac{x_{n-k}}{(n-k)!} (zq)^{n-k}. \end{aligned}$$

Thus, using the fact that $x_0 = x_1 = a_0 = a_1 = 0$,

$$\sum_{n=0}^{\infty} \frac{x_n}{n!} z^n = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x_k}{k!} (zp)^k \frac{1}{(n-k)!} (zq)^{n-k} + \sum_{k=0}^{n} \frac{1}{k!} (zp)^k \frac{x_{n-k}}{(n-k)!} (zq)^{n-k} \right).$$

Finally, we arrive at

$$x(z) = a(z) + x(zp) e^{zq} + x(zq) e^{zp},$$

where $a(z)$ is the EGF of a_n. The Poisson transform [17, 33], defined as $\tilde{X}(z) = x(z) e^{-z}$, of the above equation is

$$\tilde{X}(z) = \tilde{A}(z) + \tilde{X}(zp) + \tilde{X}(zq), \tag{22}$$

where $\tilde{A}(z) = a(z) e^{-z}$. By analytic depoissonization [17] we expect that $x_n \sim \tilde{X}(n)$ as $n \to \infty$. We refer to Theorem 10.5 of [33] to conclude that this is the case. Thus it remains to find asymptotics of $\tilde{X}(z)$ as $z \to \infty$ along the real axis.

In order to solve asymptotically the functional equation (22), we apply the Mellin transform. The reader is referred to [13, 33] for an in-depth discussion of the Mellin transform. In brief, the Mellin transform of a real-valued function $f(x)$ is defined as

$$\mathcal{M}[f(x); s] := f^*(s) = \int_0^\infty f(x)x^{s-1}dx.$$

It is defined in a strip $-\alpha < \Re(s) < -\beta$ when $f(x) = O(x^\alpha)$ for $x \to 0$ and $f(x) = O(x^\beta)$ for $x \to \infty$. Noting that

$$\mathcal{M}[f(ax), s] = a^{-s}f^*(s),$$

we transform the functional equation (22) into the following *algebraic* equation

$$X^*(s) = A^*(s) + p^{-s}X^*(s) + q^{-s}X^*(s),$$

where $X^*(s)$ and $A^*(s)$ are the Mellin transforms of $\tilde{X}(z)$ and $\tilde{A}(z)$, respectively. This leads to

$$X^*(s) = \frac{A^*(s)}{1 - p^{-s} - q^{-s}}$$

for $-2 < \Re(s) < -1$, as is easy to see under our assumption on x_0 and x_1. Observe also that

$$A^*(s) = \int_0^\infty \tilde{A}(z)z^{s-1}dz = \sum_{n\geq 2} \frac{a_n}{n!} \int_0^\infty z^n e^{-z} z^{s-1}dz = \sum_{n\geq 2} \frac{a_n}{n!}\Gamma(n+s), \tag{23}$$

and for $a_n = \lceil \log(n+1) \rceil$ the series converges for $\Re(s) < 0$.

In order to find asymptotics of x_n, we first find the inverse Mellin transform and then depoissonize as in [17, 33]. For this we need to understand the zeroes of $1-p^{-s}-q^{-s}$, that is, we study $\mathcal{Z} = \{s \in \mathbb{C} : p^{-s} + q^{-s} = 1\}$. The following lemma is basically due to Schachinger [29] and Jacquet [33] (cf. also [10]).

Lemma 12 *Suppose that $0 < p < q < 1$ with $p + q = 1$, and let*

$$\mathcal{Z} = \{s \in \mathbb{C} : p^{-s} + q^{-s} = 1\}.$$

Then
(i) *All $s \in \mathcal{Z}$ satisfy*

$$-1 \leq \Re(s) \leq \sigma_0,$$

where σ_0 is a real positive solution of $1 + q^{-s} = p^{-s}$. Furthermore, for every integer k there uniquely exists $s_k \in \mathcal{Z}$ with

$$(2k-1)\pi/\log(1/p) < \Im(s_k) < (2k+1)\pi/\log(1/p)$$

and consequently $\mathcal{Z} = \{s_k : k \in \mathbb{Z}\}$.

(ii) *If $\log q/\log p$ is irrational, then $s_0 = -1$ and $\Re(s_k) > -1$ for all $k \neq 0$.*

(iii) *If $\log q/\log p = r/d$ is rational, where $\gcd(r, d) = 1$ for integers $r, d > 0$, then $\Re(s_k) = -1$ if and only if $k \equiv 0 \bmod d$. In particular $\Re(s_1), \ldots, \Re(s_{d-1}) > -1$ and*

$$s_k = s_{k \bmod d} + \frac{2(k - k \bmod d)\pi i}{\log p},$$

that is, all $s \in \mathcal{Z}$ are uniquely determined by $s_0 = -1$ and by $s_1, s_2, \ldots, s_{d-1}$, and their imaginary parts constitute an arithmetic progression.

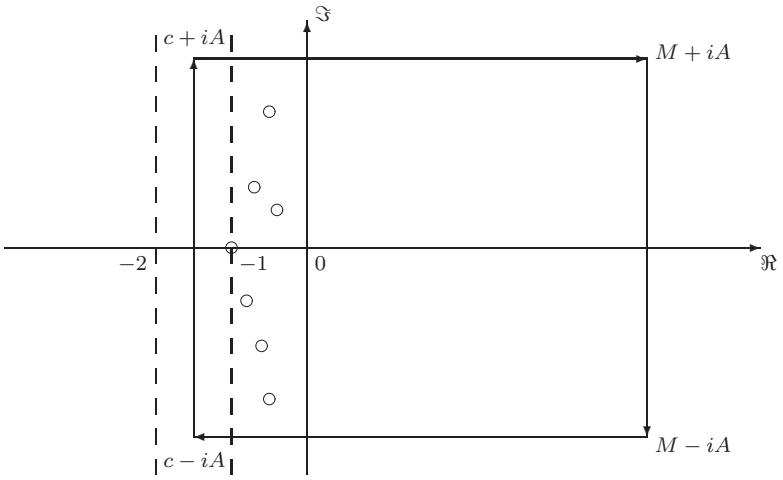


Figure 10: The integration contour (the circles represent zeroes of $p^{-s}+q^{-s}=1$.)

Using this lemma, now we find the asymptotics of $\tilde{X}(z)$ as $z\to\infty$ by the inverse Mellin transform:

$$\tilde{X}(z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}X^*(s)z^{-s}ds,$$

where $-2<c<-1$ is a constant. To compute this we apply the standard approach: consider the rectangle $\mathcal{R}$ shown in Figure 10. The integral of $X^*(s)z^{-s}$ along $\mathcal{R}$ is divided into four parts as follows:

$$\lim_{A\to\infty}\int_{\mathcal{R}}=\lim_{A\to\infty}\left(\int_{c-iA}^{c+iA}+\int_{c+iA}^{M+iA}+\int_{M+iA}^{M-iA}+\int_{M-iA}^{c-iA}\right).$$

We are interested in the first integral. We observe that $\tilde{X}(z)$ is infinitely differentiable. Thus the second and the fourth integrals contribute $O(A^{-r})$ for some large r due to the smallness property of the Mellin transform (cf. [33]). The contribution of the third integral is computed as follows:

$$\begin{aligned}\left|\int_{M+i\infty}^{M-i\infty}X^*(s)z^{-s}ds\right| &= \left|\int_{+\infty}^{-\infty}X^*(M+it)z^{-M-it}dt\right|\\ &\le |z^{-M}|\int_{+\infty}^{-\infty}|X^*(M+it)|dt \;=\; O(z^{-M}),\end{aligned}$$

since the integral above exists. Now by the Cauchy residue theorem and Lemma 12, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}X^*(s)z^{-s}ds+O(z^{-M})=-\sum_{s_k\in\mathcal{Z}}\mathrm{Res}[X^*(s)z^{-s},s=s_k].$$

We observe that the residue at $s_0=-1$ (cf. also [12]). Using these observations and analytic depoissonization [17, 33] we finally conclude the following, proving Lemma 6:
(i) If $\log p/\log q$ is *irrational*, then

$$x_n=\frac{n}{h(p)}A^*(-1)\log e+o(n), \tag{24}$$

where

$$A^*(-1) = \sum_{b\geq 2} \frac{\lceil \log(b+1) \rceil}{b(b-1)}. \tag{25}$$

(ii) If $\log p / \log q = r/d$ (*rational*) with $\gcd(r,d) = 1$, then

$$x_n = \frac{n}{h(p)} \left(A^*(-1) + \Phi(\log_p n) \right) \log e + O\left(n^{1-\eta}\right) \tag{26}$$

for some $\eta > 0$, where

$$\Phi(x) = \sum_{k\neq 0} A^*(-1 + 2k\pi r i / \log p) \exp(2k\pi r x i) \tag{27}$$

is a fluctuating function with a small amplitude. This proves Lemma 6.

6 Concluding Remarks

In this paper, we define *structure* of a graph as its unlabeled version. This allows us to introduce structural entropy, as the lower bound for a lossless structural graph compression. Furthermore, we develop a compression algorithm for structures, which we prove to be asymptotically optimal for graphs generated by the Erdős-Rényi model. We also present some experimental results that suggest our algorithm works well even for graphs not generated by the Erdős-Rényi model. While this is quite plausible, one needs a formal extension of our analysis to other graph models such as power law graphs and preferential attachment graphs. In particular, one needs to prove a proper extension of Lemma 2 for such graphs. We conjecture that such an extension does hold.

In another direction, one should extend our lossless structural compression to a lossy situation. This will require to define a distortion measure that preserves structural properties. Some possible distance graph measures are discussed in [4].

Appendix

A Proof of Lemma 2

For completeness, we prove here Lemma 2. It was established in [20] that almost surely one should alter (delete or add) $(2 - o(1))np(1-p)$ edges to obtain a symmetric graph from $G(n,p)$, which is a sufficient condition of our statement. We shall follow the footsteps of [20], except that we derive explicitly the rate of convergence.

First we need some definitions. Let $G = (V, E)$ be a graph and let $\pi : V \to V$ be a permutation of the vertices of G. For a vertex $v \in V$ we define a defect of v with respect to π to be

$$D_\pi(v) = |N(\pi(v)) \ \Delta \ \pi(N(v))|,$$

where $N(v)$ is the set of neighbors of v and Δ denotes the symmetric difference of two sets, that is, $A \Delta B = (A - B) \cup (B - A)$ for two sets A and B. Similarly, we define a defect of G with respect to π to be

$$D_\pi(G) = \max_v D_\pi(v).$$

Finally, we define a defect of a graph G to be

$$D(G) = \min_{\pi \neq identity} D_\pi(G).$$

It is easy to see that a graph G is symmetric if and only if its defect equals zero. Thus we only need to show that $D(G) > 0$ for $G \in \mathcal{G}(n,p)$ with high probability. But we shall next prove that $D(G)$ is at least $(2-o(1))np(1-p)$ with high probability.

Set $\epsilon = \epsilon(n,p)$ such that $\epsilon = o(1)$ and $\epsilon^2 np(1-p) \gg \ln n$. This is possible for all p's satisfying the conditions of the lemma (e.g., $\epsilon = \Theta\left(\sqrt[4]{\frac{\ln n}{np(1-p)}}\right)$.) Fix an arbitrary $2 \leq k \leq n$, and let π be a permutation of vertices of G which fixes all but k vertices. Let U be the set of vertices $\{u | \pi(u) \neq u\}$ and

$$X = \sum_{u \in U} D_\pi(u).$$

By definition, $D_\pi(u)$ is a binomially distributed random variable with expectation either $2(n-2)p(1-p)$ or $2(n-1)p(1-p)$, depending on whether $\pi(\pi(u)) = u$ or not. Therefore,

$$\mathbf{E}[X] = \sum_{u \in U} \mathbf{E}[D_\pi(u)] = (2-o(1))knp(1-p).$$

We prove next that X is strongly concentrated around its mean, which implies that for some vertex $u \in U$, $D_\pi(u)$ is at least $\mathbf{E}[X]/k$ with high probability. Then we conclude that $D_\pi(G)$ is at least $\mathbf{E}[X]/k$ with high probability by the definition of $D_\pi(G)$. Finally, we shall prove that, for every possible permutation π, the minimum of $D_\pi(G)$ is still at least $\mathbf{E}[X]/k$ with high probability, which implies that $D(G)$ is at least $\mathbf{E}[X]/k$ with high probability by the definition of $D(G)$.

We start with the observation that X depends only on the edges of the graph adjacent to the vertices in U. Moreover, adding or deleting any such edge, say (u,v), can change only the values of at most four terms $D_\pi(u)$, $D_\pi(v)$, $D_\pi(\pi^{-1}(u))$, and $D_\pi(\pi^{-1}(v))$ in the sum, each by at most 1. Here, X can be seen as a random variable on a probability space generated by a finite set of mutually independent 0/1 choices, indexed by i. Let p_i be the probability that choice i is 1, and let c be a constant such that changing any choice i (keeping all other choices the same) can change X by at most c. Set $\sigma^2 = c^2 \sum_i p_i(1-p_i)$. In [2] it is shown that for all positive $t < 2\sigma/c$,

$$P(|X - \mathbf{E}[X]| > t\sigma) \leq 2e^{-t^2/4}.$$

In this case, $c = 4$ and $\sigma^2 = 16(\binom{n}{2} - \binom{n-k}{2})p(1-p) = \Theta(knp(1-p))$. Therefore, for some positive constant α,

$$P(|X - \mathbf{E}[X]| > \epsilon knp(1-p)) \leq e^{-\alpha\epsilon^2 knp(1-p)}.$$

Thus, with probability at least $1 - e^{-\alpha\epsilon^2 knp(1-p)}$, there is a vertex in U with defect at least

$$\frac{1}{k}(\mathbf{E}[X] - \epsilon knp(1-p)) = (2-o(1))np(1-p).$$

Therefore,

$$P(D_\pi(G) \leq (2-\epsilon)np(1-p)) \leq e^{-\alpha\epsilon^2 knp(1-p)} = P_k.$$

Now we see that the number of permutations which fix $n-k$ vertices is at most $\binom{n}{k}k!$. Therefore, the probability that there exists a permutation such that the defect of G with respect to it is less than $(2-\epsilon)np(1-p)$ is at most

$$\begin{aligned}
\sum_{k=2}^{n}\binom{n}{k}k!P_k &\leq \sum_{k=2}^{n} n^k e^{-\alpha\epsilon^2 knp(1-p)} \\
&= \sum_{k=2}^{n}\left(e^{-\alpha\epsilon^2 np(1-p)+\ln n}\right)^k \\
&\leq \beta\left(e^{-\alpha\epsilon^2 np(1-p)+\ln n}\right)^2 \quad \text{for some constant } \beta \ (\because \epsilon^2 np(1-p) \gg \ln n) \\
&< \beta\left(e^{-\gamma\ln n+\ln n}\right)^2 \quad \text{for any positive constant } \gamma \ (\because \epsilon^2 np(1-p) \gg \ln n) \\
&= \beta\left(n^{1-\gamma}\right)^2 = O\left(n^{-w}\right) \quad \text{for any positive constant } w.
\end{aligned}$$

The last equality is obtained by setting $w = 2\gamma - 2$ and choosing $\gamma > 1$. Therefore, the probability that G is symmetric is at most $O\left(n^{-w}\right)$ for any positive constant w.

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