Online Appendix for “Dominance Solvability in Random Games”

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This Online Appendix contains the following four items. First, in Appendix A we provide proofs omitted from the main text. Second, in Appendix B we provide more details on enumerative issues arising in our general analysis and particularly on the connection between our problem and permutation patterns. Third, in Appendix C we discuss how our results extend to multiple players. Fourth, in Appendix D we include additional figures referred to in the main text.

A. OMITTED PROOFS

Lemma 4. *Consider a random game $G(2,n)$. Then,*

1. $\Pr(I(2,n)=1)$ *is strictly decreasing in n and* $\lim_{n\to\infty} 2^n \cdot n^{1/2} \cdot \Pr(I(2,n)=1) = \sqrt{\pi}$*;*
2. $\Pr(I(2,1)=2)=0$, $\Pr(I(2,2)=2)=\Pr(I(2,3)=2)=2/3$, $\Pr(I(2,n)=2)$ *is strictly decreasing in n for $n \geq 3$, and* $\lim_{n\to\infty} n^{1/2} \cdot \Pr(I(2,n)=2) = \frac{\sqrt{\pi}}{2}$*;*
3. $\Pr(I(2,1)=3)=\Pr(I(2,2)=3)=0$, $\Pr(I(2,n)=3)$ *is strictly increasing in n for $n \geq 2$, and* $\lim_{n\to\infty} n^{1/2} \cdot (1-\Pr(I(2,n)=3)) = \frac{\sqrt{\pi}}{2}$.

Proof. We prove the three statements in turn.

1. Consider $\Pr(I(2,n)=1) = \frac{\sqrt{\pi}}{2^n} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)}$, derived in Subsection 3.3. For any $n \geq 1$,

$$\Pr(I(2,n+1)=1) = \frac{\sqrt{\pi}}{2^{n+1}} \cdot \frac{\Gamma(n+1)}{\Gamma(n+3/2)} = \frac{n}{2n+1} \cdot \Pr(I(2,n)=1) < \Pr(I(2,n)=1).$$

By Stirling's formula applied to the gamma function,

$$\lim_{n\to\infty} 2^n \cdot n^{1/2} \cdot \Pr(I(2,n)=1) = \sqrt{\pi} \lim_{n\to\infty} \frac{\Gamma(n) \cdot n^{1/2}}{\Gamma(n+1)} = \sqrt{\pi}.$$

2. Similarly, consider $\Pr(I(2,n)=2) = \frac{n+2^{n-1}-2}{2^n} \cdot \sqrt{\pi} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)}$, derived in Subsection 3.3. It is straightforward to check that $\Pr(I(2,2)=2)=\Pr(I(2,3)=2)=2/3$ and $\Pr(I(2,1)=2)=0$. For any $n \geq 3$,

$$\Pr(I(2,n+1)=2) = \frac{n+2^n-1}{2n+2^n-4} \cdot \frac{n}{n+1/2} \cdot \Pr(I(2,n)=2) < \Pr(I(2,n)=2).$$

By Stirling's formula applied to the gamma function,

$$\lim_{n\to\infty} n^{1/2} \cdot \Pr(I(2,n)=2) = \frac{\sqrt{\pi}}{2} \cdot \lim_{n\to\infty} \frac{n+2^{n-1}-2}{2^{n-1}} \cdot \lim_{n\to\infty} \frac{\Gamma(n) \cdot n^{1/2}}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2}.$$

3. Finally, consider $\Pr(I(2,n)=3) = 1 - \frac{n+2^{n-1}-1}{2^n} \cdot \sqrt{\pi} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)}$, also derived in Subsection 3.3. It is straightforward to check that $\Pr(I(2,1)=3)=\Pr(I(2,2)=3)=0$.

By 1 and 2, $\Pr(I(2,n)=3)$ is strictly increasing in n for $n \geq 2$. By Stirling's formula,

$$\lim_{n\to\infty} n^{1/2}\cdot(1-\Pr(I(2,n)=3)) = \frac{\sqrt{\pi}}{2}\cdot\lim_{n\to\infty}\frac{n+2^{n-1}-1}{2^{n-1}}\cdot\lim_{n\to\infty}\frac{\Gamma(n)\cdot n^{1/2}}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2}. \qquad \square$$

Lemma 5. *Consider a random game $G(2,n)$. Then,*

1. $\lim_{n\to\infty} n^{1/2}\cdot\Pr\left(S^C(2,n)=1\right) = \frac{2}{\sqrt{\pi}}$;
2. *for any fixed* $k\geq 2$, $\lim_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\cdot\Pr\left(S^C(2,n)=k\right) = \frac{1}{(k-1)!}\cdot\left(1-\frac{1}{2^{k-1}}\right)$;
3. *for* $k(n)\sim\ln n$, $\lim_{n\to\infty}(\ln n)^{1/2}\cdot\Pr\left(S^C(2,n)=k(n)\right) = \frac{1}{\sqrt{2\pi}}$.

Proof. For these results, we use two theorems regarding the unsigned Stirling numbers of the first kind.

Hwang's Theorem (Theorem 1 for $\nu=0$ in Hwang, 1995)**.** For any $\eta>0$, the unsigned Stirling numbers of the first kind $s(n,k)$ satisfy asymptotically

$$\frac{s(n,k)}{n!} = \frac{1}{n}\cdot\frac{(\ln n+\gamma)^{k-1}}{(k-1)!} + \mathcal{O}\left(\frac{(\ln n)^k}{k!\cdot n^2}\right) \quad (n\to\infty),$$

uniformly for $1\leq k\leq \eta\ln n$.

Erdős Theorem (e.g., see p.124 in Stanley, 2011)**.** The Stirling numbers of the first kind form log-concave sequences. In addition, the signless Stirling number $s(n,k)$ is maximized at $k(n) = \arg\max_{k\in\{\lfloor H_n\rfloor,\lceil H_n\rceil\}} s(n,k)$, where $H_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$ is the n-th harmonic number. That is, $k(n)\sim\ln n$.

We now show the three statements of the lemma in sequence.

1. It follows immediately from Proposition 1.

2. It follows immediately from Hwang's theorem for fixed $k\geq 2$. In fact, the original theorem pertaining to this case was offered by Wilf (1993).

3. By Hwang's theorem,

$$\frac{s(n,k(n))}{n!} \sim \frac{1}{n}\cdot\frac{(\ln n)^{\ln n-1}}{\Gamma(\ln n)} \quad (n\to\infty,\ k(n)\sim\ln n). \tag{\dagger}$$

By applying Stirling's formula to the gamma function,

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right).$$

For $\Gamma(\ln n)$, we get

$$\Gamma(\ln n) \sim \frac{\sqrt{2\pi}}{n} \cdot (\ln n)^{\ln n - \frac{1}{2}} \quad (n \to \infty), \tag{††}$$

so that from equations † and †† we have

$$\frac{s(n, k(n))}{n!} \sim \frac{1}{\sqrt{2\pi}} \cdot (\ln n)^{-\frac{1}{2}} \quad (n \to \infty,\ k(n) \sim \ln n).$$

Thus,

$$\Pr\left(S^C(2,n) = k(n)\right) = \frac{s(n,k(n))}{n!} \cdot \left(1 - \frac{1}{2^{k(n)-1}}\right) \sim \frac{1}{\sqrt{2\pi}} \cdot (\ln n)^{-\frac{1}{2}} \quad (n \to \infty,\ k(n) \sim \ln n)$$

and $\lim_{n\to\infty} (\ln n)^{1/2} \cdot \Pr\left(S^C(2,n) = k(n)\right) = \frac{1}{\sqrt{2\pi}}$ as desired. □

Lemma 6. *Consider a random game $G(2,n)$. Then,*

1. *for any $n \geq 1$, $\mathbb{E}\left[\left|S^C(2,n)\right|\right] = W(n) \cdot \left(2 - (\psi(n+1/2) - \psi(1/2))\right) + H_n$, where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function and $W(n) = \frac{\Gamma(n+1/2)}{\Gamma(n+1)\cdot\Gamma(1/2)}$;*
2. *$\mathbb{E}\left[\left|S^C(2,n)\right|\right]$ is strictly increasing in n;*
3. *$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = \ln n + \gamma - \frac{1}{\sqrt{\pi}} \cdot \frac{\ln n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right)$, where γ is the Euler-Mascheroni constant.*

Proof. We prove each of the three statements in turn.

1. Recall that

$$\Pr\left(\left|S^C(2,n)\right| = 1\right) = \frac{1}{n!} \cdot \sum_{k=1}^{n} \frac{s(n,k)}{2^{k-1}}, \text{ and}$$
$$\Pr\left(\left|S^C(2,n)\right| = k\right) = \frac{s(n,k)}{n!} \cdot \left(1 - \frac{1}{2^{k-1}}\right) \quad \text{for any } 2 \leq k \leq n,$$

so that

$$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = \frac{1}{n!}\cdot\sum_{k=1}^{n}\frac{s(n,k)}{2^{k-1}} + \sum_{k=2}^{n} k\cdot\frac{s(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right) = \frac{1}{n!}\cdot\sum_{k=1}^{n}\frac{s(n,k)}{2^{k-1}} + \sum_{k=1}^{n} k\cdot\frac{s(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right) = 2W(n) + \sum_{k=1}^{n} k\cdot\frac{s(n,k)}{n!} - \sum_{k=1}^{n} k\cdot\frac{s(n,k)}{2^{k-1}\cdot n!}.$$

First, note that $\sum_{k=1}^{n} k\cdot\frac{s(n,k)}{n!} = H_n$ (see, e.g., Theorem 2 in Benjamin et al., 2002).

Second, we can differentiate the identity $\sum_{k=1}^{n} s(n,k)\cdot x^k = \frac{\Gamma(n+x)}{\Gamma(x)}$ to derive an explicit expression for $\sum_{k=1}^{n} k\cdot\frac{s(n,k)}{2^{k-1}\cdot n!}$. Namely,

$$\sum_{k=1}^{n} k\cdot s(n,k)\cdot x^{k-1} = \frac{d}{dx}\left(\frac{\Gamma(n+x)}{\Gamma(x)}\right) = \frac{\Gamma(n+x)}{\Gamma(x)}\cdot(\psi(n+x)-\psi(x)),$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, so that $\sum_{k=1}^{n} k\cdot\frac{s(n,k)}{2^{k-1}\cdot n!} = W(n)\cdot(\psi(n+1/2)-\psi(1/2))$.

By collecting all terms, we get

$$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = W(n)\cdot\big(2-(\psi(n+1/2)-\psi(1/2))\big) + H_n,$$

where $\psi(n+1/2) = -\gamma - 2\ln 2 + \sum_{k=1}^{n}\frac{2}{2k-1} = -\gamma + H_{n-1/2}$ and $\psi(1/2) = -\gamma - 2\ln 2$.

2. For any $n \geq 1$,

$$\mathbb{E}\left[\left|S^C(2,n+1)\right|\right] = W(n+1)\cdot\big(2-(\psi(n+3/2)-\psi(1/2)\big) + H_{n+1} \text{ and}$$
$$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = W(n)\cdot\big(2-(\psi(n+1/2)-\psi(1/2)\big) + H_n,$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\left|S^C(2,n+1)\right|\right] - \mathbb{E}\left[\left|S^C(2,n)\right|\right] = \frac{1}{n+1} \\
&+ W(n)\cdot\left(-\frac{1}{n+1} + \left(\psi(n+1/2)-\psi(1/2)\right) - \frac{n+1/2}{n+1}\cdot\left(\psi(n+3/2)-\psi(1/2)\right)\right) \\
&> W(n)\cdot\left(\sum_{k=1}^{n}\frac{2}{2k-1} - \frac{n+1/2}{n+1}\cdot\sum_{k=1}^{n+1}\frac{2}{2k-1}\right) = \frac{W(n)}{n+1}\cdot\left(\sum_{k=1}^{n}\frac{1}{2k-1}-1\right) \geq 0,
\end{aligned}$$

where the first inequality follows from $W(n) = \dfrac{\pi(2,n)}{2} < 1$ for any $n \geq 1$.

3. Note that

$$H_n = \ln n + \gamma + \mathcal{O}\left(\frac{1}{n}\right), \qquad \frac{\Gamma(n+1/2)}{\Gamma(n+1)} = \frac{1}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right), \qquad \text{and}$$
$$\psi(n+1/2) = \ln(n+1/2) + \mathcal{O}\left(\frac{1}{n}\right),$$

so that

$$\begin{aligned}
\mathbb{E}\left[\left|S^C(2,n)\right|\right] &= \frac{1}{\sqrt{\pi}}\cdot\left(\frac{1}{n^{1/2}}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right)\cdot\left(2+\gamma+2\ln 2-\ln(n+1/2)-\mathcal{O}\left(\frac{1}{n}\right)\right) \\
&\quad + \ln n + \gamma + \mathcal{O}\left(\frac{1}{n}\right) = \ln n + \gamma - \frac{1}{\sqrt{\pi}}\cdot\frac{\ln n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right).
\end{aligned}$$

In particular, $\lim_{n\to\infty}\left(\mathbb{E}\left[\left|S^C(2,n)\right|\right] - \ln n\right) = \gamma$. □

Lemma 7. *Consider a random game $G(2,n)$. Define the polygamma function $\psi^{(m)}(x)$ of order m as the m-th derivative of the digamma function $\psi(x) \equiv \psi^{(0)}(x)$. In addition, let $H_n^{(m)} \equiv 1 + \dfrac{1}{2^m} + \ldots + \dfrac{1}{n^m}$ be the generalized harmonic number of order m of n. Then,*

1. for any $n \geq 1$,

$$\begin{aligned}
\operatorname{Var}\left[S^C(2,n)\right] &= H_n - H_n^{(2)} + \frac{W(n)}{2}\cdot\Bigg(4 - 2\cdot\left(\psi(n+1/2)-\psi(1/2)\right) \\
&-\left(\left(\psi(n+1/2)-\psi(1/2)\right)^2 + \left(\psi^{(1)}(n+1/2)-\psi^{(1)}(1/2)\right)\right) - 8\cdot H_n + 4\cdot H_n\cdot\left(\psi(n+1/2)-\psi(1/2)\right)\Bigg) \\
&- \left(W(n)\cdot\left(2-\left(\psi(n+1/2)-\psi(1/2)\right)\right)\right)^2.
\end{aligned}$$

2. $\operatorname{Var}\left[S^C(2,n)\right] = \ln n + \gamma - \frac{\pi^2}{6} + \frac{3}{2\cdot\sqrt{\pi}} \cdot \frac{(\ln n)^2}{n^{1/2}} - \frac{5 - 2\ln 2 - 3\gamma}{\sqrt{\pi}} \cdot \frac{\ln n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right).$

Proof. We prove two statements in sequence.

1. Similar to the proof of Lemma 6,

$$\begin{aligned}\mathbb{E}\left[S^C(2,n)^2\right] &= \frac{1}{n!}\cdot\sum_{j=1}^{n}\frac{s(n,j)}{2^{j-1}} + \sum_{j=2}^{n} j^2\cdot\frac{s(n,j)}{n!}\cdot\left(1-\frac{1}{2^{j-1}}\right) = \frac{1}{n!}\cdot\sum_{j=1}^{n}\frac{s(n,j)}{2^{j-1}}\\ &+ \sum_{j=1}^{n} j^2\cdot\frac{s(n,j)}{n!}\cdot\left(1-\frac{1}{2^{j-1}}\right) = 2W(n) + \sum_{j=1}^{n} j^2\cdot\frac{s(n,j)}{n!} - \sum_{j=1}^{n} j^2\cdot\frac{s(n,j)}{2^{j-1}\cdot n!}\\ &= 2W(n) + \sum_{j=1}^{n} j^2\cdot\frac{s(n,j)}{n!} - \sum_{j=1}^{n} j\cdot\frac{s(n,j)}{2^{j-1}\cdot n!} - \frac{1}{2}\cdot\sum_{j=1}^{n} j(j-1)\cdot\frac{s(n,j)}{2^{j-2}\cdot n!}.\end{aligned}$$

First, we have $\sum_{j=1}^{n} j^2\cdot\frac{s(n,j)}{n!} = H_n + H_n^2 - H_n^{(2)}$ (Gontcharoff, 1944). Second, note that $\sum_{j=1}^{n} j\cdot\frac{s(n,j)}{2^{j-1}\cdot n!} = W(n)\cdot(\psi(n+1/2) - \psi(1/2))$.

Third, we can differentiate twice the identity $\sum_{j=1}^{n} s(n,j)\cdot x^j = \frac{\Gamma(n+x)}{\Gamma(x)}$ to derive an explicit expression for $\sum_{j=1}^{n} j(j-1)\cdot\frac{s(n,j)}{2^{j-2}\cdot n!}$ as follows:

$$\begin{aligned}\sum_{j=1}^{n} j(j-1)\cdot s(n,j)\cdot x^{j-2} &= \frac{d^2}{dx^2}\left(\frac{\Gamma(n+x)}{\Gamma(x)}\right) = \frac{d}{dx}\left(\frac{\Gamma(n+x)}{\Gamma(x)}\cdot(\psi(n+x)-\psi(x))\right)\\ &= \frac{\Gamma(n+x)}{\Gamma(x)}\cdot\Big(\big(\psi(n+x)-\psi(x)\big)^2 + \big(\psi^{(1)}(n+x)-\psi^{(1)}(x)\big)\Big),\end{aligned}$$

where $\psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z)$ the polygamma function of order m, so that

$$\sum_{j=1}^{n} j(j-1)\cdot\frac{s(n,j)}{2^{j-2}\cdot n!} = W(n)\cdot\Big(\big(\psi(n+1/2)-\psi(1/2)\big)^2 + \big(\psi^{(1)}(n+1/2)-\psi^{(1)}(1/2)\big)\Big).$$

Thus,

$$\begin{aligned}\mathbb{E}\left[S^C(2,n)^2\right] &= 2W(n) + \left(H_n + H_n^2 - H_n^{(2)}\right) - W(n)\cdot(\psi(n+1/2)-\psi(1/2)) \\ &\quad - \frac{W(n)}{2}\cdot\Big(\left(\psi(n+1/2)-\psi(1/2)\right)^2 + \left(\psi^{(1)}(n+1/2)-\psi^{(1)}(1/2)\right)\Big).\end{aligned}$$

By Lemma 6,

$$\mathbb{E}\left[S^C(2,n)\right] = W(n)\cdot\left(2-(\psi(n+1/2)-\psi(1/2))\right) + H_n,$$

so that by using $\mathrm{Var}\left[S^C(2,n)\right] = \mathbb{E}\left[S^C(2,n)^2\right] - \mathbb{E}\left[S^C(2,n)\right]^2$, we get the desired expression.

2. By using asymptotic expressions for each term, we get

$$\begin{aligned}\mathrm{Var}\left[S^C(2,n)\right] &= \ln n + \gamma - \frac{\pi^2}{6} + \frac{1}{2\cdot\sqrt{\pi}}\cdot\left(\frac{1}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right) \\ &\quad\times\Bigg(-(\ln(n+1/2))^2 + 4\cdot\ln(n+1/2)\cdot\ln n - 2\ln(n+1/2) \\ &\quad - (4\ln 2 + 2\gamma)\ln(n+1/2) - 8\ln n + 4\gamma\cdot\ln n \\ &\quad + (8\ln 2 + 4\gamma)\ln(n+1/2) + \mathcal{O}(1)\Bigg) \\ &= \ln n + \gamma - \frac{\pi^2}{6} + \frac{3}{2\cdot\sqrt{\pi}}\cdot\frac{(\ln n)^2}{n^{1/2}} - \frac{5-2\ln 2-3\gamma}{\sqrt{\pi}}\cdot\frac{\ln n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right),\end{aligned}$$

where $\lim_{n\to\infty} H_n^{(2)} = \dfrac{\pi^2}{6}$. □

Proposition 3. *Consider a random game $G(2,n)$. Then,*

$$\Pr\left(S^C(2,n) - \mathbb{E}\left[S^C(2,n)\right] \le x\cdot\sqrt{\mathrm{Var}\left[S^C(2,n)\right]}\right) = \Phi(x) + \mathcal{O}\left(\frac{1}{\sqrt{\ln n}}\right),$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution,

$$\mathbb{E}\left[S^C(2,n)\right] = \ln n + \gamma + o(1), \quad \textit{and} \quad \sqrt{\mathrm{Var}\left[S^C(2,n)\right]} = \sqrt{\ln n} - \frac{\pi^2-6\gamma}{12\sqrt{\ln n}} + o\left(\frac{1}{\sqrt{\ln n}}\right).$$

Proof. The proof of this statement is similar to Hwang (1998) and uses the Berry-Esseen theorem to find the convergence rate in the stated central-limit result. The difference is that the problem does not belong to the exp-log class immediately (one can observe several exp-terms below after some manipulations). Still, similar reasoning can be applied to establish the result. For notation simplicity, we let

$$\mu_n \equiv \mathbb{E}\left[S^C(2,n)\right] = \ln n + \gamma + o(1),$$
$$\sigma_n \equiv \sqrt{\mathrm{Var}\left[S^C(2,n)\right]} = \sqrt{\ln n} - \left(\frac{\pi^2}{12} - \frac{\gamma}{2}\right) \cdot \frac{1}{\sqrt{\ln n}} + o\left(\frac{1}{\sqrt{\ln n}}\right), \text{ and}$$
$$\varphi_n(t) = \sum_{j=1}^{n} \Pr\left(S^C(2,n) = j\right) \cdot e^{it(j-\mu_n)/\sigma_n},$$

where $\varphi_n(t)$ denotes the characteristic function of the normed variable $(S^C(2,n) - \mu_n)/\sigma_n$.

Berry–Esseen theorem (Theorem 2 in Petrov, 1975)**.** Let $F(x)$ be a nondecreasing function and $G(x)$ a differentiable function of bounded variation on the real line. The corresponding Fourier-Stieltjes transforms $\varphi(t)$ and $\gamma(t)$ are then:

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x), \quad \gamma(t) = \int_{-\infty}^{\infty} e^{itx} dG(x).$$

Suppose that $F(-\infty) = G(-\infty)$, $F(\infty) = G(\infty)$, T is an arbitrary positive number, and $|G'(x)| \leq A$. Then, for every $b > 1/(2\pi)$, we have

$$\sup_{-\infty<x<\infty} |F(x) - G(x)| \leq b \int_{-T}^{T} \left|\frac{\varphi(t) - \gamma(t)}{t}\right| dt + r(b)\frac{A}{T},$$

where $r(b)$ is a positive constant depending only on b.

We proceed in two steps. In step 1, we reformulate the problem by using the Berry-Esseen inequality. In step 2, we calculate the characteristic function and use it to establish the result.

Step 1. Reformulated problem

Let $G(x) = \Phi(x)$ (so that $A = 1/\sqrt{2\pi}$) and $T = T_n = c\sigma_n$, where $c > 0$ is a sufficiently small constant. By the Berry-Esseen inequality, it will be sufficient to prove that

$$J_n = \int_{-T_n}^{T_n} \left|\frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t}\right| dt = \mathcal{O}\left(\frac{1}{\sqrt{\ln n}}\right).$$

Step 2. Characteristic function

$$
\begin{aligned}
\varphi_n(t) &= \sum_{j=1}^{n} \Pr\left(S^C(2,n)=j\right)\cdot e^{it(j-\mu_n)/\sigma_n} \\
&= e^{-it\mu_n/\sigma_n}\cdot\left(\frac{1}{n!}\cdot\sum_{j=1}^{n}\frac{s(n,j)}{2^{j-1}}\cdot e^{it/\sigma_n}+\sum_{j=2}^{n}\frac{s(n,j)}{n!}\cdot\left(1-\frac{1}{2^{j-1}}\right)\cdot e^{itj/\sigma_n}\right) \\
&= e^{-it\mu_n/\sigma_n}\cdot\left(\frac{1}{n!}\cdot\sum_{j=1}^{n}\frac{s(n,j)}{2^{j-1}}\cdot e^{it/\sigma_n}+\sum_{j=1}^{n}\frac{s(n,j)}{n!}\cdot\left(1-\frac{1}{2^{j-1}}\right)\cdot e^{itj/\sigma_n}\right) \\
&= e^{-it\mu_n/\sigma_n}\cdot\left(\sum_{j=1}^{n}\frac{s(n,j)}{n!}\cdot e^{itj/\sigma_n}\right) \\
&+ 2\cdot e^{-it\mu_n/\sigma_n}\cdot\left(\frac{1}{n!}\cdot\sum_{j=1}^{n}\frac{s(n,j)}{2^{j}}\cdot e^{it/\sigma_n}-\sum_{j=1}^{n}\frac{s(n,j)}{n!}\cdot\frac{1}{2^j}\cdot e^{itj/\sigma_n}\right)=A_n(t)+B_n(t),
\end{aligned}
$$

where

$$
\begin{aligned}
A_n(t) &\equiv e^{-it\mu_n/\sigma_n}\cdot\frac{1}{\Gamma\left(e^{it/\sigma_n}\right)}\cdot\frac{\Gamma\left(n+e^{it/\sigma_n}\right)}{\Gamma(n+1)}, \text{ and} \\
B_n(t) &\equiv 2\cdot e^{-it\mu_n/\sigma_n}\cdot\left(\frac{e^{it/\sigma_n}}{\Gamma(1/2)}\cdot\frac{\Gamma(n+1/2)}{\Gamma(n+1)}-\frac{1}{\Gamma\left(e^{it/\sigma_n}/2\right)}\cdot\frac{\Gamma\left(n+e^{it/\sigma_n}/2\right)}{\Gamma(n+1)}\right).
\end{aligned}
$$

We start by finding the asymptotic expression for $A_n(t)$. By denoting $e^{it/\sigma_n}\equiv 1+\varepsilon_n$ with $\varepsilon_n=\frac{it}{\sigma_n}-\frac{t^2}{2\sigma_n^2}+\mathcal{O}\left(\frac{|t|^3}{\sigma_n^3}\right)$ and using Stirling's formula,

$$
\begin{aligned}
&\ln\frac{\Gamma\left(n+e^{it/\sigma_n}\right)}{\Gamma(n+1)} = \ln\Gamma\left(n+1+\varepsilon_n\right)-\ln\Gamma\left(n+1\right) \\
&= (n+1/2+\varepsilon_n)\cdot\ln\left(n+1+\varepsilon_n\right)-(n+1+\varepsilon_n)+\frac{1}{2}\ln 2\pi-(n+1/2)\cdot\ln\left(n+1\right)+(n+1)-\frac{1}{2}\ln 2\pi+\mathcal{O}\left(\frac{1}{n}\right) \\
&= \varepsilon_n\ln(n+1+\varepsilon_n)+(n+1/2)\ln\left(1+\frac{\varepsilon_n}{n+1}\right)-\varepsilon_n+\mathcal{O}\left(\frac{1}{n}\right)=\varepsilon_n\ln n+\mathcal{O}\left(\frac{1}{n}\right) \\
&= \left(\frac{it}{\sigma_n}-\frac{t^2}{2\sigma_n^2}+\mathcal{O}\left(\frac{|t|^3}{\sigma_n^3}\right)\right)\cdot\ln n+\mathcal{O}\left(\frac{1}{n}\right)=\left(\frac{it}{\sqrt{\ln n}}-\frac{t^2}{2\ln n}+\mathcal{O}\left(\frac{|t|^3}{(\ln n)^{3/2}}\right)\right)\cdot\ln n+\mathcal{O}\left(\frac{1}{n}\right) \\
&= it\cdot\sqrt{\ln n}-\frac{t^2}{2}+\mathcal{O}\left(\frac{|t|^3}{\sqrt{\ln n}}\right).
\end{aligned}
$$

In addition,

$$\ln\Gamma\left(e^{it/\sigma_n}\right)=\ln\Gamma\left(1+\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\right)=\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\qquad\text{and}$$

$$\frac{it\mu_n}{\sigma_n}=it\cdot\frac{\ln n+\gamma+o(1)}{\sqrt{\ln n}-\left(\frac{\pi^2}{12}-\frac{\gamma}{2}\right)\cdot\frac{1}{\sqrt{\ln n}}+o\left(\frac{1}{\sqrt{\ln n}}\right)}=it\cdot\sqrt{\ln n}+\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right).$$

Collecting all terms, we get

$$A_n(t)=e^{-\left(it\cdot\sqrt{\ln n}+\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\right)}\cdot e^{-\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)}\cdot e^{it\cdot\sqrt{\ln n}-\frac{t^2}{2}+\mathcal{O}\left(\frac{|t|^3}{\sqrt{\ln n}}\right)}=e^{-\frac{t^2}{2}+\mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\ln n}}\right)}.$$

Next, we find the asymptotic expression for $B_n(t)$. Using similar calculations,

$$\begin{aligned}
&\ln\frac{\Gamma\left(n+1/2\right)}{\Gamma(n+1)}=\ln\Gamma\left(n+1/2\right)-\ln\Gamma\left(n+1\right)\\
&=n\cdot\ln\left(n+1/2\right)-\left(n+1/2\right)+\frac{1}{2}\ln 2\pi-\left(n+1/2\right)\cdot\ln\left(n+1\right)+\left(n+1\right)-\frac{1}{2}\ln 2\pi+\mathcal{O}\left(\frac{1}{n}\right)\\
&\qquad=-\frac{1}{2}\cdot\ln\left(n+1\right)+n\ln\left(1-\frac{1}{2}\cdot\frac{1}{n+1}\right)+1/2+\mathcal{O}\left(\frac{1}{n}\right)=-\frac{1}{2}\cdot\ln n+\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\ln\frac{\Gamma\left(n+e^{it/\sigma_n}/2\right)}{\Gamma(n+1)}&=\ln\Gamma\left(n+1/2+\varepsilon_n/2\right)-\ln\Gamma\left(n+1\right)\\
&=\left(n+\varepsilon_n/2\right)\cdot\ln\left(n+1/2+\varepsilon_n/2\right)-\left(n+1/2+\varepsilon_n/2\right)+\frac{1}{2}\ln 2\pi\\
&\qquad-\left(n+1/2\right)\cdot\ln\left(n+1\right)+\left(n+1\right)-\frac{1}{2}\ln 2\pi+\mathcal{O}\left(\frac{1}{n}\right)\\
&=\left(\varepsilon_n/2-1/2\right)\ln\left(n+1\right)+\left(n+\varepsilon_n/2\right)\ln\left(1+\frac{\varepsilon_n/2-1/2}{n+1}\right)+\left(1/2-\varepsilon_n/2\right)+\mathcal{O}\left(\frac{1}{n}\right)\\
&=-\frac{1}{2}\ln n+\frac{1}{2}it\cdot\sqrt{\ln n}+\mathcal{O}\left(t^2\right).
\end{aligned}$$

Furthermore,

$$\ln\Gamma\left(e^{it/\sigma_n}/2\right) = \ln\Gamma\left(1/2 + \mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\right) = \mathcal{O}(1) \qquad \text{and}$$

$$\frac{it}{\sigma_n} = it \cdot \frac{1}{\sqrt{\ln n} - \left(\frac{\pi^2}{12} - \frac{\gamma}{2}\right)\cdot\frac{1}{\sqrt{\ln n}} + o\left(\frac{1}{\sqrt{\ln n}}\right)} = \mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right).$$

Collecting all terms, we get

$$\begin{aligned} B_n(t) = & \frac{2}{\sqrt{\pi}} \cdot e^{-\left(it\cdot\sqrt{\ln n} + \mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\right)} \cdot e^{\mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)} \cdot e^{-\frac{1}{2}\cdot\ln n + \mathcal{O}\left(\frac{1}{n}\right)} \\ & - 2\cdot e^{-\left(it\cdot\sqrt{\ln n} + \mathcal{O}\left(\frac{|t|}{\sqrt{\ln n}}\right)\right)} \cdot e^{-\mathcal{O}(1)} \cdot e^{-\frac{1}{2}\cdot\ln n + \frac{1}{2}it\cdot\sqrt{\ln n} + \mathcal{O}\left(t^2\right)}. \end{aligned}$$

Note that $B_n(0) = 0$ and $B_n(s) = \mathcal{O}\left(\frac{e^{\tau\cdot\sqrt{\ln n}}}{n^{1/2}}\right)$ uniformly for $|s| \leq \tau$, $s \in \mathcal{C}$, for some fixed $\tau > 0$. By denoting $\kappa_n \equiv \frac{n^{1/2}}{e^{\tau\cdot\sqrt{\ln n}}}$ for convenience, we can rewrite $B_n(s) = \mathcal{O}\left(\frac{1}{\kappa_n}\right)$ for $|s| \leq \tau$. Furthermore, by taking a small circle around the origin we easily obtain $B_n(s) = \mathcal{O}\left(\frac{|s|}{\kappa_n}\right)$ for $|s| \leq c < \tau$, where sufficiently small $c > 0$ can be taken less than τ. Consequently,

$$\varphi_n(t) = A_n(t) + B_n(t) = e^{-\frac{t^2}{2} + \mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\ln n}}\right)} + \mathcal{O}\left(\frac{|t|}{\kappa_n\cdot\sqrt{\ln n}}\right),$$

for $|t| \leq T_n = c\sigma_n$.

In fact, we can use Levy's convergence theorem to obtain the convergence result. However, we still need to use the Berry-Esseen inequality to find the convergence rate.

Based on the obtained approximation, we can follow the proof of Theorem 1 in Hwang (1998). That is, using the inequality $|e^w - 1| \leq |w|e^{|w|}$ for all complex w, we obtain

$$\begin{aligned} \left|\frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t}\right| &= \mathcal{O}\left(\left(\frac{1+t^2}{\sqrt{\ln n}}\right)\exp\left(-\frac{t^2}{2} + \mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\ln n}}\right)\right) + \frac{1}{\kappa_n\cdot\sqrt{\ln n}}\right) \\ &= \mathcal{O}\left(\left(\frac{1+t^2}{\sqrt{\ln n}}\right)e^{-\frac{1}{4}t^2} + \frac{1}{\kappa_n\cdot\sqrt{\ln n}}\right) \qquad (|t| \leq T_n), \end{aligned}$$

for sufficiently small $0 < c < \tau$.

Thus,

$$J_n = \int_{-T_n}^{T_n} \left| \frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t} \right| dt = \mathcal{O}\left(\frac{1}{\sqrt{\ln n}} \int_{-T_n}^{T_n} \left(1+t^2\right) e^{-\frac{1}{4}t^2} dt + \frac{1}{\kappa_n} \right)$$
$$= \mathcal{O}\left(\frac{1}{\sqrt{\ln n}} + \frac{1}{\kappa_n} \right) = \mathcal{O}\left(\frac{1}{\sqrt{\ln n}} \right),$$

because $\lim_{n\to\infty} \frac{\sqrt{\ln n}}{\kappa_n} = \lim_{n\to\infty} \frac{\sqrt{\ln n} \cdot e^{\tau \cdot \sqrt{\ln n}}}{n^{1/2}} = \lim_{n\to\infty} \frac{n \cdot e^{\tau \cdot n}}{e^{n^2/2}} = 0$. Due to Step 1, this concludes the proof. $\square$

B. Enumerative Issues

To illustrate the enumerative challenges posed by games in which both players have many agents, consider games in which Row has $m = 3$ actions. We now focus on the basic problem of finding the probability that Column has no strictly dominated actions. Fix the first row of Column's payoff matrix to be the identity permutation: $c_{1\cdot} = e_n \equiv (1, 2, \ldots, n)$. There are no strictly dominated actions for Column if and only if a pair of permutations $(c_{2\cdot}, c_{3\cdot})$ avoids the permutation pattern in which $c_{2j} > c_{2i}$ and $c_{3j} > c_{3i}$ for some $j > i$. This particular avoidance imposes restrictions on the pair of *i.i.d.* uniform $(c_{2\cdot}, c_{3\cdot})$ that has been the main object of interest for Hammett and Pittel (2008). Formally, our problem can be equivalently reformulated in terms of what is termed *parallel permutation patterns* in enumerative combinatorics (see our Lemma 8 below). This problem lies at the research frontier of that literature (Hammett and Pittel, 2008; Gunby and Pálvölgyi, 2019). To make things worse, in general, permutation patterns induced by strict dominance are different from those studied in the literature on permutation avoidance.

As a numerical demonstration, Table B.1 displays the number of possible Column's matrices with one fixed payoff row for $m = 3$ and $n \in [6]$ corresponding to exactly k strictly undominated actions, $k \in [n]$.[1] These numbers can be viewed as a generalization of the unsigned Stirling numbers of the first kind for $m = 3$. In particular, the underlined sequence corresponds to the number of incidents in which Column has no strictly dominated actions, as described above. The table suggests properties similar to those of the standard Stirling numbers. Values appear to be log-concave (and unimodal) and asymptotically normal with faster convergence rates. This hints at the qualitative similarities between the general m by n case and the particular 2 by n case studied in Section 3 of the main text.

The combinatorics community has accumulated knowledge of many number sequences summarized in the On-Line Encyclopedia of Integer Sequences (OEIS). It is worth noting that none of the sequences corresponding to any dimension of our analysis has been enumerated before in the OEIS. This suggests that our fundamental combinatorial object has not been studied previously.

To state Lemma 8, we need to introduce three additional definitions related to the literature on permutation patterns. First, we say that for $\sigma_1, \ldots, \sigma_d \in S_n$ and $\sigma'_1, \ldots, \sigma'_d \in S_m$, $(\sigma_1, \ldots, \sigma_d)$ *avoids* $(\sigma'_1, \ldots, \sigma'_d)$ if there does not exist indices $c_1 < \cdots < c_m$ such that $\sigma_i(c_1)\, \sigma_i(c_2) \cdots \sigma_i(c_m)$ is order-isomorphic to σ'_i for all i (e.g., see Gunby and Pálvölgyi,

[1] These numbers correspond to exact computations for all $(n!)^2$ possible combinations.

n \ k	1	2	3	4	5	6
1	$\underline{1}$					
2	1	$\underline{3}$				
3	4	15	$\underline{17}$			
4	36	147	242	$\underline{151}$		
5	576	2460	4775	4690	$\underline{1899}$	
6	14400	63228	134909	164193	109959	$\underline{31711}$

Table B.1: $(n!)^2 \cdot \Pr(U^C(3,n) = k)$, $k \in [n]$: exact calculations (the underlined sequence corresponds to sequence A007767 in the OEIS)

2019; Klazar, 2000). Second, we say that $\pi \preceq \sigma$ in the *weak Bruhat Order* if there is a chain $\sigma = \omega_1 \to \omega_2 \to \cdots \to \omega_s = \pi$, where each ω_t is a *simple reduction* of ω_{t-1}, i.e. obtained from ω_{t-1} by transposing two *adjacent* elements $\omega_{t-1}(i), \omega_{t-1}(i+1)$ with $\omega_{t-1}(i) > \omega_{t-1}(i+1)$. Equivalently (see Lemma 4.1 in Hammett and Pittel, 2008), $\pi \preceq \sigma$ in the *weak Bruhat Order* if $I(\pi) \subseteq I(\sigma)$, where for any $\omega \in S_n$, the inversion set $I(\omega) = \{(i,j) \mid i < j \text{ with } \omega^{-1}(i) > \omega^{-1}(j)\}$ is defined to be the set of all inversions in ω. Finally, for any $\sigma \in S_n$, let $\sigma^\star \in S_n$ denote its *complement*, i.e. $\sigma^\star(i) = n + 1 - \sigma(i)$.

Lemma 8. *Consider a random game $G(3,n)$. Then, for any $n \geq 1$,*

$$\prod_{i=1}^{n} (H_i/i) \leq \Pr(U^C(3,n) = n) = \Pr\left((c_{3\cdot}^\star)^{-1} \preceq c_{2\cdot}^{-1}\right) \leq (0.362)^n.$$

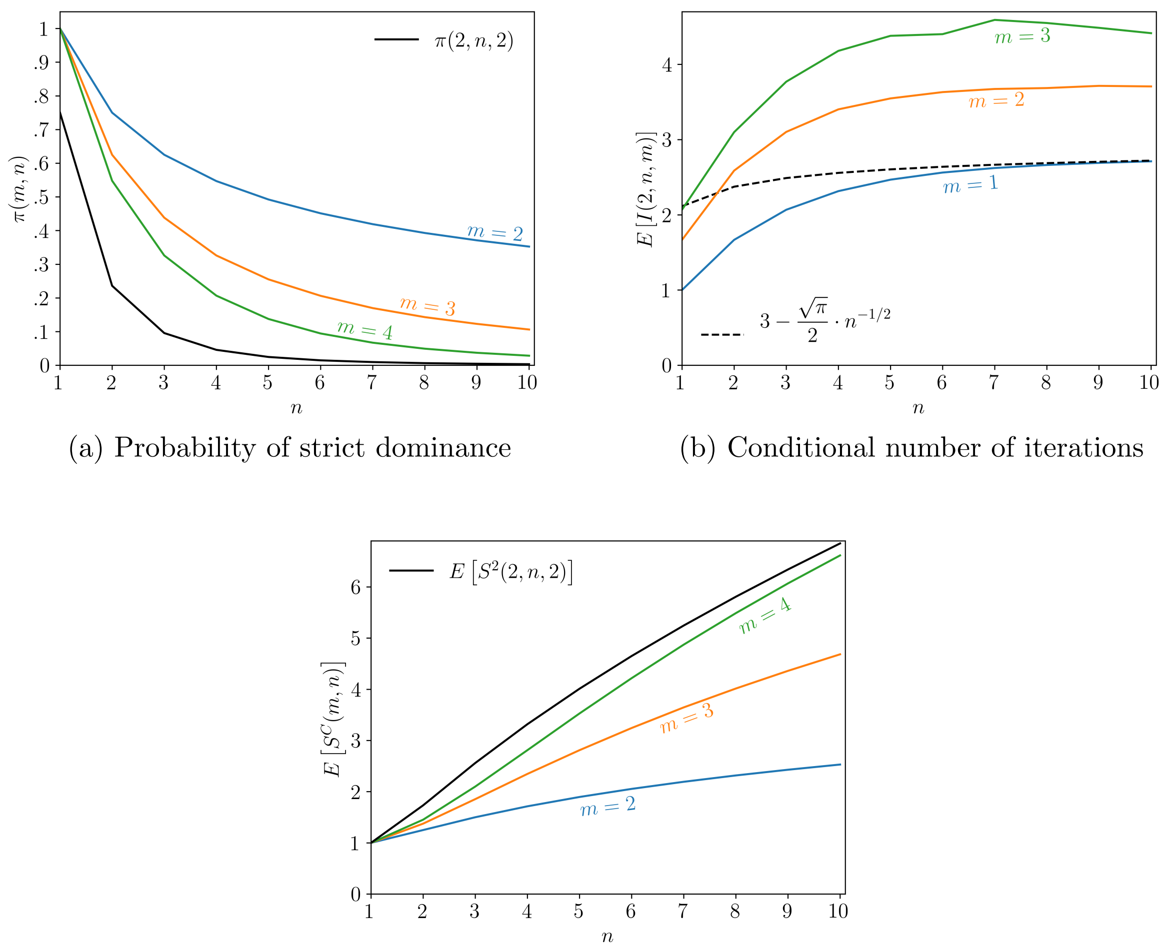
Proof. As in Lemma 1, we can set $c_{1\cdot}$ to e_n. This is without loss of generality.

All Column's actions are undominated if and only if $(c_{2\cdot}, c_{3\cdot})$ avoids $(12, 12)$. This holds whenever, for any $i < j$ with $c_{2i} < c_{2j}$, we have $c_{3i} > c_{3j}$, or equivalently $c_{3i}^\star < c_{3j}^\star$. In other words, the set $I(c_{2\cdot}^{-1})$ of inversions of $c_{2\cdot}$ contains the set $I\left((c_{3\cdot}^\star)^{-1}\right)$ of inversions of $(c_{3\cdot}^\star)^{-1}$, i.e. $I\left((c_{3\cdot}^\star)^{-1}\right) \subseteq I(c_{2\cdot}^{-1})$. This occurs if and only if $(c_{3\cdot}^\star)^{-1} \preceq c_{2\cdot}^{-1}$.

Certainly, $\Pr\left((c_{3\cdot}^\star)^{-1} \preceq c_{2\cdot}^{-1}\right) = \Pr\left(\pi \preceq \sigma\right)$, where $\sigma, \pi \in S_n$ are selected independently and uniformly at random. Probability bounds for this problem have been studied by Hammett and Pittel (2008). $\square$

C. MANY PLAYERS

Consider a random n-person game $G(m_1, m_2, \ldots, m_n)$, where player k has m_k actions, $k \in [n]$. Then, player k has the number of her undominated actions $U^k(m_1, m_2, \ldots, m_n)$ that is equivalent to $U^R\left(m_k, \prod_{i \neq k} m_i\right)$ already examined in Section 4.1. Because the number of other players' profiles $\prod_{i \neq k} m_i$ is multiplicative and becomes large even in small games, it is problematic for any player to eliminate any of her actions.



(a) Probability of strict dominance

(b) Conditional number of iterations

(c) Surviving actions

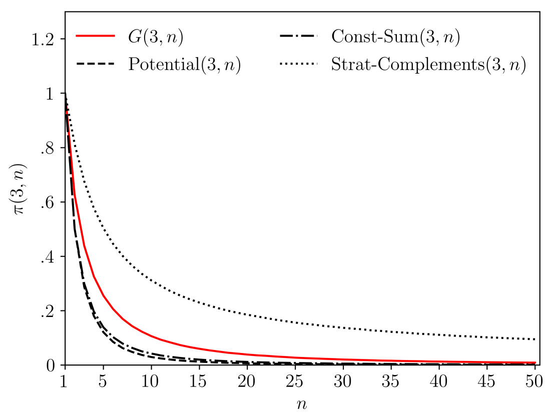
Figure C.1: Three dimensions of dominance solvability in games with three players

Therefore, results pertaining to dominance solvability features are straightforward even for small games with at least three players and closely trace results for two-player games in which players' action sets are comparably large. First, getting strict-dominance solvability is challenging even for small games since it is unlikely players can eliminate any action at all. Second, surviving games are likely to coincide with the games prior to the process of elimination—iterated elimination is statistically ineffective at simplifying games. Last, since it is even more demanding for any player to eliminate many actions at each elimination round, we expect the conditional number of iterations to be large even in small games. Figure C.1 illustrates these insights for three-player games. In particular, it shows that for random three-player games $G(2, n, m)$ with $G(2, n, 1)$ being equivalent to a two-player game $G(2, n)$, the conditional number of iterations is component-wise increasing and exceeds 3 even for small game dimensions.

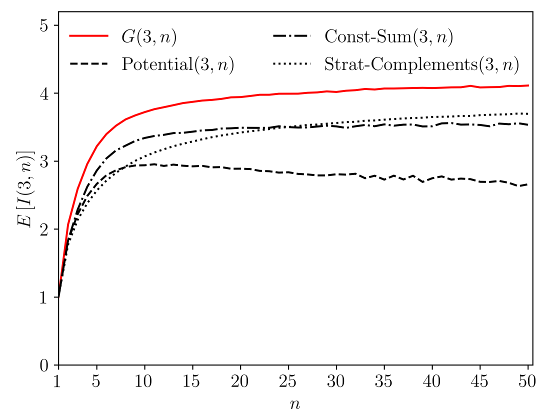
Because we have qualitative results even for small games, asymptotics is less interesting for random games with many, at least three, players. However, similar arguments to those used in the paper can be exploited to derive analogues of our main results.

D. Additional Figures

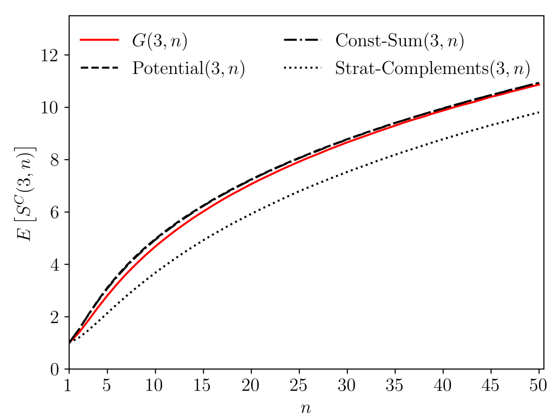
D.1. Alternative Distributional Assumptions in Imbalanced Games. Figure D.1 is the analogue of Figure 5 in the main text and displays the three dimensions of dominance solvability in $3\times n$ games, rather than $n\times n$ games, for the various distributional assumptions we inspect.



(a) Probability of strict dominance



(b) Conditional number of iterations



(c) Surviving actions

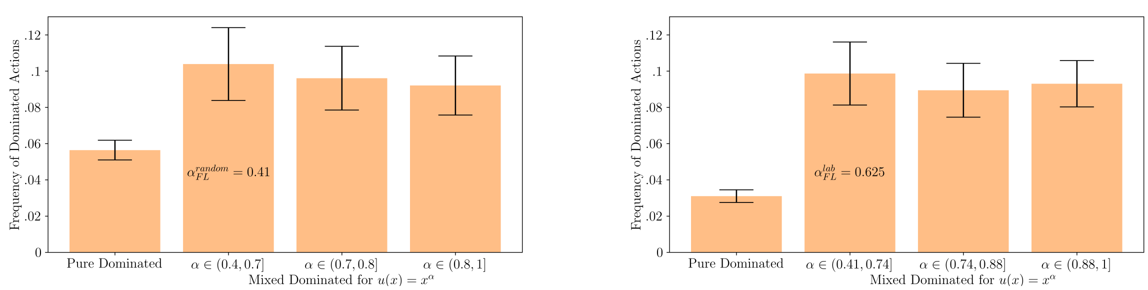
Figure D.1: Three dimensions of dominance solvability in $3 \times n$ games for alternative distributional assumptions

As can be seen, even when action sets are imbalanced, our insights regarding the efficacy of the iterative elimination procedure remain similar for uniformly random games and games governed by other distributions, corresponding to commonly-studied payoff structures.

One exception pertains to the number of iterations required conditional on dominance solvability in imbalanced potential games. As Column's action set expands, that number declines. We conjecture the underlying reason is the following. Conditional on a potential game being dominance solvable, when Column has multiple actions left after the first iterative round, Row must have some actions to eliminate—otherwise, the game would not be dominance solvable to begin with. This imposes restrictions on Column's payoffs that are absent when Row and Column's payoffs are determined independently. A deeper investigation of the features of dominance solvability in potential games is left for future research.

D.2. Dominance via Mixed Strategies in Lab Games. Figure D.2 presents the likelihood of playing an action dominated by a mixed strategy in laboratory games, using data from Fudenberg and Liang (2019), and in laboratory games, using data compiled by Wright and Leyton-Brown (2014). We consider CRRA utilities, highlighting the risk-aversion parameters estimated by Fudenberg and Liang (2019) for those games.

As can be seen, in both samples, participants have a substantially harder time eliminating actions dominated by mixed strategies.



(a) *Random* games (Fudenberg and Liang, 2019), $\alpha_{FL}^{random} = 0.41$ for *random* games is estimated by Fudenberg and Liang (2019)

(b) *Laboratory* games (Wright and Leyton-Brown, 2014), $\alpha_{FL}^{lab} = 0.625$ for *laboratory* games is estimated by Fudenberg and Liang (2019)

Figure D.2: Frequency of Row's dominated decisions in games with (1) exactly one Row's strictly dominated action and no other weakly-dominated actions and (2) exactly one Row's mixed-strategy dominated action and no weakly-dominated actions

D.3. Mixed-strategy Dominance Solvability in Imbalanced Games. Figure D.3 is the analogue of Figure 6 in the main text for imbalanced $3 \times n$ games. As can be seen, all of the insights demonstrated in the text for $n \times n$ games carry over.

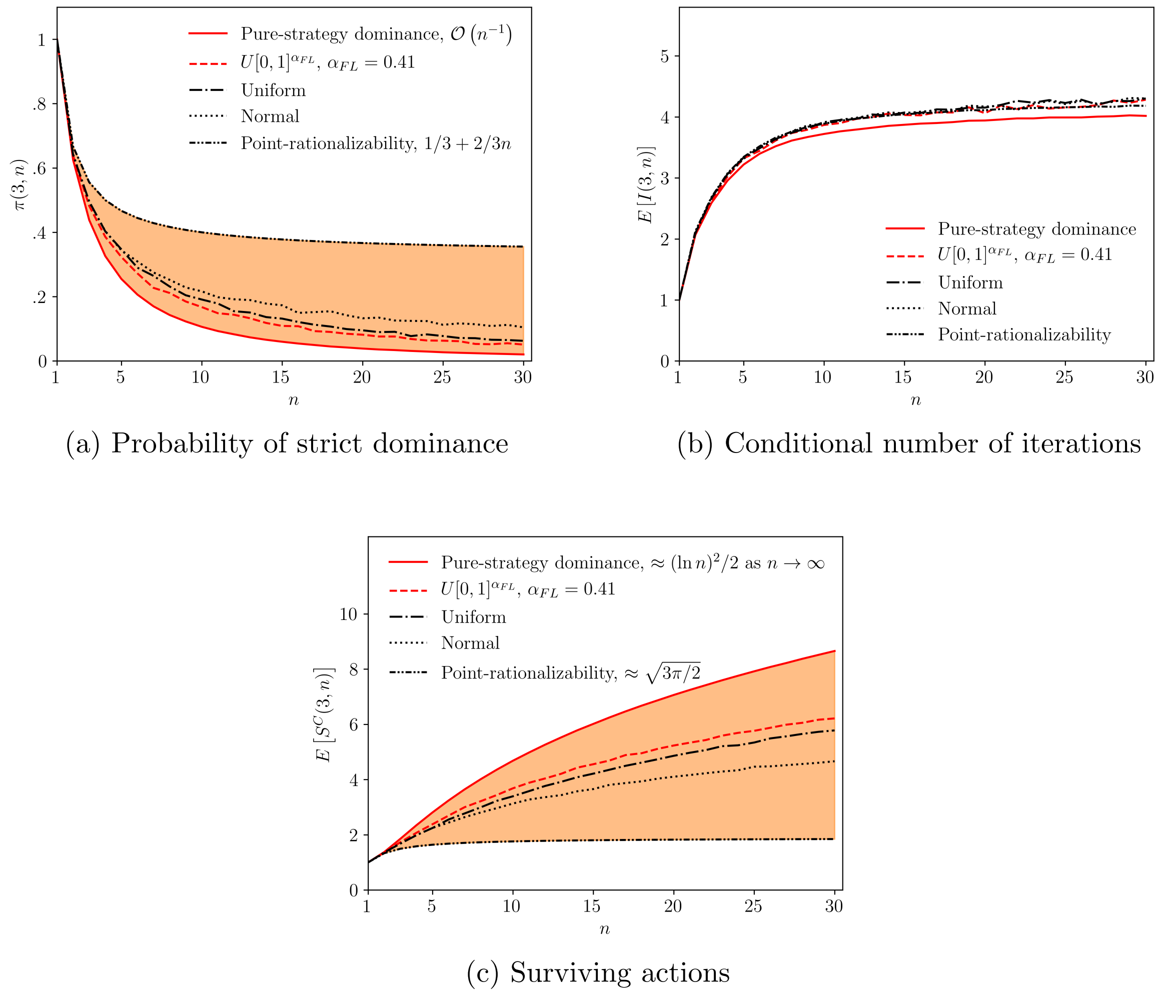


Figure D.3: Three dimensions of mixed-strategy dominance solvability in $3 \times n$ games, where $\alpha_{FL} = 0.41$ for *randomly-generated* games is estimated by Fudenberg and Liang (2019)

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