On the Structure and Generators of the Chromatic Algebra

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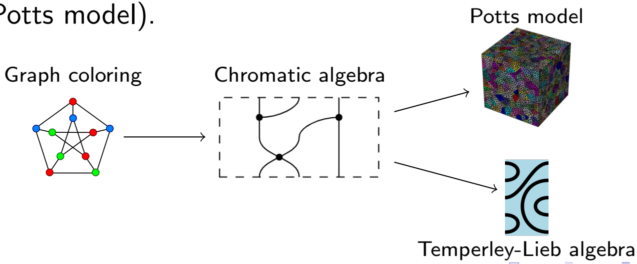
Introduction

- The chromatic algebra is a construction which combines graph colorings with algebraic binary operations to prove results in noncommutative algebra.
- It was introduced by Fendley and Krushkal to study statistical mechanics, after they noticed that the polynomial which counts graph colorings naturally arises in certain models (e.g., the Potts model).



Scope

In our project, we investigated fundamental properties of the chromatic algebra's structure, including its dimension and its generating set.

This presentation does not focus on the coloring aspect of the chromatic algebra; today, we will focus on the interactions between the operations and elements of the chromatic algebra.

For results about the trace, which heavily involves graph colorings, we recommend you read our full paper.

Associative Algebra

Definition (Associative Algebra)

An associative algebra over a field K is a set of elements which is closed under compatible operations of addition, multiplication, and multiplication by scalars from K.

It is important to note that subtraction is defined as the inverse of addition. Moreover, division is not well-defined (not every element must have a multiplicative inverse). Also, multiplication between elements is not necessarily commutative.

Example

The complex numbers form a 2-dimensional algebra over $\mathbb{R}$.

Laurent Series

In this project, our scalars come from $\mathbb{C}((Q))$, the set of complex Laurent series in the indeterminate Q. All subsequent mentions of Q refer to this indeterminate.

Definition

A **Laurent series** is a formal expression of the form $\sum_{n \geq N} a_n Q^n$, where $a_n \in \mathbb{C}$, N is an integer, and Q is an indeterminate. Informally, it is an "infinite polynomial" which can have finitely many negative power terms.

The set $\mathbb{C}((Q))$ is also the field of fractions of complex power series, which can be thought of as infinite polynomials.

Example

$$(1 - Q), \ \ (1 - Q)^{-1} = 1 + Q + Q^2 + \cdots \in \mathbb{C}((Q))$$

Chromatic Diagram

Definition (Chromatic Diagram)

An n-th order chromatic diagram is a collection of vertices and non-crossing edges inside a rectangle with n endpoints on the top border and n endpoints on the bottom border, such that each endpoint connects to exactly one edge.

Example

A 5th order chromatic diagram:

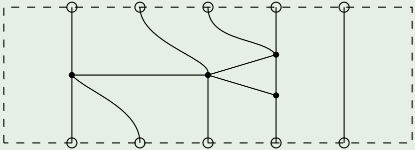
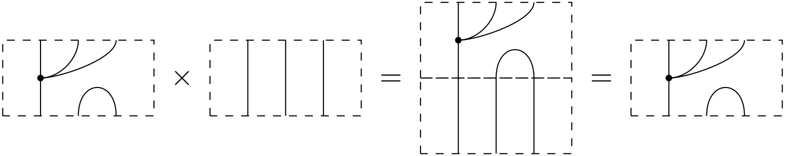
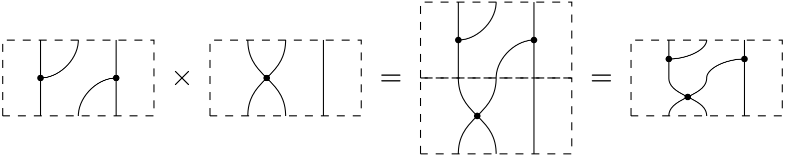


Diagram Multiplication

We can multiply two diagrams by stacking the left diagram on top of the right diagram and removing the border between the two.





Formal Diagram Addition

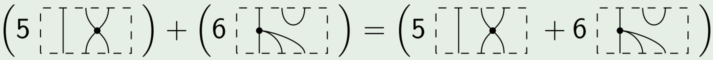
Now we know how to multiply diagrams. How do we add (and subtract) diagrams?

Definition

Formal addition of two diagrams consists of writing out one plus the other, with no further meaning or simplification.

Example

We use formal addition when adding variables; similarly, we can understand diagram addition by imagining diagrams as variables.



$$(5x) + (6y) = (5x + 6y)$$

Other Diagram Operations

Now that we have defined addition and multiplication of diagrams, what about subtraction? Division? Scalar multiplication?

Subtraction is defined as addition of the additive inverse, or the 'negative', of what is being subtracted. Therefore, it is also strictly formal.

Division is not well-defined, since many chromatic diagrams do not have multiplicative inverses.

Scalar multiplication is strictly formal.

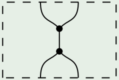
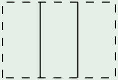
Free Algebra

Definition

The n-th order free algebra $\mathcal{F}_n$ is the algebra over $\mathbb{C}((Q))$ whose elements are linear combinations of n-th order chromatic diagrams. Multiplication of two elements is given by vertical stacking.

Example

The following is an element of the free algebra:

$\frac{1}{Q-1}$  $+ (1-Q)$  $\in \mathcal{F}_2.$

Chromatic Algebra

Now we introduce the main object of our discussion.

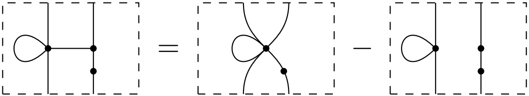
Definition

The ***n*-th order chromatic algebra** $\mathcal{C}_n$ is the algebra over $\mathbb{C}((Q))$ obtained from $\mathcal{F}_n$ by applying three equivalence relations.

These relations will be shown in the following slides.

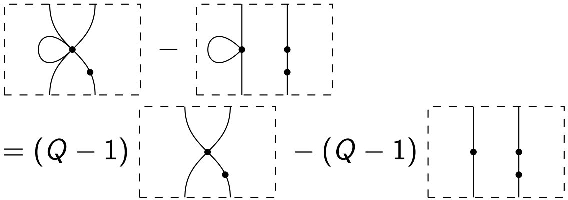
Chromatic Relation 1

If e is an inner (not connected to endpoint) edge of a diagram G which is not a loop, then $G = G/e - G \setminus e$. Here, G/e is the diagram obtained by contracting e, and $G \setminus e$ is the diagram obtained by deleting e.



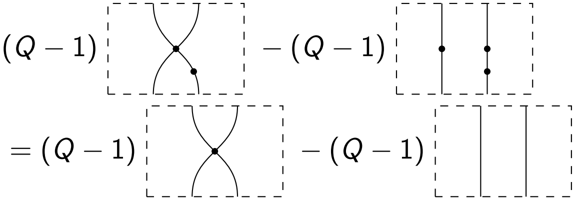
Chromatic Relation 2

If G contains an inner edge e which is a loop, then
$G = (Q-1)G \setminus e$.



Chromatic Relation 3

If G contains a 2-valent vertex v, then v can be removed by merging the two edges connected to it.



Chromatic Basis

Definition

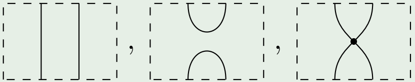
Let B_n be the set of chromatic diagrams without inner edges or self-looping edges.

Theorem (Fendley and Krushkal, 2010)

The equivalence classes of the diagrams in B_n form a basis of $\mathcal{C}_n$.

Example

Each element of $\mathcal{C}_2$ can be written as a linear combination of:

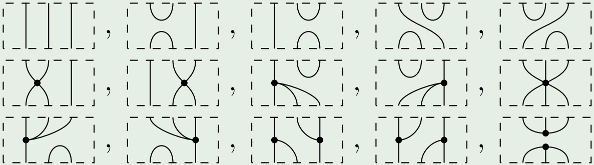
$B_2 = \{$  $\}$

The Chromatic Basis B_3

Example

Each element of $\mathcal{C}_3$ can be written as a linear combination of the following 15 diagrams:

$B_3 = \{$



$\}$

Main Result: Dimension of the Chromatic Algebra

Theorem (Main Result #1)

The n-th order chromatic algebra $\mathcal{C}_n$ is R_{2n}-dimensional, where R_i denotes the i-th Riordan number (OEIS A005043).

n	1	2	3	4	5	6	7	8
$\lvert B_n \rvert$	1	3	15	91	603	4213	30537	227475

The Riordan number is significant in combinatorics, and this result has surprising implications about the combinatorial information carried by the chromatic algebra's basis diagrams.

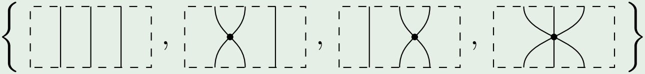
Multiplicative Generating Set

Definition

For $1 \leq i < j \leq n$, let $e_{i,j} \in B_n$ denote the diagram in which the edges at every top and bottom endpoint in columns i through j meet at one point, and the rest of the top and bottom endpoints are connected to their counterparts by vertical edges. Let E_n be the set of all such diagrams, together with the identity element.

Example

The following diagrams make up E_n for $n = 3$:

$E_3 =$ 

Main Result: Generating Set

Theorem (Main Result #2)

The set E_n generates $\mathcal{C}_n$ as an algebra over $\mathbb{C}((Q))$.

Size of the Generating Set

The regularity of the generating set is astounding, given the complexity of the chromatic basis. We illustrate this by comparing the size of these two sets: while the size of the generating set, $\binom{n}{2}+1$, exhibits quadratic growth, the size of the basis, R_{2n}, grows exponentially with n.

n	1	2	3	4	5	6	7	8
$\lvert E_n \rvert$	1	2	4	7	11	16	22	29
$\lvert B_n \rvert$	1	3	15	91	603	4213	30537	227475

Acknowledgements

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