ROOK POLYNOMIALS IN THREE AND HIGHER DIMENSIONS

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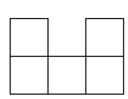
Abstract. The rook polynomial of a board counts the number of ways of placing non-attacking rooks on the board. In this paper, we describe how the properties of the 2-dimensional rook polynomials generalize to the rook polynomials of "boards" in three and higher dimensions. We also define families of 3-dimensional boards which generalize the 2-dimensional triangle boards and the boards representing the problème des rencontres. The rook coefficients of these 3-dimensional boards are shown to be related to famous number sequences such as the central factorial numbers, the number of Latin rectangles and the Genocchi numbers.

Introduction

The theory of rook polynomials provides a way of counting permutations with restricted positions. This theory was developed by Kaplansky and Riordan in [Kaplansky-Riordan 1946] and has been researched and studied quite extensively since then. Two rather comprehensive resources on it are [Riordan 1958] and [Stanley 1997]. In this paper, we generalize these properties and theorems of the 2-dimensional rook polynomials into higher dimensions, which was partially done for the 3-dimensional case in [Zindle]. A Maple program to calculate the rook numbers of a given 3-dimensional board using this generalization is included in the Appendix. In section 1 we review the 2-dimensional rook polynomials and their properties, including a discussion of famous families of boards, namely, the boards corresponding to the problème des rencontres, and the triangle boards. The results provided in this review most of the time form the basis of the proofs for the three and higher dimensional cases. In section 2 we discuss the generalization of the rook polynomials to three and higher dimensions, starting with a discussion of the 3-dimensional boards and how rooks attack in three dimensions. We provide the generalizations of the properties and theorems of 2-dimensional rook polynomials into three and higher dimensions as well as the three dimensional counterparts of the boards corresponding to the problème des rencontres and the triangle boards. In section 2.3 we introduce another family of 3-dimensional boards connected to the triangle boards. This family is named Genocchi boards due to its connection to the Genocchi numbers.

1. Overview of the Rook Theory in Two Dimensions

Given a natural number m, let $[m]$ denote the set $\{1, 2, ..., m\}$. In two dimensions, we define a *board* B with m rows and n columns to be a subset of $[m] \times [n]$. We call such a board an $m \times n$ board if m and n are the smallest such natural numbers. Each of the elements in the board is referred to as a *cell* of the board. The set $[m] \times [n]$ is called the *full* $m \times n$ board. An example of how we visualize a board is as follows:



Numbering the rows from top to bottom and columns from left to right, the above picture corresponds to the 2×3 board $B = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3)\}$. We sometimes highlight the cells missing from the board by shading them in gray.

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The *rook polynomial* $R_B(x) = r_0(B) + r_1(B)x + ... + r_k(B)x^k + ...$ of a board B represents the number of ways that one can place various numbers of non-attacking rooks on B, i.e. no two rooks can lie in the same column or row. More specifically, $r_k(B)$ is equal to the number of ways of placing k non-attacking rooks on B. For any board, $r_0(B) = 1$ and $r_1(B)$ is equal to the number of cells in B. For the above example, $r_2(B) = 4$ as there are four different ways to place 2 non-attacking rooks on the board. It is not possible to place 3 or more rooks on this board. Hence the rook polynomial of this board is $R_B(x) = 1 + 5x + 4x^2$. In general, the number of non-attacking rooks placed on an $m \times n$ board cannot exceed n and m, and hence the rook polynomial, as indicated by its name, is a polynomial of degree less than or equal to $\min\{m, n\}$. Note that the rook polynomial of a board is invariant under permuting the rows and columns of the board.

Theorem. *The number of ways of placing k non-attacking rooks, with $0 \leq k \leq \min\{m, n\}$, on the full $m \times n$ board is equal to $\binom{m}{k}\binom{n}{k}k!$.*

Proof: First choose k of the m rows and k of the n columns on which the rooks will be placed. This can be done in $\binom{m}{k}\binom{n}{k}$ ways. Once we have selected the rows and columns, we place a rook in each column and row. For the first row, there are k columns to choose from. Once the first rook is placed, for the second row, there are $k - 1$ choices left. Continuing in this way, we find that there are $k!$ ways to place the k-rooks on the chosen rows and columns. Hence we have $\binom{m}{k}\binom{n}{k}k!$ ways to place k non-attacking rooks on a full $m \times n$ board. □

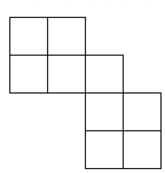
We define two boards to be *disjoint* if the boards do not share any rows or columns. If a board is composed of two disjoint sub-boards, the rook polynomial of the board can be calculated in terms of the rook polynomials of the sub-boards.

Theorem (Disjoint Board Decomposition)**.** *Let A and B be boards that share no rows or columns. Then the rook polynomial of the board $A \cup B$ consisting of the union of the cells in A and B is $R_{A\cup B}(x) = R_A(x) \times R_B(x)$.*

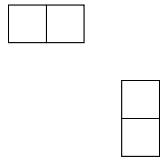
Proof: Let $R_A(x) = \sum_{k=0}^{\infty} r_k(A)x^k$ and $R_B(x) = \sum_{k=0}^{\infty} r_k(B)x^k$ be the rook polynomials of A and B. Consider the number of ways to place k rooks on $A \cup B$. We can place k rooks on A and 0 rooks on B, in $r_k(A)r_0(B)$ ways, or place $k - 1$ rooks on A and 1 rook on B, in $r_{k-1}(A)r_1(B)$ ways, and so on. Hence, the number of ways to place k rooks on $A \cup B$ is $\sum_{i=0}^{k} r_{k-i}(A)r_i(B)$, which is the coefficient of x^k in $r_A(x) \times r_B(x)$. Therefore $R_{A\cup B}(x) = R_A(x) \times R_B(x)$. □

The rook polynomial of a board which can be decomposed into two disjoint sub-boards, possibly after permuting rows and/or columns, can thus be calculated efficiently via this theorem.

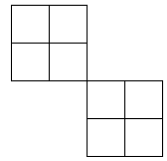
Similarly, the Cell Decomposition is another method of expressing the rook polynomial of a board in terms of smaller boards. Consider the board B shown below.



This board cannot be decomposed into two disjoint boards even if we permute the rows and columns. The Cell Decomposition method breaks the rook placements down into cases: when there is a rook in a specific position, say cell $(2, 3)$, and when there is no rook in that position. If there is a rook on cell $(2, 3)$, we cannot have another rook in row two or column three. By deleting row two and column three we create a new board B' on which the rest of the rooks can be placed. For the case that no rook is placed on $(2, 3)$, we create a board B'' by deleting the cell $(2, 3)$.



B'



B''

In order to find $r_k(B)$, the number of ways of placing k rooks on B, we add $r_{k-1}(B')$ and $r_k(B'')$ using the two cases. In terms of the rook polynomial, this implies that $R_B(x) = xR_{B'}(x)+R_{B''}(x)$. For this specific example, the Disjoint Board Decomposition can be used to compute the rook polynomials of B' and B'', making them much easier to compute than that of the board B.

The same idea of considering cases in regards to a specific cell as described above proves the more general Cell Decomposition.

Theorem (Cell Decomposition). *Let B be a board, B' be the board obtained by removing the row and column corresponding to a cell from B, and B'' be the board obtained by deleting the same cell from B. Then $R_B(x) = xR_{B'}(x) + R_{B''}(x)$.*

Another property of the rook polynomials relates the rook polynomial of a board to that of the board consisting of the missing cells. Given an $m \times n$ board B, we define the *complement* of B, denoted $\overline{B}$ to consist of all cells missing from B so that the disjoint union of B and $\overline{B}$ is the full $m \times n$ board. In other words $\overline{B} = [m] \times [n] \backslash B$. Sometimes we clearly indicate with respect to which board the complement is taken by saying that the complement is calculated inside $[m] \times [n]$.

Theorem (Complementary Board Theorem). *Let $\overline{B}$ be the complement of B inside $[m] \times [n]$ and $R_B(x) = \sum r_i(B)x^i$ the rook polynomial of B. Then the number of ways to place k non-attacking rooks on $\overline{B}$ is*

$$r_k(\overline{B}) = \sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i} \binom{n-i}{k-i} (k-i)! r_i(B)$$

taking r_i to be 0 for i greater than the degree of $R_B(x)$.

Proof: In order to find the number of ways to place k non-attacking rooks on $\overline{B}$, we consider all the placements of k non-attacking rooks on the full $m \times n$ board and remove those where one or more rooks are placed on B using the Inclusion-Exclusion Principle. We temporarily number the k rooks in our counting process, which means we will be counting $k! r_k(\overline{B})$. The total number of ways to place k numbered rooks on a full $m \times n$ board is $\binom{m}{k}\binom{n}{k}k!^2$, the additional $k!$ factor coming from the numbering of the rooks. Let A_i denote the set of placements of the rooks where the ith rook is on the board B. We have to remove these placements from the set of all placements. There are $r_1(B)$ ways to place the ith rook on B and $\binom{m-1}{k-1}\binom{n-1}{k-1}((k-1)!)^2$ ways to place the rest in the other rows and columns. Hence there are $r_1(B)\binom{m-1}{k-1}\binom{n-1}{k-1}((k-1)!)^2$ elements in A_i and there are k A_i's. Similarly, there are $r_2(B)2!\binom{m-2}{k-2}\binom{n-2}{k-2}((k-2)!)^2$ elements in $A_i \cap A_j$ for any $i \neq j$ and there are $\binom{k}{2}$ of these double intersections. There are $r_3(B)3!\binom{m-3}{k-3}\binom{n-3}{k-3}((k-3)!)^2$ elements in $\binom{k}{3}$ triple intersections $A_i \cap A_j \cap A_\ell$, and so on. Hence, using the Inclusion-Exclusion Principle, the number of ways to place k numbered rooks on $\overline{B}$ is

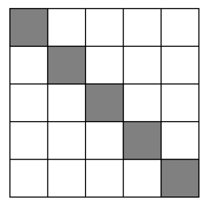
$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i! \binom{m-i}{k-i} \binom{n-i}{k-i} ((k-i)!)^2 r_i(B) \ .$$

Dividing this by $k!$ we find that

$$r_k(\overline{B}) = \sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i} \binom{n-i}{k-i} (k-i)! r_i(B) \ .$$

□

1.1. **Problème des Rencontres.** We now consider the family of boards which correspond to the famous problème des rencontres, or equivalently to derangements. An example of such a problem is as follows. Suppose that five people enter a restaurant, each person with their own hat. We want to find the number of ways that everyone can leave the restaurant without their own hat, ignoring the order in which they leave. The boards which correspond to the problème des rencontres are $m \times m$ boards with the cells along the main diagonal removed. For the hat problem, $m = 5$ and we obtain the following board B, where we highlighted the missing cells with gray.



The number of ways to place 5 rooks on B corresponds with the number of permutations of five elements where no element is in its original position. Such a permutation is equivalent to matching the owners with their hats where no owner is matched to their hat.

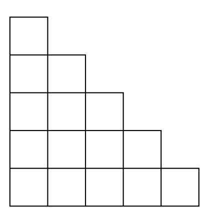
Instead of B, consider the complement. The complement $\overline{B}$ consists of 5 disjoint boards each of which is a single cell. The rook polynomial of each cell is $1 + x$. Hence, using the Disjoint Board Decomposition, we find that the rook polynomial of $\overline{B}$ is $(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$. Now, using the theorem on rook polynomials of complementary boards, we find that the number of ways to place 5 rooks on B is equal to

$$\binom{5}{5}\binom{5}{5}5! \cdot 1 - \binom{4}{4}\binom{4}{4}4! \cdot 5 + \binom{3}{3}\binom{3}{3}3! \cdot 10 - \binom{2}{2}\binom{2}{2}2! \cdot 10 + \binom{1}{1}\binom{1}{1}1! \cdot 5 - \binom{0}{0}\binom{0}{0}0! \cdot 1 = 44.$$

In general, r_k of the rook polynomial of an $m \times m$ problème des rencontres board is

$$\sum_{i=0}^{k}(-1)^i \binom{m-i}{k-i}^2 (k-i)!\binom{m}{i}.$$

1.2. **Triangle Boards.** We next consider the family of 2-dimensional boards called the triangle boards. A *triangle board* of size m consists of the cells of the form (i, j) where $j \leq i$ and $1 \leq i \leq m$. The triangle board of size 5 is shown below.



The rook numbers of this family correspond with the Stirling numbers of the second kind. Recall that the Stirling numbers of the second kind, $S(n, k)$, count the number of ways to partition a set of size n into k non-empty sets, and can be defined recursively by

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

with $S(n, 1) = 1$ and $S(n, n) = 1$.

Theorem. *The number of ways to place k non-attacking rooks on a triangle board of size m is equal to $S(m+1, m+1-k)$, where $0 \leq k \leq m$.*

Proof: We will prove this by induction on m and using the recursive definition of the Stirling numbers.

The rook polynomial of the triangle board of size 1 is equal to $1+x$, which corresponds to $S(2,1) = 1$ and $S(2,2) = 1$.

Assume now the theorem is true for some m, i.e. that the number of ways of placing k rooks on a size m triangle board is equal to $S(m+1, m+1-k)$ for $0 \leq k \leq m$. We will show that the number of ways of placing k rooks on an $(m+1) \times (m+1)$ board is equal to $S((m+1)+1, (m+1)+1-k) = S(m+2, m+2-k)$ for $0 \leq k \leq m+1$.

For $k = m+1$, there is only one way to place k non-attacking rooks on a size $m+1$ triangle board: by placing all rooks on the diagonal. This corresponds to $S(m+2, m+2-k) = S(m+2, 1) = 1$. For $k = 0$, placing k rooks on the board can be done in only one way, which corresponds to $S(m+2, m+2) = 1$. Therefore the rook numbers and the Stirling numbers agree for $k = 0$ and $k = m+1$.

We now show that these numbers agree for $0 < k < m+1$. When finding the number of ways to place k rooks on the size $m+1$ triangle board, we consider two cases. First when all k rooks are placed on the top m rows, forming a size m triangle board. There are $S(m+1, m+1-k)$ ways to do so by our inductive hypothesis. Second case is when one rook lies in the bottom row. In this case, $k-1$ rooks must be placed on the top m rows, which can be done in $S(m+1, m+1-(k-1))$ ways. We then have $m+1-(k-1)$ cells available in the last row to place our last rook, resulting in $(m+2-k)S(m+1, m+2-k)$ ways to place k rooks on the board with one rook in the last row. So there are a total of $S(m+1, m+1-k) + (m+2-k)S(m+1, m+2-k)$ ways to place k rooks on a size $m+1$ triangle board. Using the recursive definition of the Stirling numbers, this sum corresponds to $S(m+2, m+2-k)$.

Therefore, by induction, the k-th rook number for any size m triangle board is $S(m+1, m+1-k)$. □

2. Rook Polynomials in Three and Higher Dimensions

The theory of rook polynomials in two dimensions as described above can be generalized to three and higher dimensions. The theory for the three dimensions is introduced in [Zindle] and the theory we describe in this paper is a more generalized version of Zindle's theory.

In three dimensions, our boards will be subsets of $[m] \times [n] \times [p]$. We refer to such a board as an $m \times n \times p$ board. More generally, a board in d dimensions is a subset of $[m_1] \times [m_2] \times \ldots \times [m_d]$. A *full board* is again a board if the board is the whole set $[m_1] \times [m_2] \times \ldots \times [m_d]$. In three and higher dimensions, a *cell* again refers to elements of the boards. In particular, in three dimensions, a cell is a 3-tuple (i, j, k) with $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p$.

In two dimensions rows correspond to cells with the same first coordinate, and the columns correspond with the same second coordinate. We extend this idea to three dimensions to introduce new groupings of cells. All cells with the same third coordinate are said to lie in the same *layer* and we number the layers from top to bottom. All cells with the same first coordinate are said to be in the same *slab* and all cells with the same second coordinate in the same *wall*. We still have rows and columns within a layer. We also have *towers* which correspond with the cells with the same first and second coordinate. In four and higher dimensions, we use *layer* to represent cells along any hyperplane formed by fixing a coordinate.

We generalize the rook theory to three dimensions so that a rook in three dimensions will attack along walls, slabs and layers. In higher dimensions, rooks attack along hyperplanes corresponding to cells with one fixed coordinate. In three dimensions, when we place a rook in a cell, we can no longer place another rook in the same wall, slab, or layer. In higher dimensions, a rook placed in a cell means we cannot place another rook in the fixed coordinate hyperplanes that this cell belongs to. In other words, if a rook is placed in cell $(i_1, i_2, \ldots, i_d)$, then a rook may not be placed in any other cell sharing a coordinate with this cell. With this generalization, the rook polynomial of a board is invariant under permuting the layers of the board.

In another generalization of rook polynomials to three and higher dimensions, the rooks attack along lines instead of attacking over hyperplanes. For example, in three dimensions, a rook placed in cell (i, j, k) prohibits another rook to be placed in cells $(i, j, \cdot), (i, \cdot, k)$ and $(\cdot, j, k)$. This approach has possible applications as well, however, we will not pursue this generalization in this paper.

Our first theorem on the generalized rook theory deals with a three-dimensional board obtained from a two-dimensional board extended in the z-direction. In other words, if A is a two-dimensional board, the three-dimensional extension of A with p layers consists of elements of the form (i, j, k) where $(i, j) \in A$ and $1 \leq k \leq p$. It is natural that there is a relation between the rook polynomials of the two boards.

Theorem. *Let A be an $m \times n$ board and B be a three-dimensional extension of A with p layers. Then for $0 \leq k \leq \min\{m, n, p\}$,*

$$r_k(B) = \frac{p!}{(p-k)!} r_k(A) \,.$$

Proof: Given a three-dimensional rook placement on the board B consider the projection onto the board A. Since each rook can attack along either coordinate, when projected onto A no two rooks occupy the same cell in A and we get a placement of k rooks on A. There are $r_k(A)$ such placements. Given such a placement, we must distribute the k rooks among p layers. This is equivalent to k permutations of p numbers, which corresponds with $p!/(p-k)!$. So we have $r_k(A)p!/(p-k)!$ ways to place k rooks on B. Also note that for $k > \min\{m, n, p\}$, $r_k(B) = 0$ since k rooks cannot fit into the board. □

As a corollary of this theorem, we can obtain the rook numbers of the full three-dimensional boards, which are extensions of the full two-dimensional boards. However, we provide a proof similar to the two-dimensional case below which gives the idea of the proof of the general higher dimensional theorem.

Theorem. *There are $\binom{m}{k}\binom{n}{k}\binom{p}{k}(k!)^2$ ways to place k non attacking rooks on the full $m \times n \times p$ board for $0 \leq k \leq \min\{m, n, p\}$.*

Proof: Since we are placing k rooks on m slabs, n walls, and p layers, we have $\binom{m}{k}\binom{n}{k}\binom{p}{k}$ ways to choose the k slabs, walls, layers to place the rooks on. Since we have k rooks and k layers, there will be exactly one rook on each layer. For the first layer, we have k walls and k slabs that we can choose from to place the rook. After placing the first rook, on the second layer we will have $k-1$ slabs and $k-1$ walls as options. Continuing this way, we find that we have $k \cdot k \cdot (k-1) \cdot (k-1) \cdot (k-2) \cdot (k-2) \cdot \ldots \cdot 2 \cdot 2 \cdot 1 \cdot 1 = (k!)^2$ ways to place the rooks on the chosen walls, slabs and layers. So there are $\binom{m}{k}\binom{n}{k}\binom{p}{k}(k!)^2$ ways to place k non-attacking rooks on the full $m \times n \times p$ board. □

The theorem for the most general case is:

Theorem. *There are $\binom{m_1}{k}\binom{m_2}{k}\ldots\binom{m_d}{k}(k!)^{d-1}$ ways to place k non attacking rooks, with $0 \leq k \leq \min_i m_i$, on a full $m_1 \times m_2 \times \ldots \times m_d$ board in d dimensions.*

The decomposition theorems of the two-dimensional case also generalize naturally to three and higher dimensions. We define two boards in three dimensions to be *disjoint* if the boards do not share any walls, slabs or layers. In four and higher dimensions, the boards are *disjoint* if they do not share any layers. We then have the following Disjoint Board Decomposition in the general case.

Theorem (Disjoint Board Decomposition). *Let A and B be two boards in three and higher dimensions that share no layers. Then the rook polynomial of the board $A \cup B$ consisting of the union of the cells in A and B is $R_{A\cup B}(x) = R_A(x) \times R_B(x)$.*

The Disjoint Board Theorem allows easy calculation of rook polynomials of a board which can be decomposed into disjoint sub-boards, possibly after permuting layers.

The Cell Decomposition from the two-dimensional case generalizes to the three and higher dimensions as follows with the proof being a slight modification of the proof in section 1.

Theorem (Cell Decomposition). *Let B be a board, B' be the board obtained by removing the layers that correspond to a cell from B, and B'' be the board obtained by removing the same cell from B. Then $R_B(x) = xR_{B'}(x) + R_{B''}(x)$.*

The theorem on complementary boards generalizes to three and higher dimensions as

Theorem (Complementary Board Theorem). *Let $\overline{B}$ be the complement of B inside $[m_1] \times [m_2] \times [m_d]$ and $R_B(x) = \sum_i r_i(B)x^i$ the rook polynomial of B. Then the number of ways to place $0 \le k \le \min_i m_i$ non-attacking rooks on B is*

$$r_k(\overline{B}) = \sum_{i=0}^{k}(-1)^i \binom{m_1 - i}{k-i}\binom{m_2 - i}{k-i}\cdots\binom{m_d - i}{k-i}(k-i)!^{d-1} r_i(B).$$

Proof: The proof proceeds as in the two-dimensional case. We number the rooks and let A_i be the set of placements of the rooks where the ith rook is on B. There are

$$\binom{m_1}{k}\binom{m_2}{k}\cdots\binom{m_d}{k}k!^d$$

ways to place k numbered rooks on the full board. There are

$$r_1(B)\binom{m_1 - 1}{k-1}\binom{m_2 - 1}{k-1}\cdots\binom{m_3 - 1}{k-1}(k-1)!^d$$

elements in A_i and there are k A_i's. Similarly, there are

$$r_2(B)2!\binom{m_1 - 2}{k-2}\binom{m_2 - 2}{k-2}\cdots\binom{m_d - 2}{k-2}(k-2)!^d$$

elements in $A_i \cap A_j$ for any $i \neq j$ and there are $\binom{k}{2}$ of these double intersections, and so on. Hence, using the Inclusion-Exclusion Principle, the number of ways to place k numbered rooks on $\overline{B}$ is

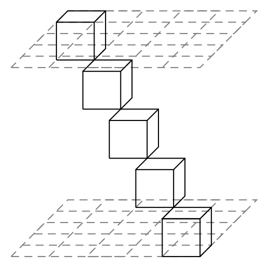
$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} i! \binom{m_1 - i}{k-i}\binom{m_2 - i}{k-i}\cdots\binom{m_d - i}{k-i}(k-i)!^d r_i(B)\ .$$

Dividing this by $k!$ we find that

$$r_k(\overline{B}) = \sum_{i=0}^{k}(-1)^i \binom{m_1 - i}{k-i}\binom{m_2 - i}{k-i}\cdots\binom{m_d - i}{k-i}(k-i)!^{d-1} r_i(B)\ .$$

□

2.1. **Problème des Rencontres in Three Dimensions.** Recall the problème des rencontres from earlier. The problème des rencontres dealt with a board with restrictions along the main diagonal. When creating a three dimensional version of the problème des rencontres board, we will again place restrictions along the diagonal. In two dimensions we explained the problème des rencontres by considering five people leaving a restaurant without their hat. For this type of problem to make sense in three dimensions we will have to alter the scenario. We will once again consider five people entering a restaurant and introduce another dimension to the story. Let each person now have their own hat and coat. We are now interested in the number of ways that the five people can leave the restaurant without both of their items. Let B be an $5 \times 5 \times 5$ board with elements (i, i, i) for $i = 1, ..., 5$ removed; we will consider placing 5 rooks on B. The visual representation of B is shown below.

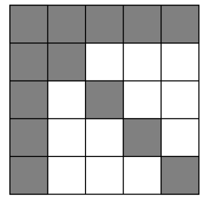


For this board we let each layer represent a person, and walls and slabs represent coats and hats, respectively. The missing cells correspond with no person leaving with both their hat and coat. We refer to this board as the problème des rencontres board of the first kind. To find the rook numbers of this board, notice that the 5 missing cells form disjoint boards. The rook polynomial for each cell is $1+x$. Hence, using the Cell Decomposition, we get $(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$ as the rook polynomial for the missing cells. Using the Complementary Board Theorem, we then find that the number of ways to place 5 rooks on B is

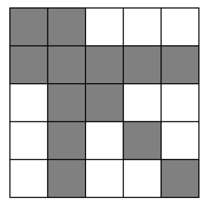
$$\binom{5}{5}\binom{5}{5}\binom{5}{5}(5!)^2 - \binom{4}{4}\binom{4}{4}\binom{4}{4}(4!)^2 \cdot 5 + \binom{3}{3}\binom{3}{3}\binom{3}{3}(3!)^2 \cdot 10 - \binom{2}{2}\binom{2}{2}\binom{2}{2}(2!)^2 \cdot 10 + \binom{1}{1}\binom{1}{1}\binom{1}{1}(1!)^2 \cdot 5 - \binom{0}{0}\binom{0}{0}\binom{0}{0}(0!)^2 \cdot 1 = 11844$$

More generally, the number of ways that we can place k rooks on an $m \times m \times m$ problème des rencontres board of this kind is $\sum_{i=0}^{k}(-1)^i \binom{m-i}{k-i}^3 (k-i)!^2 \binom{k}{j}$.

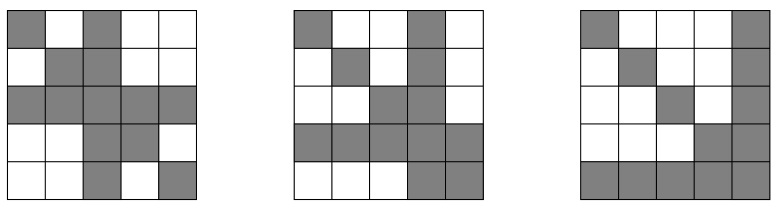
Another generalization of the problème des rencontres is to remove the rows, columns, and towers that pass through a diagonal cell, i.e. to remove cells of the form $(i, i, \cdot)$, $(i, \cdot, i)$ and $(\cdot, i, i)$. This second generalization corresponds to finding the number of ways that the five people can leave the restaurant without their coat, hat, or any proper pairing of a coat and hat. This means that each person must leave the restaurant with a hat that is not theirs, and a coat that is neither theirs nor the owner of the hat. The rook board for this problem is a bit more difficult to visualize so we will first discuss how to construct it. We again let the layers of the board represent a person, and the walls and slabs represent coats and hats, respectively. For layer 1, corresponding to the first person, we remove $(\ell, 1, 1)$ and $(1, \ell, 1)$ for $1 \leq \ell \leq 5$. This removes the column and row corresponding to the first person not leaving with their coat or hat. We will also remove all cells along the main diagonal, meaning cells of the form $(\ell, \ell, 1)$ for $1 \leq \ell \leq 5$. This corresponds with person one not leaving with another person's coat and hat. The first layer of the board will then appear as follows.



For the second layer we remove $(\ell, 2, 2)$ and $(2, \ell, 2)$ for $1 \leq \ell \leq 5$. This will remove the row and column associated with the second person leaving with their own coat or hat. We will also remove $(\ell, \ell, 2)$ for $1 \leq \ell \leq 5$. This corresponds with the second person not leaving with another person's coat and hat. This layer will appear as follows:



Continuing this method for the final three layers we get:



The problème des rencontres board of the second kind of any size m is constructed in a similar fashion.

We use a Maple program to compute the rook polynomials of this type board of various sizes. The program is included in the Appendix. The rook numbers of boards of size from 3 up to 7 are given in the following table:

$m \backslash k$	0	1	2	3	4	5	6	7
3	1	6	6	2				
4	1	24	132	176	24			
5	1	60	960	4580	5040	552		
6	1	120	4260	52960	213000	206592	21280	
7	1	210	14070	368830	3762360	13109712	11404960	1073160

Notice from the table that the rook numbers for $k = m$ correspond to the number of $3 \times m$ Latin rectangles. In fact, the correspondence between these rook placements and the Latin rectangles is very natural.

Theorem. *The number of ways to place m rooks on the size m problème des rencontres board of the second kind is equal to the number of $3 \times m$ Latin rectangles in which the first row is in order.*

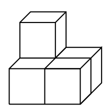
Proof: A $3 \times m$ Latin rectangle consists of three rows each of which is a permutation of the numbers in $[m]$ and where in each of the m columns no number is repeated. Given such a rectangle, each column can be represented by an ordered triple (r_1, r_2, r_3) in which no two entries are the same. These are exactly the cells missing from the problème des rencontres board of the second kind. We then take these m ordered triples and place rooks in the corresponding cells of this board. Because each number appears in each row of the Latin rectangle exactly once, we have exactly one rook per slab, wall, and layer. Therefore, the rooks are non-attacking. This shows that any $3 \times m$ Latin rectangle corresponds with a valid placement of m rooks on the size m problème des rencontres board of the second kind.

Now consider an arbitrary placement of m rooks on the size m problème des rencontres board of the second kind. Since there are m rooks, there is a rook in each slab. We read the positions of the rooks starting with the rook in the first slab, and record these into the columns of a $3 \times m$ array. In this way, the first row is arranged from 1 to m in an increasing order and as explained above, each row is a permutation of $[m]$ and that no two entries in each column are the same. This also shows that the correspondence is one-to-one. □

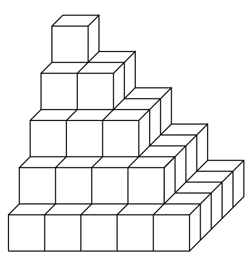
This second kind of the problème des rencontres board can be generalized to dimensions higher than three as follows: a size m problème des rencontres board in d dimensions is a subset of the set $[m]^d$ where the cells with two equal coordinates are removed. With this generalization, using a method similar to the proof of the above theorem, we obtain the following theorem:

Theorem. *The number of ways to place m rooks on a size m problème des rencontres board in d dimensions is equal to the number of $d \times m$ Latin rectangles in which the first row is in order.*

2.2. **Triangle Boards in Three Dimensions.** In two dimensions the triangle board of size m contains the cells of the form (i, j) with $j \leq i$ and $1 \leq i \leq m$. This board has the property that there is only one way to place m rooks on a size m triangle board. Another property of the triangle board is that removing both the row and column corresponding to a diagonal cell of a size m triangle board results in a size $m-1$ triangle board. We want to replicate these aspects of a triangle board in three dimensions, and this is how the three dimensional triangle board evolved. In three dimensions a size 1 triangle board is simply one cell. The size 2 triangle board is obtained by placing a 2×2 layer below the size 1 triangle board as follows.



We build the larger triangle boards recursively in a similar way, by adding an $(m+1) \times (m+1)$ layer at the bottom of a size m triangle board. The cells included in the size m triangle are (i, j, k) with $1 \leq i, j \leq k$ and $1 \leq k \leq m$. With this definition, there is only one way to place m rooks on a size m triangle board. Additionally, removing the wall, slab and layer including a diagonal cell of a size m triangle board results in a size $m-1$ triangle board. The size 5 triangle board is depicted below.



The rook numbers of the triangle boards up to size 8 are calculated using Maple and are shown in the table on the next page. The numbers turn out to be the *central factorial numbers* defined recursively by

$$T(n, k) = T(n-1, k-1) + k^2 T(n-1, k)$$

with $T(n, 1) = 1$ and $T(n, n) = 1$.

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	5	1						
3	1	21	14	1					
4	1	85	147	30	1				
5	1	341	1408	627	55	1			
6	1	1365	13013	11440	2002	91	1		
7	1	5461	118482	196053	61490	5278	140	1	
8	1	21845	1071799	3255330	1733303	251498	12138	204	1

Sequence A008957 in [Sloane 2009], Triangle of central factorial numbers

Theorem. *The number of ways to place k rooks on a size m triangle board in three dimensions is equal to $T(m+1, m+1-k)$, where $0 \leq k \leq m$.*

Proof: We will prove this theorem by induction on m.

For the base case, $m = 1$, the rook polynomial is $1 + x$ and the corresponding central factorial numbers are $T(2, 2) = T(2, 1) = 1$. Hence the result is true for $m = 1$.

Assume now the theorem is true for some m, i.e. that the number of ways of placing k rooks on a size m triangle board is equal to $T(m + 1, m + 1 - k)$ for $0 \leq k \leq m$. We will show that the number of ways of placing k rooks on an $(m + 1) \times (m + 1)$ board is equal to $T((m + 1) + 1, (m + 1) + 1 - k) = T(m + 2, m + 2 - k)$ for $0 \leq k \leq m + 1$.

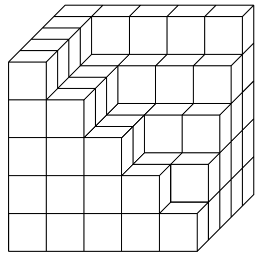
We know that there is only one way to place no rooks, which corresponds to $T(m+2, m+2) = 1$. We also know that there is only one way to place the maximum number of rooks, $m + 1$ rooks, which corresponds to $T(m + 2, m + 2 - (m + 1)) = T(m + 2, 1) = 1$.

Now let $0 < k < m + 1$. Similar to the two-dimensional case, we consider two cases, when all rooks are on the top m layers and when one of the rooks is on the bottom layer. The top m layers form a triangle board of size m and hence the number of ways to place k rooks on the top m layers is $T(m + 1, m + 1 - k)$. If one rook is on the bottom layer, the rest of the rooks will be on the top m layers, which can be done in $T(m + 1, m + 1 - (k - 1))$ ways. Once these $k - 1$ rooks are placed, the corresponding $k - 1$ rows and columns in the bottom layer are restricted for the last rook, leaving $(m + 1 - (k - 1))^2$ cells available for that rook. Hence, there are a total of $(m+2-k)^2 T(m+1, m+2-k)$ ways to have k rooks on the board with one being on the bottom layer. Adding the two cases, we obtain

$$T(m + 1, m + 1 - k) + (m + 2 - k)^2 T(m + 1, m + 2 - k)$$

ways of placing the k rooks on the size $m + 1$ triangle board. Using the recursive definition of the central factorial numbers, this sum corresponds to $T(m + 2, m + 2 - k)$, proving the theorem by induction. □

2.3. **Genocchi Board.** Another possible three-dimensional generalization of the triangle boards is obtained by generalizing the following property of the two-dimensional triangle boards. The number of cells in each row of a two-dimensional triangle board is equal to the row number. We generalize this property by letting the number of cells in a tower over a fixed row and column be equal to the maximum of the row and column number. In terms of the coordinates, the cells in the size m three-dimensional triangle board are of the form (i, j, k) with $1 \leq k \leq \max\{i, j\}$ and $1 \leq i, j \leq m$. The rook numbers of these boards are related to the Genocchi numbers, hence we call this family the *Genocchi boards*. Below is the depiction of the size 5 Genocchi board turned upside down and rotated for clarity. From the picture, we can see that the complement of the size m Genocchi board inside the $m \times m \times m$ cube is the size $m-1$ triangle board.



Using Maple, we generated rook numbers for various Genocchi boards. We found that the number of ways to place m rooks on a board of size m corresponds with the unsigned $(m+1)$th Genocchi number.

m	0	1	2	3	4	5	6	7
r_m	1	1	3	17	155	2073	38227	929569
G_{m+1}	-1	1	-3	17	-155	2073	-38227	929569

Sequence A001469 in [Sloane 2009], Genocchi numbers (of first kind)

Theorem. *The number of ways to place m non-attacking rooks on a size m Genocchi board is the unsigned $(m+1)$th Genocchi number.*

Proof: Recall that the complement of a size m Genocchi board in an $m \times m \times m$ cube is a size $m-1$ triangle board. Hence, using the theorem of complementary boards, we can calculate the number of ways to place m rooks on a size m Genocchi board in terms of the rook numbers of the triangle board. Recall that r_k for a size m triangle board is $T(m+1, m+1-k)$ and that the number of ways to place k rooks on the complement of a three-dimensional board B in the $m \times m \times m$ is

$$\sum_{i=0}^{k} \binom{m-i}{k-i}^3 (k-i)!^2 r_i(B)\,.$$

Using these two formulas, we find that the number of ways to place $k = m$ rooks on a size m Genocchi board is

$$\sum_{i=0}^{m-1} \binom{m-i}{m-i}^3 (m-i)!^2 T(m, m-i)\,.$$

We omitted the term corresponding to $i = m$ in the summation because $r_m(B) = 0$ for the triangle board of size $m-1$. This last summation can be rewritten via a change of variables $j = m - i$ as

$$(-1)^{m+1} \sum_{j=1}^{m} (-1)^{j+1} j!^2 T(m, j)$$

which is shown to equal $(-1)^{m+1} G_{m+1}$ in [Dumont 1973], thus the number of ways to place m rooks on a size m Genocchi board is the unsigned $(m+1)$th Genocchi number.

□

Appendix

```
Rook:=proc(A,m,n,p,B,k,rem)
local C,i,j,h,g,l,count,v;
count:=0;
if k=1 then
 for i from 1 to m do
  for j from 1 to n do
   for g from 1 to p do
    if 'not'('in'([i,j,g],B)) then
     if add(add(A[i,a1,a2],a1=1..n),a2=1..p)=0 then
      if add(add(A[b1,j,b2],b1=1..m),b2=1..p)=0 then
       if add(add(A[c1,c2,g],c1=1..m),c2=1..n)=0 then
        count:=count+1
       end if
      end if
     end if
    end if
   end do
  end do
 end do
else
C:=Array(1..m,1..n,1..p);
 for i from 1 to m do
  for j from 1 to n do
   for g from 1 to p do
    if 'not'('in'([i,j,g],B)) then
     if add(add(A[i,a1,a2],a1=1..n),a2=1..p)=0 then
      if add(add(A[b1,j,b2],b1=1..m),b2=1..p)=0 then
       if add(add(A[c1,c2,g],c1=1..m),c2=1..n)=0 then
        for h from 1 to m do
         for l from 1 to n do
          for v from 1 to p do
           C[h,l,v]:=A[h,l,v]
          end do
         end do
        end do
        C[i,j,g]:=1;
        count:=count+Rook(C,m,n,p,B,k-1);
        C[i,j,g]:=0
       end if
      end if
     end if
    end if
   end do
  end do
 end do
end if
count:=count/k
end proc:
```

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